

## An Explicit Formula for the Euler Polynomials of Higher Order

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We obtain an explicit formula for the Euler polynomials of higher order in terms of the Gaussian hypergeometric function. The corresponding new formulas of the Euler polynomials and numbers are also deduced. Finally, we also remark some possible applications of these results in the number theory and information science fields.

**Keywords:** Euler numbers of higher order, Euler polynomials of higher order, Gaussian hypergeometric function, Stirling numbers of the second kind, difference operator, coding theory and cryptography.

### 1 Introduction

The classical Bernoulli polynomials  $B_n(x)$  and Euler polynomials  $E_n(x)$  together with their familiar generalizations  $B_n^{(\alpha)}(x)$  and  $E_n^{(\alpha)}(x)$  of (real or complex) order  $\alpha$ , are usually defined by means of the following generating functions (see, for details, [1, 3, 8]):

$$\left(\frac{z}{e^z - 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < 2\pi; 1^\alpha := 1), \quad (1.1)$$

and

$$\left(\frac{2}{e^z + 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < \pi; 1^\alpha := 1), \quad (1.2)$$

respectively. The classical Bernoulli and Euler polynomials are respectively found from (1.1) and (1.2) as follows:

$$B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x) \quad (n \in \mathbb{N}_0). \quad (1.3)$$

Further the classical Bernoulli numbers  $B_n$  and Euler numbers  $E_n$  with their forms of higher order are

$$\begin{aligned} B_n &:= B_n(0) \quad \text{and} \quad E_n := 2^n E_n\left(\frac{1}{2}\right), \\ B_n^{(\alpha)} &:= B_n^{(\alpha)}(0) \quad \text{and} \quad E_n^{(\alpha)} := 2^n E_n^{(\alpha)}\left(\frac{\alpha}{2}\right), \end{aligned} \quad (1.4)$$

respectively.

Recently, Luo *et al.* [6] gave certain new recursive formulas for the Euler numbers and polynomials of higher order; Luo [7] further obtained several formulas involving the Stirling numbers of the second kind. Srivastava and Todorov [9] gave some elegant formulas for the Bernoulli numbers and polynomials of higher order as follows

$$B_n^{(\alpha)} = \sum_{k=0}^n (-1)^k \binom{\alpha+n}{n-k} \binom{\alpha+k-1}{k} \binom{n+k}{k}^{-1} S(n+k, k) \quad (1.5)$$

and

$$\begin{aligned} B_n^{(\alpha)}(x) &= \sum_{k=0}^n \binom{n}{k} \binom{\alpha+k-1}{k} \frac{k!}{(2k)!} \sum_{j=0}^k (-1)^j \binom{k}{j} j^{2k} (x+j)^{n-k} \\ &\quad \times {}_2F_1\left[k-n, k-\alpha; 2k+1; \frac{j}{x+j}\right]. \end{aligned} \quad (1.6)$$

Here  ${}_2F_1[a, b; c; z]$  denotes the Gaussian hypergeometric function defined by (see [1, 15.1.1])

$${}_2F_1[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (1.7)$$

where  $(\lambda)_0 = 1$ ,  $(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) = \Gamma(\lambda+n)/\Gamma(\lambda)$  ( $n \geq 1$ ).

## 2 An Explicit Formula of $E_n^{(\alpha)}(x)$ in Terms of the Gaussian Hypergeometric Function

In the present section, we will prove an explicit formula for the Euler polynomials of higher order which is an analogue of the formula (1.6). We need the following lemmas before proving main result.

**Lemma 2.1.** *For  $n = 0, 1, \dots$ ,  $x \in \mathbb{R}$ ,  $\alpha \in \mathbb{C}$ , the following formulas are true*

$$E_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n}{k} \frac{E_k^{(\alpha)}}{2^k} \left(x - \frac{\alpha}{2}\right)^{n-k},$$

$$E_n^{(\alpha)}(x) = (-1)^n E_n^{(\alpha)}(\alpha - x),$$

$$E_n^{(\alpha)}\left(\frac{x}{2}\right) = \sum_{k=0}^n \binom{n}{k} \frac{(-x)^{n-k}}{2^{n-k}} E_k^{(\alpha)}(x).$$

*Proof.* From (1.2), we can readily obtain above formulas.  $\square$

Clearly, by Lemma 2.1 we have

$$E_n^{(\alpha)}(\alpha) = (-1)^n E_n^{(\alpha)}(0),$$

$$E_n^{(\alpha)}\left(\frac{\alpha}{2}\right) = \sum_{k=0}^n \binom{n}{k} \frac{(-\alpha)^{n-k}}{2^{n-k}} E_k^{(\alpha)}(\alpha). \quad (2.1)$$

**Theorem 2.1.** For  $n = 0, 1, \dots$ ,  $x \in \mathbb{R}$ ,  $\alpha \in \mathbb{C}$ , the following formula in terms of the Gaussian hypergeometric function is true

$$E_n^{(\alpha)}(x) = \sum_{k=0}^n \frac{1}{2^k} \binom{n}{k} \binom{\alpha + k - 1}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} j^k (x + j)^{n-k}$$

$$\times {}_2F_1\left[k - n, k; k + 1; \frac{j}{x + j}\right]. \quad (2.2)$$

*Proof.* By Taylor's expansion and Leibniz's rule, the generating relation (1.2) yields

$$E_n^{(\alpha)}(x) = \sum_{s=0}^n \binom{n}{s} x^{n-s} D_z^s \left\{ \left( \frac{2}{e^z + 1} \right)^\alpha \right\} \Big|_{z=0}, \quad D_z = \frac{d}{dz}. \quad (2.3)$$

