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Recursive Construction of the Bosonic Bogoliubov Vacuum State

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Abstract: In this work we derive a novel procedure for obtaining the bosonic Bogoliubov vacuum states by using a recursive scheme. The vacuum state for the new creation and annihilation operators is explicitly constructed in terms of the number states of the old operators, which are connected by a Bogoliubov transformation. The coefficients of the ground state in Fock basis are thus obtained as exclusive functions of the parameters of the Bogoliubov transformations.

Keywords: quantum physics, bosonic quantum systems, bosonic vacuum states, Bogoliubov transformation

1 Introduction

The Bogoliubov transformation has been used as a powerful tool in studying the properties of various quantum systems [1,2]. One of the advantages in using this method is that linear canonical transformations provide the exact diagonalization of quadratic multidimensional Hamiltonians [3,4]. It is well known that the problem of finding the Hamiltonian eigenvalues and eigenvectors for a set of coupled harmonic oscillators can be solved in analogy with the classical case, when a Bogoliubov transformation is performed. In order to obtain the complete solution, one needs the quantum state transformations from the base states of the old operator to those of the new one. Ultimately, this is equivalent to obtain the relations between linear canonical transformations and the related unitary operators in Hilbert space, which were extensively studied for bosonic and fermionic operators by many authors [5,6,7].

The unitary operator corresponding to the Bogoliubov transformation has no classical equivalent and its study contributes to understanding remarkable quantum aspects. Particularly, it is theoretically relevant the bosonic Bogoliubov vacuum, or bosonic ground states. For example, in Bose-Einstein condensate state [8,9], and in the study of vacuum structure of de Sitter space [10].

In the coherent state representation, the ground state is given by a Gaussian function [13]. However, in some situations it turns to be convenient to have the ground states in the Fock space, given in terms of the number states of the old creation and annihilation operators (i. e., before the Bogoliubov transformation is carried out). In principle, it is possible to construct the Bogoliubov vacuum state in the occupation number representation by simply changing the basis. However this is a very cumbersome task which demands a huge work.

In this work we give explicit formulas for the coefficients $c_{n_1,n_2,\dots} = \langle n_1,n_2,\dots,0\rangle_b$, appearing in the expansion of the Bogoliubov transformed vacuum with respect to the original number of particles basis. The formulas we propose allow a recursive computation of the coefficients $c_{n_1,n_2,\dots}$ based only on the parameters characterizing the Bogoliubov transformation. The proposed new formulas are an alternative calculation of standard formulas based on coherent states and Gaussian integration.

The work is organized as follows. In Section 2, we briefly discuss the properties of the Bogoliubov transformation of creation and annihilation bosonic operators. In Section 3, we present a recursive method of calculation of the coefficients of the Bogoliubov transformation. In Section 4 we present the conclusions.

2 The Bogoliubov transformation

Let us consider a bosonic Fock space, with creation and annihilation operators, $\mathbf{a}_j^\dagger$ and $\mathbf{a}_j$, satisfying canonical

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commutation relations. An orthonormal basis for the Fock space is given by vectors $|n_1, n_2, \ldots\rangle$, where the integers n_j specify how many particles occupy the j-th one-particle state.

The linear canonical transformation acting on creation and annihilation bosonic operators was first introduced by N. Bogoliubov in 1947 [11,12]. In such transformation the new creation ($\mathbf{b}_i$) and annihilation ($\mathbf{b}_i^\dagger$) operators are related to the corresponding old operators $\mathbf{a}_i$ and $\mathbf{a}_i^\dagger$ through,

$$\mathbf{b}_i = \sum_{j=1}^{s} \left(\mu_{ij}\mathbf{a}_j + \nu_{ij}\mathbf{a}_j^\dagger \right), \tag{1}$$

$$\mathbf{b}_i^\dagger = \sum_{j=1}^{s} \left(\bar{\nu}_{ij}\mathbf{a}_j + \bar{\mu}_{ij}\mathbf{a}_j^\dagger \right), \tag{2}$$

where $i = 1, \ldots, s$, being s the number of oscillators. Using matrix notation, we may write

$$\mathbf{b} = \mu\mathbf{a} + \nu\mathbf{a}^\dagger, \tag{3}$$

$$\mathbf{b}^\dagger = \bar{\nu}\mathbf{a} + \bar{\mu}\mathbf{a}^\dagger, \tag{4}$$

