

The Application of Optimal Homotopy Asymptotic Method for One-Dimensional Heat and Advection-Diffusion Equations

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Abstract: Aim of the paper is to investigate approximate analytical solution of time-dependent partial differential equation using a semi-analytical method, the Optimal Homotopy Asymptotic Method (OHAM). To show the efficiency of the proposed method, we consider one-dimensional heat and advection-diffusion equations. OHAM uses simple computations with pretty good enough approximate solution, which has an excellent agreement with the exact solution available in open literature. OHAM is not only reliable in obtaining series solution for such problems with high accuracy but it also saving the volume and time as compared to other analytical methods.

Keywords: Optimal Homotopy Asymptotic Method, Heat Equation, Advection-Diffusion Equation

1 Introduction

Consider the one dimensional advection-diffusion equation [4]:

$$u_t + \beta u_x = \alpha u_{xx}, \quad a \leq x \leq b, \quad t \geq 0. \quad (1)$$

subject to the initial condition:

$$u(x, 0) = \phi(x), \quad x \in [a, b] \quad (2)$$

and the boundary conditions are:

$$\begin{cases} u(a, t) = g_0(t) \\ u(b, t) = g_1(t), \quad t \in [0, T] \end{cases} \quad (3)$$

where the subscripts t and x denote differentiation with respect to time and space respectively, and are supposed to be smooth functions. In case of g_0, g_1 the advection-diffusion equation will reduced into one-dimensional heat equation is considered as thermal diffusion.

In case of $\beta = 0$, the advection-diffusion equation will reduced into one-dimensional heat equation is considered as thermal diffusion.

The Advection-Diffusion Equation (ADE) is of primary importance in many physical systems, especially those involving fluid flow [1], one-dimensional version of the partial differential equations which describe advection-diffusion equation arise frequently in transferring mass, heat, energy and vorticity in chemistry and engineering [2]. Parlange [3] used ADE is to model water transport in soils, Caglar et al. [4] have utilized third-degree B-Spline function for the numerical solution of one dimensional heat equation, Mohebbi and Dehghan [5] have presented finite difference approximation and cubic C1-spline collocation method for the solution with fourth-order accuracy in both space and time variables $O(h^4, k^4)$. Cubic B-Spline Collocation Method for the numerical solution of one dimensional heat and advection-diffusion equations are well reported by Goh et al. [6].

A newly developed analytical method namely the optimal homotopy asymptotic method has recently been used to solve a wide class of physical problems. Marinca and Nicolae [7,8,9] used OHAM for solving nonlinear equations related to different physical phenomena. Also Marinca et al. [10] studied the thin film flow using OHAM. Iqbal et al. [11] provided the OHAM solutions of

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the linear and nonlinear Klein-Gordon equations, Islam et al. [12] applied OHAM for the asymptotic solutions of Couette and Poiseuille flows of a third grade fluid whilst Idrees et al. [13,14] and Mabood et al. [15,16] have utilized the proposed method (OHAM) effectively for different higher order boundary values problems.

According to the best of author's knowledge the heat modeling problem mentioned above has not been yet studied by optimal homotopy asymptotic method (OHAM).

2 Basic Formulation of OHAM

We review the basic principles of OHAM as developed by Marinca et al. [8]. Consider the following differential equation and boundary condition:

$$L((v(z,t)) + g(z,t) + N(v(z,t)) = 0, \quad z \in \Omega \quad (4)$$

$$B\left(v, \frac{dv}{dt}\right) = 0 \quad (5)$$

where L, N are linear and nonlinear operators, z, t denote the spatial and time variables respectively, Ω is the problem domain and $v(z,t)$ is an unknown function, $g(z,t)$ is a known function and B is a boundary operator.

