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On The Numerical Solution of Differential-Algebraic Equations(DAES) with Index-3 by Pade Approximation

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Abstract: In this paper, Numerical solution of Differential-Algebraic equations (DAEs) with index-3 is considered by Pade approximation. We applied this method to two examples. First differential-algebraic equations (DAEs) with index-3 has been converted to power series by one-dimensional differential transformation, Then the numerical solution of equation was put into Pade series form. Thus we obtained numerical solution of differential-algebraic equations (DAEs) with index-3.

Keywords: Differential-Algebraic equations(DAEs), index-3, one-dimensional differential transformation, Power series, Pade Approximation

1 Introduction

Differential-Algebraic Equations (DAEs) can be found in a wide variety of scientific and engineering applications, including circuit analysis, computer-aided design and real-time simulation of mechanical (multibody) systems, power systems, chemical process simulation, and optimal control. Many important mathematical models can be expressed in terms of Differential-Algebraic Equations (DAEs). Many physical problems are most easily initially modeled as a system of differential-algebraic equations (DAEs) [4]. Some numerical methods have been developed, using both backward differential formula [1,3,4,5] and implicit Runge-Kutta methods [2,4], Pade Approximations method [8,9]. These methods are only directly suitable for low index problems and often require that the problem to have special structure. Although many important applications can be solved by these methods there is a need for more general approaches. The most general form of a DAE is given by

$$F(t,x,x') = 0, \tag{1.1}$$

where $\partial f/\partial x'$ may be singular. The rank and structure of this jacobian matrix may depend, in general, on the solution $x(t)$, and for simplicity we will always assume that it is independent of t. The important special case of a semi-explicit DAE or an ODE with constraints,

$$x' = f(t,x,z), \tag{1.2a}$$

$$0 = g(t,x,z). \tag{1.2b}$$

This is a special case of (1.1). The index is 1 if $\partial g/\partial z$ is nonsingular, because then one differentiation of (1.2b) yields z' in principle. For the semi-explicit index-1 DAE we can distinguish between differential variables $x(t)$ and algebraic variables $z(t)$ [4]. The algebraic variables may be less smooth than the differential variables by one derivative. In the general case, each component of x may contain a mix of differential and algebraic components, which makes the numerical solution of such high-index problems much harder and riskier. The semi-explicit form is decoupled in this sense. The Pade approximation method used to accelerate the convergence of the power series solution.Thus we obtained numerical solution of Differential-Algebraic equations(DAEs) with index-3.

2 Special Differential-Algebraic Equations(DAEs) Forms

Most of the higher-index problems encountered in practice can be expressed as a combination of more

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restrictive structures of ODEs coupled with constraints. In such systems the Algebraic and differential variables are explicitly identified for higher-index DAEs as well, and the algebraic variables may all be eliminated using the same number of differentiations. These are called Hessenberg forms of the DAE and are given belong. In this article, the Pade approximation method has proposed for solving differential-algebraic equations with index-3. The presented solutions are performed using MAPLE computer algebra systems [7].

2.1 Hessenberg Index-1

$$x' = f(t,x,z), \tag{2.1a}$$

$$0 = g(t,x,z). \tag{2.1b}$$

Here the Jacobian matrix function g_z is assumed to be nonsingular for all t. This is also often referred to as a semi-explicit index-1 system. Semiexplicit index-1 DAEs are very closely related to implicit ODEs.

2.2 Hessenberg Index-2

$$x' = f(t,x,z), \tag{2.2a}$$

$$0 = g(t,x). \tag{2.2b}$$

Here the product of Jacobians $g_x f_z$ is nonsingular for all t. Note the absence of the algebraic variables z from the constraints (2.2b). This is a pure index-2 DAE, and all algebraic variables play the role of index-2 variables.

2.3 Hessenberg Index-3

$$x' = f(t,x,y,z), \tag{2.3a}$$

$$y' = g(t,x,y), \tag{2.3b}$$

$$0 = h(t,y). \tag{2.3c}$$

Here the product of three matrix functions $h_y g_x f_z$ is nonsingular. The index of a Hessenberg DAE is found, as in the general case, by differentiation. However, here only algebraic constraints must be differentiated [2].