where we have denoted the matrices $\mu = [\mu_{jk}]_{s\times s}$, $\nu = [\nu_{jk}]_{s\times s}$, $\mathbf{a} = [\mathbf{a}_i]_{s\times 1}$, etc. The commutation relations satisfied by the bosonic operators

$$[\mathbf{b}_j, \mathbf{b}_k^\dagger] = \delta_{jk}, \quad [\mathbf{b}_j, \mathbf{b}_k] = 0, \tag{5}$$

yield the following two matrix relations:

$$\mu\mu^\dagger - \nu\nu^\dagger = 1, \tag{6}$$

and

$$\mu\nu^T - \nu\mu^T = 0. \tag{7}$$

The number states in the Fock basis can be written in terms of the old vacuum states, namely

$$|n_1 \ldots n_s\rangle_a = \prod_{j=1}^{s} \frac{1}{\sqrt{n_j!}} \left(\mathbf{a}_j^\dagger \right)^{n_j} |0, \ldots, 0\rangle_a, \tag{8}$$

where $\{|n_1, \ldots, n_s\rangle_a\}$ represent the eigenvectors of the old number operator $\mathbf{a}^\dagger \mathbf{a}_i$.

In the same way, the construction of the number states in the Fock basis for the new number operator $\mathbf{b}^\dagger \mathbf{b}_j$ can be expressed in terms of the vacuum state $|0\rangle_b = |0, \ldots, 0\rangle_b$. Notice that $|0\rangle_b$ are not necessarily the ground states. By definition, the minimum energy state correspond to the vacuum state associated with a transformation that diagonalizes the system Hamiltonian. In fact, any quadratic multidimensional Hamiltonian can be written as

$$\mathbf{H} = \sum_i E_i \mathbf{b}_i^+ \mathbf{b}_i, \tag{9}$$

from an appropriate Bogoliubov transformation.

In view of the discussion in the previous section, we can ask how to obtain the vacuum state $|0\rangle_b$ in terms of the old number states $|n_1, \ldots, n_s\rangle_a$ given in Eq. (8). From the formal point of view, this may be written in the form

$$|0\rangle_b = \sum_{n_1 \ldots n_s} c_{n_1 \ldots n_s} |n_1, \ldots, n_s\rangle_a. \tag{10}$$

The explicit form of $|0\rangle_b$ obtained in the coherent representation is given by [13],

$$_a\langle\alpha_1, \ldots, \alpha_s|0\rangle_b = \left[\det\left(\mu^\dagger\mu\right)\right]^{-1/4} \times \exp\left(-\frac{1}{2}\sum_i \bar{\alpha}_i\alpha_i + \frac{1}{2}\sum_{ij} \sigma_{ij}\bar{\alpha}_i\bar{\alpha}_j \right), \tag{11}$$

where $|\alpha_i\rangle_a$ represent the coherent states associated with the operator $\mathbf{a}_i$ ($\mathbf{a}_i|\alpha_i\rangle = \alpha_i|\alpha_i\rangle$), where the matrix σ is defined as

$$\sigma = -\mu^{-1}\nu. \tag{12}$$

Notice that the matrix μ is invertible, since from Eq. (6) follows that $\det(\mu) \neq 0$. It is not difficult to obtain $|0\rangle_a$, if we assume $\mu = 1$ and $\nu = 0$ (with $b = a$) in Eq. (11). Thus, we get

$$_a\langle\alpha_1, \ldots, \alpha_s|0\rangle_a = \exp\left(-\frac{1}{2}\sum_i \bar{\alpha}_i\alpha_i \right) \tag{13}$$

In terms of the number states $|n_1, \ldots, n_s\rangle_a$ relative to the old creation and annihilation operators, the change of basis of Eq. (13) yields

$$c_{n_1,\ldots,n_s} = \left[\det\left(\mu^\dagger\mu\right)\right]^{-1/4} \int \Pi_i \left(\frac{d^2\alpha_i}{\pi} \frac{\bar{\alpha}_i^{n_i}}{\sqrt{n_i!}} \right) \times \exp\left[-\sum \bar{\alpha}_i\alpha_i - \frac{1}{2}\sum_{ij} \sigma_{ij}\bar{\alpha}_i\bar{\alpha}_j \right], \tag{14}$$

where the integral in the right hand side of the last equation is carried out over the whole complex plane.

