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New Generalization of Eulerian Polynomials and their Applications

Serkan Araci[1,*], *Mehmet Acikgoz*[1,*] *and Erdoğan Şen*[2,*]

[1] Department of Mathematics, Faculty of Science and Arts, University of Gaziantep, 27310 Gaziantep, Turkey
[2] Department of Mathematics, Faculty of Science and Letters, Namık Kemal University, 59030 Tekirdağ, Turkey

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Abstract: In the present paper, we introduce Eulerian polynomials with parameters a and b and give the definition of them. By using the definition of generating function for our polynomials, we derive some new identities in Analytic Numbers Theory. Also, we give relations between Eulerian polynomials with parameters a and b, Bernstein polynomials, Poly-logarithm functions, Bernoulli and Euler numbers. Moreover, we see that our polynomials at $a=-1$ are related to Euler-Zeta function at negative inetegers. Finally, we get Witt's formula for new generalization of Eulerian polynomials which we express in this paper.

Keywords: Eulerian polynomials, Poly-logarithm functions, Stirling numbers of the second kind, Bernstein polynomials, Bernoulli numbers, Euler numbers and Euler-Zeta function, p-adic fermionic integral on $\mathbb{Z}_p$.
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1 Introduction

The Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, Stirling numbers of the second kind, Bernstein polynomials and Eulerian polynomials possess many interesting properties not only in complex analysis, and analytic numbers theory but also in mathematical physics related to knot theory and ζ-function, and p-adic analysis. These polynomials have been studied by many mathematicians for a long time (for details, see [1-30]).

Eulerian polynomial sequence $\{\mathscr{A}_n(x)\}_{n\geq 0}$ is given by the following summation:

$$\sum_{l=0}^{\infty} l^n x^l = \frac{\mathscr{A}_n(x)}{(1-x)^{n+1}}, \quad |x|<1. \tag{1}$$

It is well-known that the Eulerian polynomial, $\mathscr{A}_n(x)$, of degree n can be introduced as

$$\mathscr{A}_n(x) = \sum_{k=1}^{n} \mathscr{A}(n,k)x^k, \quad \mathscr{A}_0(z)=1 \tag{2}$$

where $\mathscr{A}(n,k)$ are called the Eulerian numbers that can be computed by using

$$\mathscr{A}(n,k) = \sum_{j=0}^{k} \binom{n+1}{j} (-1)^j (k-j)^n, \quad 1\leq k\leq n, \tag{3}$$

where $\mathscr{A}(n,0)=1$. Eulerian polynomials, $\mathscr{A}_n(x)$, are also given by means of the following exponential generating function:

$$e^{\mathscr{A}(x)t} = \sum_{n=0}^{\infty} \mathscr{A}_n(x)\frac{t^n}{n!} = \frac{1-x}{e^{t(1-x)}-x} \tag{4}$$

where $\mathscr{A}^n(x) := \mathscr{A}_n(x)$, symbolically. Eulerian polynomials can be found via the following recurrence relation:

$$(\mathscr{A}(t)+(t-1))^n - t\mathscr{A}_n(t) = \begin{cases} 1-t, & \text{if } n=0 \\ 0, & \text{if } n\neq 0, \end{cases} \tag{5}$$

(for details, see [5], [6], [25], [9] and [10]).

Now also, we give the definition of Eulerian fraction, $\alpha_n(x)$, can be expressed as

$$\alpha_n(x) := \frac{\mathscr{A}_n(x)}{(1-x)^{n+1}}. \tag{6}$$

* Corresponding author e-mail: mtsrkn@hotmail.com, acikgoz@gantep.edu.tr, erdogan.math@gmail.com

We want to note that Eulerian fraction is very useful in the study of the Eulerian numbers, Eulerian polynomials, Euler function and its generalization, Jordan function in Number Theory (for details, see [16]). Firstly, Acikgoz and Araci introduced the generating function of Bernstein polynomials as follows:

$$\sum_{n=k}^{\infty} B_{k,n}(x)\frac{t^n}{n!}=\frac{(tx)^k}{k!}e^{t(1-x)},\ t\in\mathbb{C}, \tag{7}$$

where $B_{k,n}(x)$ are called Bernstein polynomials, which are defined by

$$B_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k},\ 0\le x\le 1, \tag{8}$$

(for details on this subject, see [8]).