3 One-Dimensional Differential Transform

Differential transform of function $y(x)$ is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}, \tag{2.1}$$

In equation (2.1), $y(x)$ is the original function and $Y(k)$ is the transformed function, which is called the T-function. Differential inverse transform of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k), \tag{2.2}$$

from equation (2.1) and (2.2), we obtain

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}, \tag{2.3}$$

Equation (2.3) implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which are described by the transformed equations of the original functions. In this study we use the lower case letter to represent the original function and upper case letter represent the transformed function. From the definitions of equations (2.1) and (2.2), it is easily proven that the transformed functions comply with the basic mathematics operations shown in Table 1. In actual applications, the function $y(x)$ is expressed by a finite series and equation (2.2) can be written as

$$y(x) = \sum_{k=0}^{m} x^k Y(k), \tag{2.4}$$

Equation (2.3) implies that $y(x) = \sum_{k=m+1}^{\infty} x^k Y(k)$ is negligibly small. In fact, m is decided by the convergence of natural frequency in this study.

Definition 2.1. Let $\otimes$ denote convolution

$$y(x) = u(x)v(x), u(x) = D^{-1}\left[U(k)\right], v(x) = D^{-1}\left[V(k)\right]$$

so we have

$$D\left[y(x)\right] = D\left[u(x)v(x)\right] = U(k) \otimes V(k) = \sum_{r=0}^{k} U(r)V(k-r)$$

Table 1 The fundamental operations of one-dimensional differential transform method

Original function	Transformed function
$y(x)=u(x)+v(x)$	$Y(k)=U(k)+V(k)$
$y(x)=cw(x)$	$Y(k)=cW(k)$
$y(x)=\frac{dw(x)}{dx}$	$Y(k)=(k+1)W(k+1)$
$y(x)=\frac{d^jw(x)}{dx^j}$	$Y(k)=(k+1)(k+2)...(k+j)W(k+j)$
$y(x)=u(x)v(x)$	$Y(k)=\sum_{r=0}^{k}U(r)V(k-r)$
$y(x)=x^j$	$Y(k)=\delta(k-j)=\begin{cases}1, & k=j\\ 0, & k\neq j\end{cases}$

4 Pade Approximation

Suppose that we are given a power series $\sum_{i=0}^{\infty}a_ix^i$, representing a function $f(x)$, so that

$$f(x)=\sum_{i=0}^{\infty}a_ix^i, \tag{3.1}$$

A Pade approximation is a rational fraction

$$[L/M]=\frac{p_0+p_1x+...+p_Lx^L}{q_0+q_1x+...+q_Mx^M}, \tag{3.2}$$

which has a Maclaurin expansion which agress with (3.1) as for as possible, Notice that in (3.2) there are $L+1$ numerator coefficients and $M+1$ denominator coefficients. There is a more or less irrelevant common factor between them, and for definitenees we take $q_0=1$. This choice turns out to be an essential part of the precise definition and (3.2) is our conventional notation with this choice for q_0. So there are $L+1$ independent numerator coefficients and M independent numerator coefficients, making $L+M+1$ unknown coefficients in all. This number suggest that normally the $[L/M]$ ought to fit the power series (3.1) through the orders $1,x,x^2,...,x^{L+M}$ in the notation of formal power series.

$$\sum_{i=0}^{\infty}a_ix^i=\frac{p_0+p_1x+...+p_Lx^L}{q_0+q_1x+...+q_Mx^M}+O(x^{L+M+1}). \tag{3.3}$$

Multiply the both side of (3.3) by the denominator of right side in (3.3) and compare the coefficients of both sides (3.3), we have

$$a_l+\sum_{k=1}^{M}a_{l-k}q_k=p_l,\ (l=0,...,M), \tag{3.4}$$

$$a_l+\sum_{k=1}^{L}a_{l-k}q_k=p_l,\ (l=M+1,...,M+L). \tag{3.5}$$

Solve the linear equation in (3.5), we have $q_k,(k=1,...,L)$. And substitute q_k into (3.4), we have $p_l,(L=0,...,M)$. Therefore, we have constructed a $[L/M]$ Pade approximation, which agress with $\sum_{i=0}^{\infty}a_ix^i$ through order x^{L+M}. if $M\leq L\leq M+2$, where M and L are the degree of numerator and denominator in Pade series, respectively, then Pade series gives an A-stable formula for an ordinary differential equation.