So, we have a close relationship for the coefficients. However, calculating the integral on the right side of the last equation, using analytical or numerical methods, it is usually a difficult task to be performed. So is the determination of values for $c_{n_1,\ldots,n_s}$. In particular, for $n_i = 0$, $i = 1, \ldots, s$, the last equation provides

$$c_{0,\ldots,0} = \left[\det\left(\mu^\dagger\mu\right)\right]^{-1/4}. \tag{15}$$

It is useful to obtain the vacuum $|0\rangle_b$ independent representation, which is given by

$$|0\rangle_b = \left[\det\left(\mu^\dagger\mu\right)\right]^{-1/4} \exp\left(\frac{1}{2}\sum_{ij} \sigma_{ij}\mathbf{a}_i^+\mathbf{a}_j^+ \right) |0\rangle_a. \tag{16}$$

In order to verify Eq. (16), we just multiply it by $_a\langle\alpha_1, \ldots, \alpha_s|$ and then use Eq. (13). It is also possible to obtain the coefficients $\{c_{n_1,\ldots,n_s}\}$ by means of a expansion in power series of Eq. (16). For example, for $s = 1$, the direct calculation leads to

$$c_n = \langle n|0\rangle_b = \frac{1}{\sqrt{|\mu|}} \left(-\frac{\nu}{2\mu} \right)^{n/2} \frac{\sqrt{n!}}{(n/2)!}. \tag{17}$$

Nevertheless, for $s > 1$ the calculation of the coefficients $\{c_{n_1,\ldots,n_s}\}$ by expansion in power series of Eq. (16) is not easy to carry out, and this justifies the search for an alternative method for calculating the coefficients.

3 Description of the method

We want to show that the vacuum state $|0\rangle_b$ given in the Fock basis can be constructed avoiding the use of Eq. (14). We then take the expression

$$ {}_b\langle 0|\mathbf{b}_j^\dagger|n_1,\ldots,n_s\rangle_a = 0. \tag{18}$$

But from Eqs. (1) and (2), this equation can be put in the form

$$\sum_j {}_b\langle 0|\left(\bar{\mu}_{ij}\mathbf{a}_j^\dagger + \bar{\nu}_{ij}\mathbf{a}_j\right)|n_1,\ldots,n_s\rangle_a = 0, \tag{19}$$

from which we get

$$\sum_j \Big(\mu_{ij}\sqrt{n_j+1}c_{n_1,\ldots,n_j+1,\ldots,n_s} + \nu_{ij}\sqrt{n_j}c_{n_1,\ldots,n_j-1,\ldots,n_s}\Big) = 0. \tag{20}$$

The last equation can also be written in the matrix form

$$\begin{pmatrix} c_{n_1+1,\ldots,n_s}\sqrt{n_1+1} \\ \vdots \\ c_{n_1,\ldots,n_s+1}\sqrt{n_s+1} \end{pmatrix} = \sigma \begin{pmatrix} c_{n_1-1,\ldots,n_s}\sqrt{n_1} \\ \vdots \\ c_{n_1,\ldots,n_s-1}\sqrt{n_s} \end{pmatrix}, \tag{21}$$

From Eq. (7) follows

$$\nu = \mu\nu^T\left(\mu^T\right)^{-1}, \tag{22}$$

and

$$\mu^{-1}\nu = \left(\mu^{-1}\nu\right)^T. \tag{23}$$

So, from Eq. (12) together with the two last equations, one shows that σ is a symmetric matrix. Thus, Eq. (21) provides s equations in the form of

$$c_{n_1,\ldots,n_i+1,\ldots,n_s} = \sum_{j=1}^{s}\sigma_{ij}\sqrt{\frac{n_j}{n_i+1}}c_{n_1,\ldots,n_j-1,\ldots,n_s} \quad (i=1,\ldots,s), \tag{24}$$

which yields the recursive equations

$$c_{n_1,\ldots,n_s} = \sigma_{ii}\sqrt{\frac{n_i-1}{n_i}}c_{n_1,\ldots,n_i-2,\ldots,n_s} + \sum_{j\neq i}^{s}\sigma_{ij}\sqrt{\frac{n_j}{n_i}}c_{n_1,\ldots,n_i-1,\ldots,n_j-1,\ldots,n_s}. \tag{25}$$