An equation known as optimal homotopy equation is constructed:

$$(1-r)[L(\phi(z,t;r) + g(z,t)] = H(r)[L(\phi(z,t;r) + g(z,t) + N(\phi(z,t;r))] \quad (6)$$

where $0 \leq r \leq 1$ is an embedding parameter, $H(r)$ is auxiliary function such that $H(r) \neq 0$ for $r \neq 0$ and $H(0) = 0$, we have from Eq.(6)

$$r = 0 \Rightarrow [L(\phi(z,t;0) + g(z,0))] = 0 \quad (7)$$

$$r = 1 \Rightarrow [L(\phi(z,t;1) + g(z,1) + N(\phi(z,t;1)))] = 0 \quad (8)$$

Thus, for $r = 0$ and $r = 1$ we obtain, $\phi(z,t;0) = v_0(z,t)$ and $\phi(z,t;1) = v(z,t)$ respectively. Hence, as r varies from 0 to 1 the solution $\phi(z,t;r)$ varies from $v_0(z,t)$ to the solution $v(z,t)$, where $v_0(z,t)$ is obtained from Eq. (6) set $r = 0$

$$L((v_0(z,t)) + g(z,t) = 0, \quad B\left(v_0, \frac{dv_0}{dt}\right) = 0 \quad (9)$$

The auxiliary function $H(r)$ is chosen of the form:

$$H(r) = \sum_{k=1}^n r^k C_k \quad (10)$$

where $C_i, i \in N$ are constants which are to be determined later [8].

For solution, expand $\phi(z,t;r,C_i)$ in Taylor's series about r and written as:

$$\phi(z,t;r,C_i) = v_0(z,t) + \sum_{k=1}^{\infty} v_k(z,t;C_i), \quad i = 1, 2, \dots \quad (11)$$

Substituting equation (11) into equation (6), and equating the coefficients of the like powers of r equal to zero, gives the linear equations as described below:

The zeroth order problem is given by equation (9), and the first and second order problems are given by the equations (12) and (13), respectively, while the general governing equation for $v_k(z,t)$ is given in equation (14):

$$L(v_1(z,t)) = C_1 N_0(v_0(z,t)), \quad B\left(v_1, \frac{dv_1}{dt}\right) = 0 \quad (12)$$

$$L(v_2(z,t)) - L(v_1(z,t)) = C_2 N_0(v_0(z,t)) + C_1 [L(v_1(z,t)) + N_1(v_0(z,t), v_1(z,t))] \quad (13)$$

$$B\left(v_2, \frac{dv_2}{dt}\right) = 0$$

$$L(v_k(z,t)) - L(v_{k-1}(z,t)) = C_k N_0(v_0(z,t)) + \sum_{i=1}^{k-1} C_i [L(v_{k-i}(z,t)) + N_{k-i}(v_0(z,t), v_1(z,t), \dots, v_{k-1}(z,t))] \quad (14)$$

$$B\left(v_k, \frac{dv_k}{dt}\right) = 0, \quad k = 1, 2, \dots$$

where $N_m(v_0(z,t), v_1(z,t), v_2(z,t), \dots, v_m(z,t))$ is the coefficient of r^m in the expansion of $N(\phi(z,t;r,C_i))$ about the embedding parameter.

$$N(\phi(z,t;r,C_i)) = N_0(v_0(z,t)) + \sum_{k \geq 1} N_m(v_0, v_1, v_2, \dots, v_m) r^m \quad (15)$$

The convergence of the series (11) is dependent upon the auxiliary constants C_1, C_2, \dots . If it is convergent at $r = 1$, one has:

$$\tilde{v}(z,t;C_i) = v_0(z,t) + \sum_{i=1}^m v_i(z,t;C_i) \quad (16)$$

Substituting equation (16) into equation (4), the general problem results in the following residual:

$$R(z,t;C_i) = L(\tilde{v}(z,t;C_i)) + g(z,t) + N(\tilde{v}(z,t;C_i)) \quad (17)$$

If $R(z,t;C_i) = 0$, then \tilde{v} will be the exact solution. For nonlinear problems, generally this will not be the case. For determining $C_i (i = 1, 2, \dots)$, a and b are chosen such that the optimum values for C_i are obtained, using the method of least squares:

$$J(C_i) = \int_0^t \int_{\Omega} R^2(z,t,C_i) dz dt \quad (18)$$

where R is the residual,

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \dots = \frac{\partial J}{\partial C_m} = 0 \quad (19)$$

3 Solution of Heat Equation via OHAM

Consider the heat equation is as follow [6]:

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0 \quad (20)$$

with initial condition is

$$u(x, 0) = \sin(\pi x) \quad (21)$$

and boundary conditions are

$$u(0, t) = u(1, t) = 0 \quad (22)$$

The exact solution is $u(x, t) = e^{-\pi^2 t} \sin(\pi x)$ which satisfies equation (20).

