Applied Mathematics & Information Sciences Letters
An International Journal

Fuzzy Topological Properties on Fuzzy Function Spaces

A. I. Aggour[1], *and F. E. Attounsi*[2]

[1]Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt.
[2]Department of Mathematics, Faculty of Science, 7th University, Bani Walid, Libya.

Received: 28 Jan. 2013, Revised: 10 Mar. 2013, Accepted: 19 Mar. 2013
Published online: 1 May. 2013

Abstract: In this paper, we study the fuzzy continuous convergence of fuzzy nets on the set $FC(X,Y)$ of all fuzzy continuous functions of a fuzzy topological space X into another Y. Also, we introduce fuzzy topologies on fuzzy function spaces.

Keywords: Fuzzy upper limit of fuzzy nets, Fuzzy function spaces, Fuzzy continuously converges nets, Fuzzy jointly continuous topologies, Fuzzy splitting topologies.

1 Introduction and Preliminaries

The notion of convergence is one of the basic notion in analysis. In this paper, fuzzy continuous convergence theory of fuzzy nets on the set $FC(X,Y)$ of fuzzy continuous functions of an fts X into another Y is presented. In 1976, the concept of fuzzy topology was introduced by R. Lowen [4].

In 1980, Pu and Liu introduced the notions of fuzzy nets and Q-neighborhoods. The concept of the Q-neighborhood reflect the features of the neighborhood structure in fuzzy topological spaces. By this new neighborhood structure the Moore-Smith convergence theory was established [6]. In this paper, we will give new concepts of fuzzy continuous convergence of fuzzy nets on the set $FC(X,Y)$. Also, we introduce the notions of fuzzy splitting topologies and fuzzy jointly continuous topologies on the fuzzy functions spaces.

Let X be an arbitrary nonempty set. A fuzzy set in X is a mapping from X to the closed unit interval $I=[0,1]$, that is, an element of I^X. A fuzzy point x_t is a fuzzy set in X defined by $x_t(x)=t$ and $x_t(y)=0$ for all $y\neq x$, whose support is the single point x and whose value is $t\in(0,1]$ [6]. We denote by $FP(X)$ the collection of all fuzzy points in X.

Definition 1. *[12] Let $\mu,\eta\in I^X$. We define the following fuzzy sets:*

i)$\mu\wedge\eta\in I^X$, by $(\mu\wedge\eta)(x)=\min\{\mu(x),\eta(x)\}$, for each $x\in X$.
ii)$\mu\vee\eta\in I^X$, by $(\mu\vee\eta)(x)=\max\{\mu(x),\eta(x)\}$, for each $x\in X$.
iii)$\mu'\in I^X$, by $\mu'(x)=1-\mu(x)$, for each $x\in X$.

Definition 2. *[4] A fuzzy topology $\Im$ on a non empty set X is a family of fuzzy subsets of X such that:*

i)$\Im$ contains all constant fuzzy subsets of X,
ii)$\mu\wedge\eta\in\Im$, for each $\mu,\eta\in\Im$,
iii)If $\{\mu_\lambda\}_{\lambda\in\Lambda}$ is subfamily of $\Im$, then $\bigvee_{\lambda\in\Lambda}\mu_\lambda\in\Im$.

The pair $(X,\Im)$ is called a fuzzy topological space denoted by fts. Each member of $\Im$ is called fuzzy open set and its complement is called fuzzy closed set.

Definition 3. *[6] Let $(X,\Im)$ be an fts and $\mu,\eta\in I^X$. Then:*

i)A fuzzy point x_t is said to be quasi-coincident with μ denoted by $x_t q\mu$ iff $t>\mu'(x)$ or $t+\mu(x)>1$.
ii)μ is called quasi-coincident with η, denoted by $\mu q\eta$, if there exists $x\in X$ such that $\mu(x)+\eta(x)>1$. If μ is not quasi-coincident with η, then we write $\mu\not{q}\eta$.
iii)A fuzzy subset μ of X is called a neighborhood (or a nbd, for short) of a fuzzy point x_t iff there exists a fuzzy open set ν of X such that $x_t\in\nu\subseteq\mu$. The family N_{x_t} of all nbds of x_t is called the system of nbds of x_t.
iv)μ is called Q-neighborhood of a fuzzy point $x_t\in FP(X)$ if there exists a fuzzy open set $\eta\in\Im$ such that $x_t q\eta$ and $\eta\leq\mu$. The class of all open Q-neighborhoods of x_t is denoted by $N^Q_{x_t}$.