Eq. (24) is the natural way to obtain the vacuum $|0\rangle_b$ in Fock basis. Our goal in this work is to obtain, from Eq. (24), an explicit expression for any of the coefficients $c_{n_1,\ldots,n_s}$ given as a function of the parameters of the Bogoliubov transformation. Initially, we will distribute the coefficients into sets $C_N = \{c_{n_1,\ldots,n_s}|n_1+\ldots+n_s = N\}$, where N is a nonnegative integer. We point out that the coefficients $c_{n_1\ldots n_s} \in C_N$ are generated from the coefficients $c_{n_1'\ldots n_s'} \in C_{N-2}$. So we must conclude that we have two independent groups of coefficients: $c_{n_1\ldots n_s} \in C_N$, for N an even integer, and $c_{n_1\ldots n_s} \in C_N$, for N an odd integer. It is very important to note that none of the members of a group relates to the elements of another group.

The starting values of the odd group are s coefficients $c_{0,\ldots,1,\ldots,0} \in C_1$, which can be determined from Eq. (20):

$$\mu\begin{pmatrix} c_{1,0\ldots,0} \\ \vdots \\ c_{0,\ldots,0,1}\end{pmatrix} = 0. \tag{26}$$

Since the matrix μ is always invertible, we have

$$c_{0,\ldots,n_i=1,\ldots,0} = 0 \tag{27}$$

for every i. Therefore, for $n_1+\ldots+n_s$ an odd integer number one has

$$c_{n_1,\ldots,n_s} = 0. \tag{28}$$

When considering the even group, we see that the coefficient $c_{0,\ldots,0}$ is the only element of the set C_0, and so it is taken as the initial value. However, it can not be determined by Eq. (20), and its value yet determined by Eq. (15). Thus, it is possible obtaining $c_{n_1\ldots n_s} \in C_N$, for every even N from Eq. (25) using the starting value $c_{0,\ldots,0} = \left[\det\left(\mu^\dagger\mu\right)\right]^{-1/4}$.

In order to derive the general expression, we need first define a permutation group particularly useful in solving the proposed problem. So, suppose the product

$$\sigma_{11}^{k_{11}}\sigma_{12}^{k_{12}}\ldots\sigma_{ss}^{k_{ss}}, \tag{29}$$

consisting of $N/2$ factors, where k_{ij} $(k_{ij} = k_{ji})$ are non-negative integers satisfying the condition

$$2k_{ii} + \sum_{j\neq i}^{s}k_{ij} = n_i. \tag{30}$$

Imposing the condition given by the last equation is equivalent to requiring that each index i must appear n_i times in the product (29).

Let us call $K_{n_1,\ldots,n_s}$ the set of all different configurations of k_{ij} satisfying Eq. (30). Formally, we can write

$$K_{n_1,\ldots,n_s} = \{k_{11},k_{12},\ldots,k_{ss})\mid 2k_{ii}+\sum_{j\neq i}^{s}k_{ij} = n_i, \quad k_{ij}\in\mathbb{N}\ \text{and} k_{ij} = k_{ji}\}. \tag{31}$$

The set $K_{n_1,\ldots,n_s}$ is equivalent to get all different products in (29) obtained by index permutation, preserving $j \geq i$ in σ_{ij}. For example, for $s=3$, with $n_1 = 4$, $n_2 = 1$, and $n_3 = 3$, we have the possibilities

$\{\sigma_{11}^2\sigma_{23}\sigma_{33}, \sigma_{11}\sigma_{12}\sigma_{13}\sigma_{33}, \sigma_{11}\sigma_{13}^2\sigma_{23}, \sigma_{13}^3\sigma_{12}\}$, which is equivalent to write

$$K_{4,1,3} = \{(k_{11}=2,\ k_{23}=1,\ k_{33}=1,\ k_{ij}=0,\ \text{in all other cases}),\ldots\}. \tag{32}$$

From $K_{n_1,\ldots,n_s}$ we can obtain directly $K_{n_1,\ldots,n_i-2,\ldots,n_s}$, not considering elements with $k_{ii}=0$ and, for $k_{ii}>0$, replace $k_{ii}\to k_{ii}-1$. Similarly, $K_{n_1,\ldots,n_i-1,\ldots,n_j-1,\ldots n_s}$ is obtained by deleting the elements $k_{ij}=0$ and, for $k_{ij}>0$, replace $k_{ij}\to k_{ij}-1$.