Applying the proposed method (OHAM) mentioned in Section 2, on equation (20) leads to the following:

Zeroth order problem:

$$\frac{\partial u_0(x, t)}{\partial t} = 0 \quad (23)$$

with initial condition: $u_0(x, 0) = \sin(\pi x)$

Its solution is

$$u_0(x, t) = \sin(\pi x) \quad (24)$$

First order problem:

$$\frac{\partial u_1(x, t, C_1)}{\partial t} = (1 + C_1) \frac{\partial u_0(x, t)}{\partial t} - C_1 \frac{\partial^2 u_0(x, t)}{\partial x^2} \quad (25)$$

with initial condition: $u_1(x, 0) = 0$

Its solution is

$$u_1(x, t, C_1) = C_1 \pi^2 t \sin(\pi x) \quad (26)$$

Second order problem:

$$\begin{aligned} \frac{\partial u_2(x, t, C_1, C_2)}{\partial t} &= (1 + C_1) \frac{\partial u_1(x, t, C_1)}{\partial t} \\ &- C_1 \frac{\partial^2 u_1(x, t, C_1)}{\partial x^2} + C_2 \frac{\partial u_0(x, t)}{\partial t} + C_2 \frac{\partial u_0(x, t)}{\partial x} \end{aligned} \quad (27)$$

with initial condition: $u_2(x, 0) = 0$

Its solution is

$$\begin{aligned} u_2(x, t, C_1, C_2) &= \frac{1}{2} [2C_1 \pi^2 t \sin(\pi x) + 2C_1^2 \pi^2 t \sin(\pi x) \\ &+ C_1^2 \pi^4 t^2 \sin(\pi x) + 2C_2 \pi t \cos(\pi x)] \end{aligned} \quad (28)$$

Third order problem:

$$\begin{aligned} \frac{\partial u_3(x, t, C_1, C_2, C_3)}{\partial t} &= (1 + C_1) \frac{\partial u_2(x, t, C_1, C_2)}{\partial t} \\ &+ C_3 \frac{\partial u_0(x, t)}{\partial t} + C_2 \frac{\partial u_1(x, t, C_1)}{\partial t} - C_3 \frac{\partial^2 u_0(x, t)}{\partial x^2} \\ &- C_2 \frac{\partial^2 u_1(x, t, C_1)}{\partial x^2} - C_1 \frac{\partial^2 u_2(x, t, C_1, C_2)}{\partial x^2} \end{aligned} \quad (29)$$

with initial condition: $u_3(x, 0) = 0$

Its solution is

$$\begin{aligned} u_3(x, t, C_1, C_2, C_3) &= \frac{1}{6} [6C_1 \pi^2 t \sin(\pi x) + 12C_1^2 \pi^2 t \sin(\pi x) \\ &+ 6C_1^2 \pi^4 t^2 \sin(\pi x) + 6C_1^3 \pi^2 t \sin(\pi x) \\ &+ 6C_1^3 \pi^4 t^2 \sin(\pi x) + C_1^3 \pi^6 t^3 \sin(\pi x) \\ &+ 6C_2 \pi t \cos(\pi x) + 6C_1 C_2 \pi t \cos(\pi x) \\ &+ 3C_1 C_2 \pi^3 t^2 \cos(\pi x) + 6C_3 \pi^2 t \sin(\pi x) \\ &+ 3C_1 C_2 \pi^4 t^2 \sin(\pi x) \\ &+ 6C_1 C_2 \pi^2 t \sin(\pi x)] \end{aligned} \quad (30)$$