The Poly-logarithms can be defined by the series:

$$Li_n(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^n} \tag{9}$$

for $n\ge 0$, and $|z|<1$. We easily see that if $n=0$

$$Li_0(z)=\frac{z}{1-z}.$$

Also, Poly-logarithms can be given by the integral representation, as follows:

$$Li_n(z)=\int_0^z\frac{Li_n(z)}{z}dz$$

in $\mathbb{C}\backslash[1,\infty)$. We note that $Li_1(z)=-\log(1-z)$ is the usual logarithm (see [27]).

In [28], [29], Luo *et al*. defined the generalization of the Bernoulli and Euler polynomials with parameters a,b,c as follows:

$$\frac{tc^{xt}}{b^t-a^t}=\sum_{n=0}^{\infty}\frac{B_n(x;a,b,c)}{n!}t^n,\ \left(\left|t\log\frac{b}{a}\right|<2\pi\right), \tag{10}$$

$$\frac{2c^{xt}}{b^t+a^t}=\sum_{n=0}^{\infty}\frac{E_n(x;a,b,c)}{n!}t^n,\ \left(\left|t\log\frac{b}{a}\right|<\pi\right). \tag{11}$$

So that, obviously,

$$B_n(x;1,e,e):=B_n(x)\ \text{and}\ E_n(x;1,e,e):=E_n(x). \tag{12}$$

Here $B_n(x)$ and $E_n(x)$ are the classical Bernoulli polynomials and the classical Euler polynomials, respectively.

Next, for the classical Bernoulli numbers, B_n and the classical Euler numbers, E_n we have

$$B_n(0):=B_n\ \text{and}\ E_n(0):=E_n. \tag{13}$$

By the same motivation of all the above generalizations, we consider, in this paper, the generalization of Eulerian polynomials and derive some new theoretical properties for them. Also, we show that our polynomials are related to poly-logarithm function, the Bernstein polynomials, Bernoulli numbers, Euler numbers, Genocchi numbers, Euler-Zeta function and Stirling numbers of the second kind. Finally, we get Witt's formula for new generalization of Eulerian polynomials which seems to be interesting for further work in p-adic analysis.

2 On the new generalization of Eulerian polynomials

In this section, we start by giving the following definition of new generalization of Eulerian polynomials.

Definition 1. *Let $b\in\mathbb{R}^+$(positive real numbers) and $a\in\mathbb{C}$(field of complex numbers), then we define the following:*

$$e^{t\mathscr{A}(a,b)}=\sum_{n=0}^{\infty}\mathscr{A}_n(a,b)\frac{t^n}{n!}=\frac{1-a}{b^{t(1-a)}-a} \tag{14}$$

where $\mathscr{A}_n(a,b)$ are called the generalization of Eulerian polynomials (or Eulerian polynomials with parameters a and b). Also, $\mathscr{A}^n(a,b):=\mathscr{A}_n(a,b)$, symbolically.

So that, obviously,

$$\mathscr{A}_n(x,e):=\mathscr{A}_n(x).$$

By (14), we have the following recurrence relation for the Eulerian polynomials with parameters a and b:

$$e^{t\mathscr{A}(a,b)}=\frac{1-a}{e^{t(1-a)\ln b}-a}.$$

By applying combinatorial techniques to the above equality, then we easily derive the following theorem:

Theorem 1. *The following recurrence relation holds:*

$$[\mathscr{A}_n(a,b)+(1-a)\ln b]^n-a\mathscr{A}_n(a,b)=(1-a)\delta_{n,0} \tag{15}$$

where $\delta_{n,0}$ is the Kronecker's symbol.