5 Applications

Example 1. We first considered the following differential-algebraic equations with index-3

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\times\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix}+\begin{pmatrix}1&1&x\\e^x&x+1&0\\0&x^2&0\end{pmatrix}\times\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \tag{4.1}$$
$$=\begin{pmatrix}2x\\x^2+x+2\\x^3\end{pmatrix},x\in[0,\infty],$$

with initial conditions

$$x(0)=\begin{pmatrix}1\\0\\1\end{pmatrix},$$

The exact solutions are $x_1(x)=e^{-x}$, $x_2(x)=x$, $x_3(x)=1$. By using the basic properties of differential transform method and taking the transform of differential-algebraic equations given (4.1), we obtained

$$(k+1)X_1(k+1)+X_1(k)+X_2(k)+\sum_{r=0}^{k}\delta(r-1)X_3(k-r) \tag{4.2}$$
$$=2\delta(k-1),$$

$$(k+1)X_2(k+1)+\sum_{r=0}^{k}\frac{1}{r!}X_1(k-r)+\sum_{r=0}^{k}\delta(r-1)X_2(k-r) \tag{4.3}$$
$$+X_2(k)=\delta(k-2)+\delta(k-1)+2\delta(k),$$

$$\sum_{r=0}^{k}\delta(r-2)X_2(k-r)=\delta(k-3), \tag{4.4}$$

Equations (4.2) and (4.3) can be simplified as

$$X_1(k+1)=\frac{1}{k+1}[2\delta(k-1)-X_1(k)-X_2(k) \tag{4.5}$$
$$-\sum_{r=0}^{k}\delta(r-1)X_3(k-r)],$$

$$X_2(k+1)=\frac{1}{k+1}[\delta(k-2)+\delta(k-1)+2\delta(k) \quad (4.6)$$
$$-\sum_{r=0}^{k}\frac{1}{r!}X_1(k-r)-\sum_{r=0}^{k}\delta(r-1)X_2(k-r)-X_2(k)],$$

$$\sum_{r=0}^{k}\delta(r-2)X_2(k-r)=\delta(k-3), \quad (4.7)$$

The initial conditions can be transformed at $x_0=0$, as

$$X_1(0)=1,\ X_2(0)=0,\ X_3(0)=1,$$

For $k=0,1,2,...,$ $X_1(k),X_2(k),X_3(k)$ coefficients can be calculated from equations (4.5)-(4.7)

$$X_1(1)=-1,\ X_1(2)=\frac{1}{2},\ X_1(3)=-\frac{1}{6},\ X_1(4)=\frac{1}{24},$$
$$X_1(5)=-\frac{1}{120},\ X_1(6)=\frac{1}{720},\ X_1(7)=-\frac{1}{5040},$$
$$X_1(8)=\frac{1}{40320},X_1(9)=-\frac{1}{362880},\ X_2(1)=1,$$
$$X_2(2)=0,\ X_2(3)=0,\ X_2(4)=0,\ X_2(5)=0,$$
$$X_3(1)=0,X_3(2)=0,\ X_3(3)=0,\ X_3(4)=0,$$
$$X_3(5)=0,....$$

By substituting the values of $X_1(k),\ X_2(k),\ X_3(k),...$into equation (4.3), the solutions can be written as

$$x_1^*(x)=1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5$$
$$+\frac{1}{720}x^6-\frac{1}{5040}x^7+\frac{1}{40320}x^8-\frac{1}{362880}x^9+...,$$
$$x_2^*(x)=x,$$
$$x_3^*(x)=1.$$

Power series $x_1^*(x)$ can be transformed into Pade series

$$x_1^{**}(x)=P[5/4]$$
$$=(1-0.5555555556x+0.1388888889x^2$$
$$-0.01984126984x^3+0.1653439153\times10^{-2}x^4$$
$$+0.6613756614\times10^{-4}x^5)/(1+0.4444444444x$$
$$+0.08333333333x^2+0.7936507936\times10^{-2}x^3$$
$$+0.3306878307\times10^{-3}x^4)$$

Example 2.