Using equations (24), (26), (28) and (30), the third order approximate solution via OHAM for $r = 1$ is

$$\begin{aligned} \tilde{u}(x, t, C_1, C_2, C_3) &= u_0(x, t) + u_1(x, t, C_1) + u_2(x, t, C_1, C_2) \\ &+ u_3(x, t, C_1, C_2, C_3) \end{aligned} \quad (31)$$

With the help of least square method, we can obtain the values of unknown constants, for $t = 0.0001$ the values of $C_1 = 0.003639$, $C_2 = 0.00007461$, $C_3 = 0.001725$ and substituting the values of C_1, C_2, C_3 in equation (31), we obtain the approximate solution of heat equation as follow:

$$\begin{aligned} \tilde{u}(x, t) &= [0.04194 - 0.02048t] t \cos(\pi x) [1 + t(-4.03304 \\ &+ (4.10617 - 0.9361011t)t)] \sin(\pi x) \end{aligned} \quad (32)$$

Figs.1, 2 and 3 have been prepared for the comparative picture of the series solution obtained using OHAM with the existing exact solution for different assigned values of t .

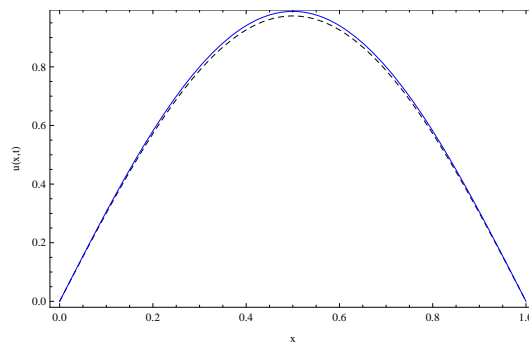


Fig. 1: Comparison of solution using OHAM (solid line) with exact solution (dashed line) for $t = 0.001$

4 Solution of Advection-Diffusion Equation via OHAM

The advection-diffusion equation with $\beta = 1, \alpha = 0.1$ in equation (1) is as follow [4]:

$$u_t + u_x = 0.1u_{xx}, \quad 0 < x < 1, \quad t > 0 \quad (33)$$

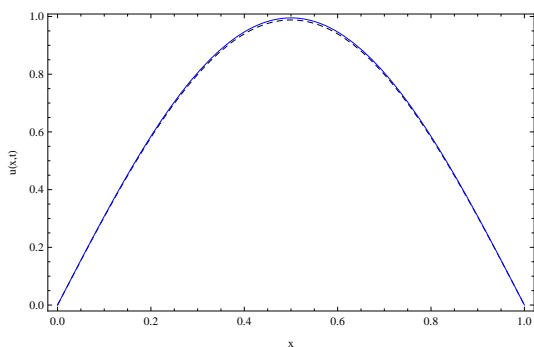


Fig. 2: Comparison of solution using OHAM (solid line) with exact solution (dashed line) for $t = 0.00125$

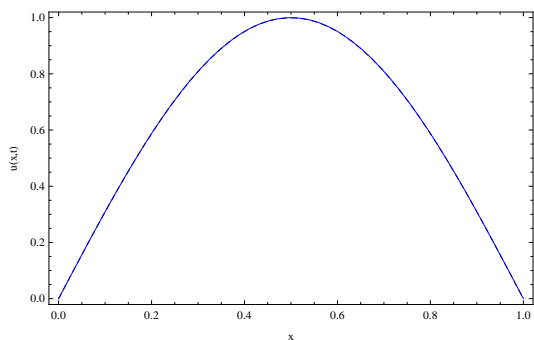


Fig. 3: Comparison of solution using OHAM (solid line) with exact solution (dashed line) for $t = 0.0001$