Definition 4. *[7] A map $f:X\to Y$ is called fuzzy continuous if the inverse image of every fuzzy open subset of Y is fuzzy open subset of X.*

* Corresponding author e-mail: e-mail:atifaggour@yahoo.com

Theorem 1. *[7] A map $f : X \to Y$ is fuzzy continuous iff for each fuzzy point x_t in X and each fuzzy open nbd V of $f(x_t)$, there exists fuzzy open nbd U of x_t such that $f(U) \subseteq V$.*

Definition 5. *[6] Let $(X,\Im)$ be an fts, $x_t \in FP(X)$ and $\mu \in I^X$. The closure of μ, denoted by $cl(\mu)$ is defined by: $x_t \in cl(\mu)$ iff for each $\eta \in N^Q_{x_t}$, we have $\eta q \mu$. The fuzzy set μ is called closed if $\mu = cl(\mu)$.*

Definition 6. *[5] Let (X,τ_1) and (Y,τ_2) be fuzzy topological spaces, then the fuzzy topology $\tau = \tau_1 \times \tau_2$ on the set $X \times Y$ is defined as the initial fuzzy topology on $X \times Y$ making the projection mappings $P_1 : X \times Y \to X$ and $P_2 : X \times Y \to Y$ fuzzy continuous.*

Definition 7. *[6] A mapping $S : D \to FP(X)$ is called a fuzzy net in X and is denoted by $\{S(n) : n \in D\}$ or $\{S_n : n \in D\}$, where D is a directed set.*

Definition 8. *[6] A fuzzy net $\{\xi(m) : m \in M\}$ in X is called a fuzzy subnet of a fuzzy net $\{S(n) : n \in D\}$ iff there is a mapping $f : M \to D$ such that:*

i)$\xi_m = S_{f(m)}$, for each $m \in M$.
ii)For each $n \in D$ there exists some $m \in M$ such that, if $\rho \in M$ with $\rho \geq m$, then $f(\rho) \geq n$.

Definition 9. *[9] A fuzzy net $\{S(n) : n \in D\}$ in an fts X is said to be fuzzy converges to x_t if for each fuzzy open nbd ν of x_t there is some $n_0 \in D$ such that $n \geq n_0$ implies $S(n) \in \nu$.*

Definition 10. *[2] A fuzzy net $\{f_m : m \in M\}$ in $FC(X,Y)$ is said to be fuzzy continuously converges to $f \in FC(X,Y)$ iff for every x_t in X and for every fuzzy open nbd V of $f(x_t)$ in Y there exists an element $m_0 \in M$ and a fuzzy open nbd U of x_t in X such that $f_m(U) \subseteq V$, for every $m \in M$, $m \geq m_0$.*

2 Fuzzy Continuously Convergence of Fuzzy Nets

Definition 11. *Let I^X be the set of all fuzzy subsets of the fuzzy topological space X. If D is a directed set, then by $\underline{\lim}_D(\mu_\lambda)$, where $\mu_\lambda \in I^X$, we denote the fuzzy upper limit of the fuzzy net $\{\mu_\lambda : \lambda \in D\}$ in I^X, that is $x_t q \overline{\lim}_D(\mu_\lambda)$ iff for every $\lambda_0 \in D$ and for every fuzzy open nbd V of x_t in X there exists an element $\lambda \in D$ for which $\lambda \geq \lambda_0$ and $\mu_\lambda q V$.*

Definition 12. *Let D be a directed set, and for each $m \in D$ there are a directed set E_m and fuzzy net $\{f_m(n) : n \in E_m\}$ in $FC(X,Y)$. Then, for the directed set $T = D \times \prod_{m\in D} E_m$ (ordered by $(n_2,g) \geq (n_1,h)$ iff $n_2 \geq n_1$, and $g(n) \geq h(n)$ for each $n \in D$), we have a fuzzy net $f_{(m,g)} : T \to FC(X,Y)$ defined by $f_{(m,g)} = f_m(g(n))$, $n \in D$, $g \in \prod_{m\in D} E_m$. The fuzzy net $f_{(m,g)}$ is called the induced fuzzy net in $FC(X,Y)$.*

Definition 13. *Let C be a class of pairs (S, f), where S is a fuzzy net in $FC(X,Y)$ and $f \in FC(X,Y)$. We say that C is a continuously convergence class for $FC(X,Y)$ iff the following axioms listed below are satisfied. For convenience, we write SC-converges to f whenever $(S, f) \in C$:*