$$\begin{pmatrix}0&1&0\\0&x&1\\0&0&0\end{pmatrix}\times\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix}+\begin{pmatrix}1&0&0\\0&2&0\\0&x&1\end{pmatrix}\times\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\2x\\e^x\end{pmatrix},x\in[0,\infty] \quad (4.8)$$

Table 2 Numerical solution of $x_1(x)$

x	$x_1(x)$	$x_1^{**}(x)$	$\lvert x_1(x)-x_1^{**}(x)\rvert$
0.0	1.0000000000	1.0000000000	0
0.1	0.9048374180	0.9048374182	2×10^{-10}
0.2	0.8187307531	0.8187307533	2×10^{-10}
0.3	0.7408182207	0.7408182204	3×10^{-10}
0.4	0.6703200460	0.6703200456	4×10^{-10}
0.5	0.6065306597	0.6065306602	5×10^{-10}
0.6	0.5488116361	0.5488116359	2×10^{-10}
0.7	0.4965853038	0.4965853040	2×10^{-10}
0.8	0.4493289641	0.4493289639	2×10^{-10}
0.9	0.4065696597	0.4065696595	2×10^{-10}
1.0	0.3678794412	0.3678794404	8×10^{-10}

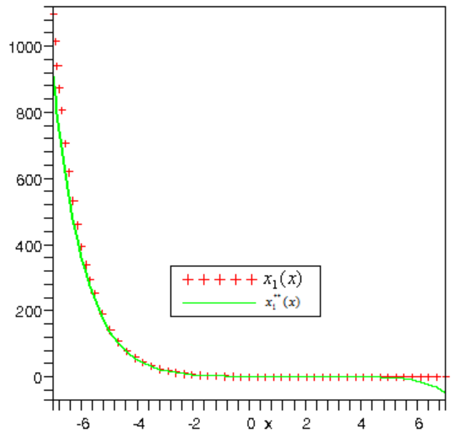

Fig. 1 values of $x_1(x)$ and its $x_1^{**}(x)$ Pade approximant

with initial conditions

$$\begin{pmatrix}x_1(0)\\x_2(0)\\x_3(0)\end{pmatrix}=\begin{pmatrix}0\\-1\\1\end{pmatrix},$$

The exact solutions are

$$x_1(x)=e^x-1,\ x_2(x)=2x-e^x,\ x_1(x)=(1+x)e^x-2x^2,$$

By using the basic properties of differential transform method and taking the transform of differential-algebraic equations given (4.8) it is obtained that

$$X_2(k+1) = \frac{1}{k+1}[\delta(k) - X_1(k)], \tag{4.9}$$

$$X_3(k+1) = \frac{1}{k+1}[2\delta(k-1) - 2X_2(k) - \sum_{r=0}^{k} \delta(r-1)(k-r+1)X_2(k-r+1)],$$

$$\sum_{r=0}^{k} \delta(r-1)X_2(k-r) + X_3(k) = \frac{1}{k!},$$

The initial conditions can be transformed at $x_0 = 0$, as

$$X_1(0) = 0,\ X_2(0) = -1,\ X_3(0) = 1,$$

For $k = 0,1,2,...,\ X_1(k),X_2(k),X_3(k)$ coefficients can be calculated from equations (4.9)

$$X_1(1) = 1,\ X_1(2) = \frac{1}{2},\ X_1(3) = \frac{1}{6},\ X_1(4) = \frac{1}{24},$$

$$X_1(5) = \frac{1}{120},\ X_2(0) = -1,\ X_2(1) = 1,\ X_2(2) = -\frac{1}{2},$$

$$X_2(3) = -\frac{1}{6},\ X_2(4) = -\frac{1}{24},\ X_2(5) = -\frac{1}{120},$$

$$X_3(0) = 1,\ X_3(1) = 2, X_3(2) = -\frac{1}{2},\ X_3(3) = \frac{2}{3},$$

$$X_3(4) = \frac{5}{24},\ X_3(5) = \frac{1}{20},....$$

By substituting the values of $X_1(k),X_2(k),X_3(k)$ into equation (4.3), the solutions can be written as

$$x_1^*(x) = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9,$$

$$x_2^*(x) = -1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 - \frac{1}{5040}x^7 + \frac{1}{40320}x^8 - \frac{1}{362880}x^9,$$

$$x_3^*(x) = 1 + 2x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{5}{24}x^4 + \frac{1}{20}x^5 + \frac{7}{720}x^6 + \frac{1}{630}x^7 + \frac{1}{4480}x^8 + \frac{1}{36288}x^9,$$