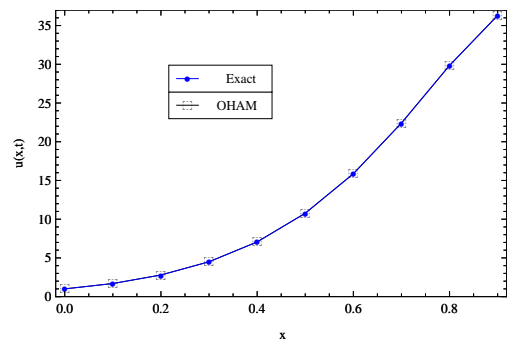


Fig. 4: Comparison of solution using OHAM with exact solution

with initial condition is

$$u(x, 0) = e^{5x} \left[\cos\left(\frac{\pi x}{2}\right) + 0.25 \sin\left(\frac{\pi x}{2}\right) \right] \quad (34)$$

The exact solution of equation (33) is

$$u(x, t) = e^{5(x-\frac{t}{2})} e^{\left(\frac{-\pi^2 t}{40}\right)} \left[\cos\left(\frac{\pi x}{2}\right) + 0.25 \sin\left(\frac{\pi x}{2}\right) \right] \quad (35)$$

The boundary conditions can be obtained easily at $x = 0$ and $x = 1$ from the exact solution. Applying the proposed

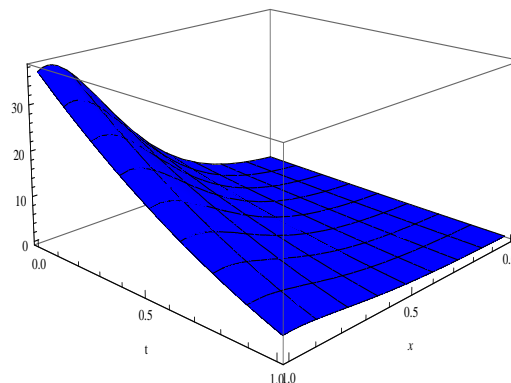


Fig. 5: Spatial-time approximation for Advection-Diffusion Equation over a time period $t \in [0, 1.2]$

method (OHAM) on Eq. (33), the **zeroth, first and second** order problem are given as:

$$\frac{\partial u_0(x, t)}{\partial t} = 0 \quad (36)$$

with boundary condition:
 $u_0(x, 0) = e^{5x} \left[\cos\left(\frac{\pi x}{2}\right) + 0.25 \sin\left(\frac{\pi x}{2}\right) \right]$

$$\begin{aligned} \frac{\partial u_1(x, t, C_1)}{\partial t} &= (1 + C_1) \frac{\partial u_0(x, t)}{\partial t} + C_1 \frac{\partial u_0(x, t)}{\partial x} \\ &\quad - 0.1 C_1 \frac{\partial^2 u_0(x, t)}{\partial x^2} \end{aligned} \quad (37)$$

with initial condition: $u_1(x, 0) = 0$

$$\begin{aligned} \frac{\partial u_2(x, t, C_1, C_2)}{\partial t} &= (1 + C_1) \frac{\partial u_1(x, t, C_1)}{\partial t} + C_1 \frac{\partial u_1(x, t, C_1)}{\partial x} \\ &\quad + C_2 \frac{\partial u_0(x, t)}{\partial t} - 0.1 C_1 \frac{\partial^2 u_1(x, t)}{\partial x^2} \\ &\quad + C_2 \frac{\partial u_0(x, t)}{\partial x} - 0.1 C_2 \frac{\partial^2 u_0(x, t)}{\partial x^2} \end{aligned} \quad (38)$$

with initial condition: $u_2(x, 0) = 0$

Solving equations (36), (37) and (38), we can obtain second order approximate solution using OHAM for $r = 1$ is

$$\tilde{u}(x, t, C_1, C_2) = u_0(x, t) + u_1(x, t, C_1) + u_2(x, t, C_1, C_2) \quad (39)$$

Using the method of least square, the values of unknown constants for $t = 0.0025$ are $C_1 = -0.36295, C_2 = 0.06096$.