With $K_{n_1,\ldots,n_s}$ determined, we can now try to show that the recursive relation Eq. (25), together with Eq. (15), result in the following expression for the coefficients $c_{n_1,\ldots,n_s}$:

$$c_{n_1,\ldots,n_s} = \frac{\sqrt{n_1!\ldots n_s!}}{[\det(\mu^\dagger\mu)]^{1/4}} \times \sum_{K_{n_1,\ldots,n_s}} \prod_{i=1}^{s}\left(\frac{\sigma_{ii}^{k_{ii}}}{(2^{k_{ii}})\,k_{ii}!}\prod_{l>k}^{s}\frac{\sigma_{ij}^{k_{ij}}}{k_{ij}!}\right), \tag{33}$$

where the summation is carried out over all elements of $K_{n_1,\ldots,n_s}$. We have thus

$$c_{n_1,\ldots,n_i-2,\ldots,n_s} = \frac{\sqrt{n_1!\ldots n_s!}}{[\det(\mu^\dagger\mu)]^{1/4}\sqrt{n_i(n_i-1)}} \times \sum_{K_{n_1,\ldots,n_s}} \frac{2k_{ii}\sigma_{ii}^{k_{ii}-1}}{(2^{k_{ii}})\,k_{ii}!}\prod_{p\neq i}^{s}\left(\frac{\sigma_{pq}^{k_{pq}}}{(2^{k_{pq}})\,k_{pq}!}\prod_{q>p}^{s}\frac{\sigma_{pq}^{k_{pq}}}{k_{pq}!}\right). \tag{34}$$

Notice that $\sum_{K_{n_1,\ldots,n_s}}$ and $\sum_{K_{n_1,\ldots,n_l-2,\ldots,n_s}}$ represent different summations, but multiplication by k_{ii} will exclude any additional terms in $\sum_{K_{n_1,\ldots,n_s}}$. Thus, we arrive at

$$\sigma_{ii}\frac{\sqrt{n_i-1}}{\sqrt{n_i}}c_{n_1,\ldots,n_i-2,\ldots,n_s} = \frac{\sqrt{n_1!\ldots n_s!}}{[\det(\mu^\dagger\mu)]^{1/4}} \times \sum_{K_{n_1,\ldots,n_s}} \frac{2k_{ii}}{n_i}\prod_{p=1}^{s}\left(\frac{\sigma_{pq}^{k_{pq}}}{(2^{k_{pq}})\,k_{pq}!}\prod_{q>p}^{s}\frac{\sigma_{pq}^{k_{pq}}}{k_{pq}!}\right). \tag{35}$$

Analogously, we have

$$\sigma_{ij}\frac{\sqrt{n_j}}{\sqrt{n_i}}c_{n_1,\ldots,n_i-1,\ldots,n_j-1,\ldots,n_s} = \frac{\sqrt{n_1!\ldots n_s!}}{[\det(\mu^\dagger\mu)]^{1/4}} \times \sum_{K_{n_1,\ldots,n_s}} \frac{k_{ij}}{n_i}\prod_{p=1}^{s}\left(\frac{\sigma_{pq}^{k_{pq}}}{(2^{k_{pq}})\,k_{pq}!}\prod_{q>p}^{s}\frac{\sigma_{pq}^{k_{pq}}}{k_{pq}!}\right), \tag{36}$$

which leads directly to Eq. (25), since

$$\frac{1}{n_i}\left(2k_{ii}+\sum_{j\neq i}^{s}k_{ij}\right)=1, \tag{37}$$

concluding the proof.

4 Conclusions

In this work we derive a new alternative procedure where the vacuum state for the new creation and annihilation operators is constructed in terms of the number states of the old operators. The new and old creation and annihilation operators are connected by a Bogoliubov transformation, given by Eqs. (1) and (2). Formally, the new vacuum state can be written as in Eq. (10), which requires knowing the complete set of coefficients $\{c_{n_1,\ldots,n_s}\}$. We have shown that such coefficients can be obtained recursively from Eq. (33). It is worth to mention that the coefficients of the ground state in Fock basis are written as exclusive functions of the parameters of the Bogoliubov transformation.

Finally, it should be mentioned that formula (34) can be understood as a method to calculate the integral defined in the right-hand side of equation (14).

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