Power series $x_1^*(x)$, $x_2^*(x)$ and $x_3^*(x)$ can be transformed into Pade series

$$\begin{aligned} x_1^{**}(x) &= P_1[5/4] \\ &= (-x + 0.5555555556x^2 - 0.02777777778x^3 \\ &+ 0.001322751323x^4 - 0.6613756614 \times 10^{-4}x^5) \\ &/(1 + 0.4444444444x + 0.08333333333x^2 \\ &+ 0.007936507936x^3 + 0.3306878307 \times 10^{-3}x^4), \end{aligned}$$

$$\begin{aligned} x_2^{**}(x) &= P_2[5/4] \\ &= (-1 + 1.444444444x - 1.027777778x^2 \\ &+ 0.1468253968x^3 - 0.01752645503x^4 \\ &+ 0.5952380952 \times 10^{-3}x^5)/(1 - 0.4444444444x \\ &+ 0.0833333333x^2 - 0.7936507936 \times 10^{-2}x^3 \\ &+ 0.3306878307 \times 10^{-3}x^4), \end{aligned}$$

$$\begin{aligned} x_3^{**}(x) &= P_3[5/4] \\ &= (1 + 1.613112133x - 1.214826536x^2 \\ &+ 0.9740730863x^3 - 0.08686838489x^4 \\ &+ 0.01080566945x^5)/(1 - 0.3868878669x \\ &+ 0.05894919782x^2 - 0.3935909473 \times 10^{-2}x^3 \\ &+ 0.6994421809 \times 10^{-4}x^4), \end{aligned}$$