Since

$$(1 + w)^{-\alpha} = \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} (-w)^k \quad (|w| < 1),$$

setting  $1 + w = (e^z + 1)/2$  and applying the binomial theorem, we find from (2.3) that

$$E_n^{(\alpha)}(x) = \sum_{s=0}^n \binom{n}{s} x^{n-s} \sum_{k=0}^s \frac{(-1)^k}{2^k} \binom{\alpha + k - 1}{k} D_z^s \{(e^z - 1)^k\} \Big|_{z=0}. \quad (2.4)$$

Now we make use of the well-know formula (see [1, 24.1.4])

$$(e^z - 1)^k = \sum_{r=k}^{\infty} \frac{z^r}{r!} \Delta^k 0^r, \quad S(r, k) = \frac{1}{k!} \Delta^k 0^r, \quad (2.5)$$

where  $S(r, k)$  denotes the Stirling number of the second kind defined by

$$x^r = \sum_{k=0}^r \binom{x}{k} k! S(r, k).$$

For convenience we write

$$\Delta^k a^r = \Delta^k x^r \Big|_{x=a} = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (a + j)^r, \quad (2.6)$$

where  $\Delta$  is the difference operator defined by (see [1, pp. 822, III])

$$\Delta f(x) = f(x+1) - f(x).$$

So, in general (see [1, pp. 823, 24.1.1]),

$$\Delta^k f(x) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(x+j).$$

From (2.5) we find that

$$D_z^s \{(e^z - 1)^k\}|_{z=0} = \Delta^k 0^s = k! S(s, k). \quad (2.7)$$

Substituting this value into (2.4) yields

$$E_n^{(\alpha)}(x) = \sum_{s=0}^n \binom{n}{s} x^{n-s} \sum_{k=0}^s \frac{(-1)^k k!}{2^k} \binom{\alpha + k - 1}{k} S(s, k), \quad (2.8)$$

or

$$E_n^{(\alpha)}(x) = \sum_{s=0}^n \binom{n}{s} x^{n-s} \sum_{k=0}^s \frac{(-1)^k}{2^k} \binom{\alpha + k - 1}{k} \Delta^k 0^s. \quad (2.9)$$

If we rearrange the resulting duple series (2.9), we have

$$E_n^{(\alpha)}(x) = \sum_{k=0}^n \frac{(-1)^k}{2^k} \binom{\alpha + k - 1}{k} \sum_{s=0}^{n-k} \binom{n}{s+k} x^{n-s-k} \Delta^k 0^{s+k}. \quad (2.10)$$

Further substituting for  $\Delta^k 0^{s+k}$  from the definition (2.6) with  $a = 0$  into (2.10), we get that

$$\begin{aligned} E_n^{(\alpha)}(x) &= \sum_{k=0}^n \frac{1}{2^k} \binom{n}{k} \binom{\alpha + k - 1}{k} x^{n-k} \sum_{j=0}^k (-1)^j \binom{k}{j} j^k \\ &\quad \times {}_2F_1 \left[ k - n, 1; k + 1; \frac{-j}{x} \right]. \end{aligned} \quad (2.11)$$

Finally, we apply the known transformation [1, 15.3.4]

$${}_2F_1[a, b; c; z] = (1-z)^{-a} {}_2F_1 \left[ a, c-b; c; \frac{z}{z-1} \right]$$

and (2.11) leads immediately to the desired (2.2).  $\square$

### 3 Further Remarks

**Remark 3.1.** Taking  $\alpha = 1$  in (2.2), we obtain the following new formula for the Euler polynomials

$$E_n(x) = \sum_{k=0}^n \frac{1}{2^k} \binom{n}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} j^k (x+j)^{n-k} {}_2F_1 \left[ k - n, k; k + 1; \frac{j}{x+j} \right].$$

**Remark 3.2.** By [1, 15.1.20], we have

$${}_2F_1[a, b; c; 1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c \neq 0, -1, -2, \dots, \Re(c-a-b) > 0),$$

which (for  $a = k - n, b = k$ , and  $c = k + 1$ ) readily yields

$${}_2F_1[k - n, k; k + 1; 1] = \binom{n}{k}^{-1}, \quad (0 \leq k \leq n). \quad (3.1)$$

In view of (3.1) and the special case when  $x = 0$  in formula (2.2), we get that

$$E_n^{(\alpha)}(0) = \sum_{k=0}^n \frac{1}{2^k} \binom{\alpha + k - 1}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} j^n, \quad (3.2)$$

or equivalently,

$$E_n^{(\alpha)}(0) = \sum_{k=0}^n \frac{(-1)^k k!}{2^k} \binom{\alpha + k - 1}{k} S(n, k). \quad (3.3)$$

Moreover, we note that, with  $E_n^{(\alpha)} := 2^n E_n^{(\alpha)}(\alpha/2)$  and (2.1), the Euler numbers of higher order can be written as

$$E_n^{(\alpha)} = \sum_{k=0}^n \binom{n}{k} 2^k \alpha^{n-k} \sum_{j=0}^k \frac{(-1)^j j!}{2^j} \binom{\alpha + j - 1}{j} S(k, j). \quad (3.4)$$

Further, setting  $\alpha = 1$  in formula (3.4), we deduce easily the following formula for the Euler numbers

$$E_n = \sum_{k=0}^n \binom{n}{k} 2^k \sum_{j=0}^k \frac{(-1)^j j!}{2^j} S(k, j). \quad (3.5)$$

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