We now consider for $n>0$ in (15), becomes

$$\mathscr{A}_n(a,b)=\frac{1}{a-1}\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{A}_k(a,b)(1-a)^{n-k}(\ln b)^{n-k}. \tag{16}$$

We want to note that taking $a=x$ and $b=e$ in (16) reduces to

$$\mathscr{A}_n(x)=\frac{1}{x-1}\sum_{k=0}^{n-1}\binom{n}{k}\mathscr{A}_k(x)(1-x)^{n-k} \tag{17}$$

(see [5] and [25]). We see that (17) is proportional with Bernstein polynomials which we state in the following theorem:

Theorem 2.*The following identity*

$$\mathscr{A}_n(x)=\sum_{k=0}^{n-1}\frac{\mathscr{A}_k(x)B_{k,n}(x)}{x^{k+1}-x^k}$$

is true.

Let us now consider $\lim_{t\to 0}\frac{d^k}{dt^k}$ in (14), then we readily arrive at the following theorem.

Theorem 3.*Let $b\in\mathbb{R}^+$ and $a\in\mathbb{C}$, then we have*

$$\mathscr{A}_k(a,b)=\lim_{t\to 0}\left[\frac{d^k}{dt^k}\left(\frac{1-a}{b^{t(1-a)}-a}\right)\right]. \tag{18}$$

By (18), we easily conclude the following corollary.

Corollary 1.*The following Cauchy-type integral holds true:*

$$\frac{1}{1-a}\mathscr{A}_k(a,b)=\frac{k!}{2\pi i}\int_C\frac{t^{-k-1}}{b^{t(1-a)}-a}dt$$

where C is a loop which starts at $-\infty$, encircles the origin once in the positive direction, and the returns $-\infty$.

By (14), we discover the following:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{A}_n(a^2,b^2)\frac{t^n}{n!}&=\left[\frac{1-a}{b^{t(1+a)(1-a)}-a}\right]\left[\frac{1+a}{b^{t(1-a)(1+a)}-a}\right]\\&=\left[\sum_{n=0}^{\infty}(1+a)^n\mathscr{A}_n(a,b)\frac{t^n}{n!}\right]\left[\sum_{n=0}^{\infty}(1-a)^n\mathscr{A}_n(-a,b)\frac{t^n}{n!}\right].\end{aligned}$$

By using Cauchy product on the above equality, then we get the following theorem.

Theorem 4.*The following equality*

$$\mathscr{A}_n(a^2,b^2)=\sum_{k=0}^{n}\binom{n}{k}(1+a)^k\mathscr{A}_k(a,b)\mathscr{A}_{n-k}(-a,b)(1-a)^{n-k} \tag{19}$$

is true.

After the basic operations in (19), we discover the following corollary.

Corollary 2.*The following property holds:*

$$\mathscr{A}_n(a^2,b^2)=\sum_{k=0}^{n}\left(1+\frac{1}{a}\right)^k B_{k,n}(a)\mathscr{A}_k(a,b)\mathscr{A}_{n-k}(-a,b).$$

Now also, we consider geometric series in (14), then we compute as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{A}_n(a,b)\frac{t^n}{n!}&=\frac{1-a}{e^{t(1-a)\ln b}-a}=\frac{1-a^{-1}}{1-a^{-1}e^{t(1-a)\ln b}}\\&=\left(1-\frac{1}{a}\right)\sum_{j=0}^{\infty}a^{-j}e^{jt(1-a)\ln b}\\&=\left(1-\frac{1}{a}\right)\sum_{j=0}^{\infty}a^{-j}\sum_{n=0}^{\infty}j^n(1-a)^n(\ln b)^n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left[\left(1-\frac{1}{a}\right)\sum_{j=0}^{\infty}a^{-j}j^n(1-a)^n(\ln b)^n\right]\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the above equation, then we readily derive the following theorem.

Theorem 5.*The following*

$$\left(\frac{1}{a-1}\right)^n\mathscr{A}_n(a,b)=\left[\frac{(\ln b)^n}{a}-(\ln b)^n\right]\sum_{j=1}^{\infty}\frac{a^{-j}}{j^{-n}}$$

is true.