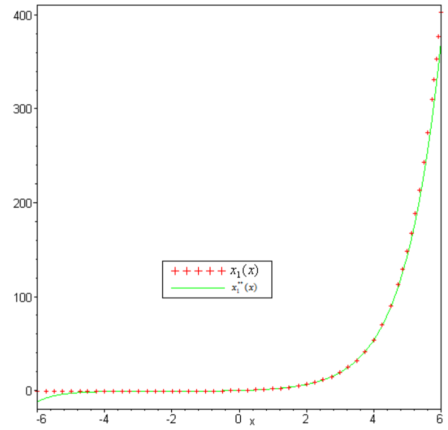

Fig. 2 Values of $x_1(x)$ and its $x_1^{**}(x)$ Pade approximant

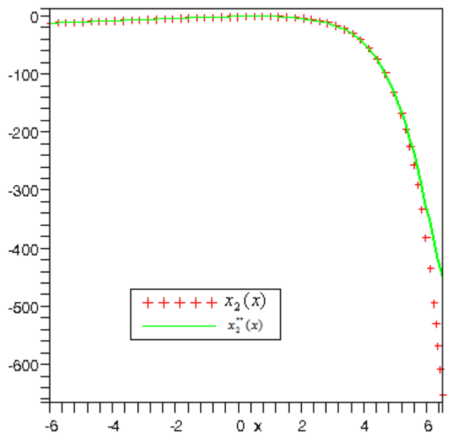

Fig. 3 Values of $x_2(x)$ and its $x_2^{**}(x)$ Pade approximant

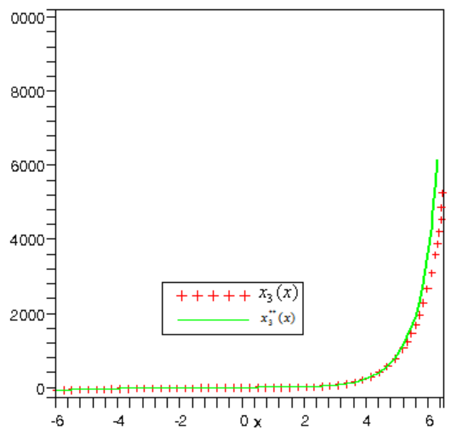

Fig. 4 Values of $x_3(x)$ and its $x_3^{**}(x)$ Pade approximant

Table 3 Numerical solution of $x_1(x)$

x	$x_1(x)$	$x_1^{**}(x)$	$\|x_1(x)-x_1^{**}(x)\|$
0.0	0.0000000000	0.0000000000	0
0.1	−0.0951625820	−0.09516258200	0
0.2	−0.1812692469	−0.1812692470	1×10^{-10}
0.3	−0.2591817793	−0.2591817792	1×10^{-10}
0.4	−0.3296799540	−0.3296799537	3×10^{-10}
0.5	−0.3934693403	−0.3934693406	3×10^{-10}
0.6	−0.4511883639	−0.4511883638	1×10^{-10}
0.7	−0.5034146962	−0.5034146965	3×10^{-10}
0.8	−0.5506710359	−0.5506710359	0
0.9	−0.5934303403	−0.5934303408	5×10^{-10}
1.0	−0.6321205588	−0.6321205600	1.2×10^{-9}

Table 4 Numerical solution of $x_2(x)$

x	$x_2(x)$	$x_2^{**}(x)$	$\|x_2(x)-x_1^{**}(x)\|$
0.0	−1.000000000	−1.000000000	0
0.1	−0.9051709180	−0.9051709180	0
0.2	−0.821402758	−0.8214027581	1×10^{-10}
0.3	−0.749858808	−0.7498588076	4×10^{-10}
0.4	−0.691824698	−0.6918246976	4×10^{-10}
0.5	−0.648721271	−0.6487212706	4×10^{-10}
0.6	−0.622118800	−0.6221188006	6×10^{-10}
0.7	−0.613752707	−0.6137527076	6×10^{-10}
0.8	−0.625540928	−0.6255409276	4×10^{-10}
0.9	−0.659603111	−0.6596031094	1.6×10^{-9}
1.0	−0.718281828	−0.7182818240	4×10^{-9}

Table 5 Numerical solution of $x_3(x)$

x	$x_3(x)$	$x_3^{**}(x)$	$\|x_3(x)-x_3^{**}(x)\|$
0.0	1.00000000	1.0000000000	0
0.1	1.195688010	1.195688010	0
0.2	1.385683310	1.385683312	2×10^{-9}
0.3	1.574816450	1.574816450	0
0.4	1.768554577	1.768554576	1×10^{-9}
0.5	1.973081906	1.973081907	1×10^{-9}
0.6	2.195390080	2.195390081	1×10^{-9}
0.7	2.443379602	2.443379605	3×10^{-9}
0.8	2.725973670	2.725973679	9×10^{-9}
0.9	3.053245911	3.053245941	3×10^{-8}
1.0	3.436563656	3.436563744	8.8×10^{-8}

6 Conclusion

The method has proposed for solving differential - algebraic equations with index-3. Results show the advantages of the method. Table 2, Table 3, Table 4, Table 5 and Fig.1, Fig.2, Fig.3 and Fig.4 shows that the numerical solution approximates the exact solution very well in accordance with above method. The Pade approximation method used to accelerate the convergence of the power series solution.The presented solutions in this article are performed using MAPLE computer algebra systems [7].

References

[1] U.M. Ascher, SIAM J. Sci. Stat. Comput **10**, 937-949 (1989).
[2] U.M. Ascher and L. R. Petzold, SIAM J. Numer. Anal **28**, 1097-1120 (1991).
[3] U.M. Ascher and R.J. Spiter **15**, 938-952 (1994).
[4] K.E. Brenan, S.L. Campbell, L.R. Petzold, Elsevier, New York, 1989.
[5] C.W. Gear, L.R. Petzold, SIAM J. Numer. Anal **21**, 716-728(1984).
[6] U. M. Ascher and L. R. Petzold, Philadelphia, 1998.
[7] G. Frank, MAPLE V, CRC Pres, Boca Raton, FL 33431, 1996.
[8] E. Celik, E. Karaduman and M. Bayram, International Journal of Quantum Chemistry **89** , 447-451(2002).
[9] E. Celik and M. Bayram, Applied Mathematics and Computation **137**, 151-160 (2003).
[10] M. Bayram and E. Celik, journal of Applied Mathematics Statistics(IJAMAS), 29-39(2004).