Substituting the values of C_1, C_2 in equation (39) one can obtain the three terms approximate analytical solution via OHAM for advection-diffusion equation. Fig. 4 has been presented for comparison of OHAM solution with exact solution of advection-diffusion equation whilst in Fig. 5 we have shown the spatial-time approximation.

5 Conclusion

A series solution based on Optimal Homotopy Asymptotic Method had been described in section 3 and 4 for solving one-dimensional heat and advection-diffusion equations. The obtained solution using OHAM is then compared with the exact solutions. The results are in good agreement with the existing exact results and therefore elucidate the reliability and efficiency of OHAM. The comparisons made suggest that the OHAM could be a useful and effective tool for solving one-dimensional heat and advection-diffusion equations accurately.

References

- [1] M. Dehghan, Weighted finite difference techniques for the one-dimensional advection-diffusion equation, *Applied Mathematics and Computation*. **147**, 307-319 (2004).
- [2] B.J. Noye, Numerical solution of partial differential equations, Lecture Notes (1990).
- [3] H. Caglar, M. Ozer and N. Caglar, The numerical solution of the one-dimensional heat equation by using third degree B-spline functions, *Chaos, Solitons and Fractals*. **38**, 1197-1201 (2008).
- [4] A. Mohebbi and M. Dehghan, High-order compact solution of the one-dimensional heat and advection-diffusion equations, *Applied Mathematical Modelling*. **34**, 3071-3084 (2010).
- [5] J.Y. Parlange, Water transport in soils, *Ann Rev Fluids Mech.*, **2**, 77-102 (1980).
- [6] J. Goh, Ahmad Abd. Majid and Ahmad Izani Md Ismail, Cubic B-Spline Collocation Method for One-Dimensional Heat and Advection-Diffusion Equations, *Journal of Applied Mathematics*. Article ID **458701**, 8 (2012).
- [7] V. Marinca and N. Herisanu, Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer, *Int. Commun. Heat Mass Transfer*, **35**, 710-715 (2008).
- [8] V. Marinca and N. Herisanu, Optimal homotopy perturbation method for strongly nonlinear differential equations, *Nonlinear Sci. Lett. A*, **1**, 273-280 (2010).
- [9] N. Herisanu and V. Marinca, Accurate analytical solutions to oscillators with discontinuities and fractional-power restoring force by means of the optimal homotopy asymptotic method, *Computers and Mathematics with Applications*, **60**, 1607-1615 (2010).
- [10] V. Marinca, N. Herisanu and I. Nemes, Optimal homotopy asymptotic method with application to thin film flow, *Cent. Eur. J. Phys.* **6**, 648-653 (2008).
- [11] S. Iqbal, M. Idrees, A.M. Siddiqui, A.R. Ansari, Some solutions of the linear and nonlinear Klein-Gordon equations using the optimal homotopy asymptotic method, *Applied Mathematics and Computation*. **216**, 2898-2909 (2010).
- [12] S. Islam, R. A. Shah, I. Ali, Optimal homotopy asymptotic solutions of Couette and Poiseuille flows of a third grade fluid with heat transfer analysis, *Int. J. Nonlinear Sci. Numer. Simul.* **11**, 389-400 (2010).
- [13] M. Idrees, S.I Haq, S. Islam, Application of optimal homotopy asymptotic method to Fourth Order Boundary Values Problems. *World Applied Sciences Journal*. **9**, 131-137 (2010).
- [14] M. Idrees, S. Haq, S. Islam, Application of Optimal Homotopy Asymptotic Method to special Sixth Order Boundary Values Problems, *World Applied Sciences Journal*. **9**, 138-143 (2010).
- [15] F. Mabood, W. A. Khan, A. I. M. Ismail, Solution of Fifth Order Boundary Values Problems via Optimal Homotopy Asymptotic Method, *Wulfenia Journal*. **19**, 103-115 (2012).
- [16] F. Mabood, A.I.M. Ismail, I. Hashim. The application of Optimal Homotopy Asymptotic Method for the Approximate Solution of Riccati Equation, *Sains Malaysiana*. **42**, (2013) (In-Press).