The above theorem is related to Poly-logarithm function, as follows:

$$\left(\frac{1}{a-1}\right)^n\mathscr{A}_n(a,b)=\left[\frac{(\ln b)^n}{a}-(\ln b)^n\right]Li_{-n}\left(a^{-1}\right). \tag{20}$$

In [27], it is well-known that

$$Li_{-n}(x)=\left(x\frac{d}{dx}\right)^n\frac{x}{1-x}=\sum_{k=0}^{n}k!S(n+1,k+1)\left(\frac{x}{1-x}\right)^{k+1} \tag{21}$$

where $S(n,k)$ are the Stirling numbers of the second kind. By (20) and (21), we have the following interesting theorem.

Theorem 6.*The following holds true:*

$$a\mathscr{A}_n(a,b)=-(\ln b)^n\sum_{k=0}^{n}k!S(n+1,k+1)\left(\frac{1}{a-1}\right)^{k-n}.$$

3 Further Remarks

Now, we consider (14) for evaluating at $a=-1$, as follows:

$$\sum_{n=0}^{\infty}\mathscr{A}_n(-1,b)\frac{t^n}{n!}=\frac{2}{b^{2t}+1} \tag{22}$$

where $\mathscr{A}_n(-1,b)$ are called Eulerian polynomials with parameter b.

By (22), we derive the following equality in complex plane:

$$\sum_{n=0}^{\infty}i^n\mathscr{A}_n(-1,b)\frac{t^n}{n!}=\frac{2}{b^{2it}+1}=\frac{2}{e^{2it\ln b}+1}.$$

From this, we discover the following:

$$\sum_{n=0}^{\infty}i^n\mathscr{A}_n(-1,b)\frac{t^n}{n!}=\sum_{n=0}^{\infty}E_n2^ni^n(\ln b)\frac{t^n}{n!} \tag{23}$$

where E_n are n-th Euler numbers which are defined by the following exponential generating function:

$$\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}=\frac{2}{e^t+1},\ |t|<\pi. \tag{24}$$

By (23) and (24), we have the following theorem.

Theorem 7.*Let $n\in\mathbb{N}$ (field of natural numbers) and $b\in\mathbb{C}$, then we get*

$$\mathscr{A}_n(-1,b)=2^nE_n(\ln b)^n.$$

We now give the definition of Bernoulli numbers for sequel of this paper via the following exponential generating function:

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}, \ |t| < 2\pi. \tag{25}$$

By using (22) and (25), we see that

$$\sum_{n=0}^{\infty} \mathscr{A}_n(-1,b) \frac{t^n}{n!} = \frac{2}{e^{2t\ln b}+1} = \frac{1}{t}\left[\frac{2t}{e^{2t\ln b}-1} - \frac{4t}{e^{4t\ln b}-1}\right].$$

So from above

$$\sum_{n=0}^{\infty} \mathscr{A}_n(-1,b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[2^n (\ln b)^n B_n - 4^n (\ln b)^n B_n\right] \frac{t^{n-1}}{n!}.$$

By comparing the coefficients of t^n on the above equation, then we can state the following theorem.

Theorem 8.*The following identity*

$$\mathscr{A}_n(-1,b) = \frac{2^{n+1}(\ln b)^{n+1}\left(1-2^{n+1}\right)B_{n+1}}{n+1}$$

holds true.

By (22), we obtain the following:

$$\Sigma_{n=0}^{\infty} \mathscr{A}_n(-1,b) \frac{t^n}{n!} = \frac{1}{t}\Sigma_{n=0}^{\infty} 2^n (\ln b)^n G_n \frac{t^n}{n!} = \Sigma_{n=0}^{\infty} 2^n (\ln b)^n G_n \frac{t^{n-1}}{n!}.$$

That is, we reach the following theorem.

Theorem 9.*The following holds true:*

$$\mathscr{A}_n(-1,b) = \frac{2^{n+1}(\ln b)^{n+1} G_{n+1}}{n+1}$$

where G_n are the familiar Genocchi numbers which is defined by

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t+1}.$$

We reconsider (22) and using definition of geometric series, then we compute as follows:

$$\sum_{n=0}^{\infty} \mathscr{A}_n(-1,b) \frac{\left(\frac{t}{2}\right)^n}{n!} = 2\sum_{j=0}^{\infty} (-1)^j e^{jt\ln b} = \sum_{n=0}^{\infty} \left(2(\ln b)^n \sum_{j=0}^{\infty} (-1)^j j^n\right) \frac{t^n}{n!}.$$

Therefore, we obtain the following theorem

Theorem 10.*For $n > 0$, then we have*

$$\mathscr{A}_n(-1,b) = 2^{n+1}(\ln b)^n \sum_{j=1}^{\infty} (-1)^j j^n. \tag{26}$$

As is well known, Euler-zeta function is defined by

$$\zeta_E(s) = 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j^s}, \ s \in \mathbb{C} \ (see[3]). \tag{27}$$

From (26) and (27), we obtain the interpolation function of new generalization of Eulerian polynomials at $a = -1$, as follow:

$$\mathscr{A}_n(-1,b) = 2^n (\ln b)^n \zeta_E(-n). \tag{28}$$

Equation (28) seems to be interpolation function at negative integers for Eulerian polynomials with parameter b.

Let us now consider Witt's formula for our polynomials at $a = -1$, so we need the following notations:

Imagine that p be a fixed odd prime number. Throughout this paper, we use the following notations. By $\mathbb{Z}_p$, we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

The normalized p-adic absolute value is defined by

$$|p|_p = \frac{1}{p}.$$

Let q be an indeterminate with $|q-1|_p < 1$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For a positive integer d with $(d,p) = 1$, let

$$X = X_d = \lim_{\overleftarrow{n}} \mathbb{Z}/dp^n\mathbb{Z} = \bigcup_{a=0}^{dp-1} (a + dp\mathbb{Z}_p)$$

with

$$a + dp^n\mathbb{Z}_p = \{x \in X \mid x \equiv a \,(mod\, dp^n)\}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^n$ and let $\sigma : X \to \mathbb{Z}_p$ be the transformation introduced by the inverse limit of the natural transformation

$$\mathbb{Z}/dp^n\mathbb{Z} \mapsto \mathbb{Z}/p^n\mathbb{Z}.$$

If f is a function on $\mathbb{Z}_p$, then we will utilize the same notation to indicate the function $f \circ \sigma$.

For a continuous function $f : X \to \mathbb{C}_p$, the p-adic fermionic integral on $\mathbb{Z}_p$ is defined by T. Kim in [2] and [3], as follows:

$$I_{-1}(f) = \int_X f(\upsilon)\, d\mu_{-1}(\upsilon) = \int_{\mathbb{Z}_p} f(\upsilon)\, d\mu_{-1}(\upsilon) = \lim_{n\to\infty} \sum_{\upsilon=0}^{p^n-1} (-1)^{\upsilon} f(\upsilon). \tag{29}$$

By (29), it is well-known that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0) \tag{30}$$

where $f_1(\upsilon) := f(\upsilon+1)$. Substituting $f(\upsilon) = b^{2\upsilon t}$ into (30), we get the following:

$$\int_X e^{2t\upsilon \ln b} d\mu_{-1}(\upsilon) = \frac{2}{b^{2t}+1} = \sum_{n=0}^{\infty} \mathscr{A}_n(-1,b)\frac{t^n}{n!}. \quad (31)$$

By (31) and using Taylor expansion of $e^{2t\upsilon \ln b}$, we obtain Witt's formula for our polynomials at $a=-1$, as follows:

Theorem 11.*The following holds true:*

$$\mathscr{A}_n(-1,b) = (\ln b)^n 2^n \int_X \upsilon^n d\mu_{-1}(\upsilon). \quad (32)$$

Equation (32) seems to be interesting for our further works in the concept of p-adic integrals.

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