1.If $S = \{f_n : n \in D\}$ is a fuzzy net in $FC(X,Y)$ such that $f_n = f$ for each n, then $(S, f) \in C$;
2.If $(S, f) \in C$, then for every subnet T of S, $(T, f) \in C$;
3.If S does not C-converges to f, then there is a subnet of S, no subnet of which C-converges to f.
4.Let D be a directed set. For each $m \in D$, let E_m be a directed set and $f_m = \{f_m(n) : n \in E_m\}$ be a fuzzy net C-converges to $f(m)$ and let the fuzzy net $\{f(m) : m \in D\}$ C-converges to f. Then, the induced net $\{f_{(m,g)} = f_m(g(n)) : n \in D, g \in \prod_{m\in D} E_m\}$ C-converges to f.

Theorem 2. *A fuzzy net $\{f_n : n \in D\}$ in $FC(X,Y)$ fuzzy continuously converges to $f \in FC(X,Y)$ iff for every fuzzy net $\{\eta_m : m \in M\}$ in X which fuzzy converges to x_t in X we have that the fuzzy net $\{f_n(\eta_m) : (n,m) \in D \times M\}$ fuzzy converges to $f(x_t)$ in Y.*

*Proof.*Let x_t in X and let V be a fuzzy open nbd of $f(x_t)$ in Y such that for every $m_0 \in D$ and for every fuzzy open nbd U of x_t in X there exists $m \geq m_0$, $m \in D$ such that $f_m(U) \not\subseteq V$. Then, for every fuzzy open nbd U of x_t in X we can choose a fuzzy point $x^U_t \in U$ such that $f_m(x^U_t) \notin V$. It is clear that the fuzzy net $\{x^U_t : U \in N(x_t)\}$ fuzzy converges to x_t but the fuzzy net $\{f_n(x^U_t) : (U,n) \in N(x_t) \times D\}$ does not fuzzy converges to $f(x_t)$ in Y.

Conversely, let $\{S(n) : n \in \Lambda\}$ be a fuzzy net in X which fuzzy converges to x_t in X and let V be a fuzzy open nbd of $f(x_t)$ in Y. Then, there exists a fuzzy open nbd U of x_t in X and an element $n_0 \in D$ such that $f_n(U) \subseteq V$, for every $n \geq n_0$, $n \in D$. Since the fuzzy net $\{S(n) : n \in \Lambda\}$ fuzzy converges to x_t in X. There exists $n_0 \in \Lambda$ such that $S(n) \in U$, for every $n \in \Lambda$, $n \geq n_0$. Let $(n_0,m_0) \in \Lambda \times D$. Then, for every $(n,m) \in \Lambda \times D$, $n \geq n_0$, $m \geq m_0$ we have that, $f_m(S(n)) \in f_m(U) \subseteq V$. Thus, the fuzzy net $\{f_m(S(n)) : (m,n) \in D \times \Lambda\}$ fuzzy converges to $f(x_t)$ in Y.

Theorem 3. *A fuzzy net $\{f_m : m \in M\}$ in $F\underline{C}(X,Y)$ fuzzy continuously converges to $f \in FC(X,Y)$ iff $\underline{\lim}_M(f^{-1}_m(K)) \subseteq f^{-1}(K)$, for every fuzzy closed subset K of Y.*

Proof. Let $\{f_m : m \in M\}$ be a fuzzy net in $FC(X,Y)$, which fuzzy continuously converges to f and let K be arbitrary fuzzy closed subset of Y. Let $x_t q \underline{\lim}_M(f^{-1}_m(K))$ and let w be an arbitrary fuzzy open nbd of $f(x_t)$ in Y. Since the fuzzy net $\{f_m : m \in M\}$ fuzzy continuously converges to f, there exists a fuzzy open nbd V of x_t in X and an element $m_0 \in M$ such that $f_m(V) \subseteq w$, for every $m \in M$, $m \geq m_0$. Then $V q f^{-1}_m(K)$. Hence, $f_m(V) q f_m(f^{-1}_m(K)) \subseteq K$. So, wqK. This means that $f(x_t) q cl(K) = K$. Thus $x_t q f^{-1}(K)$.

Conversely, let $\{f_m : m \in M\}$ be a fuzzy net in $FC(X,Y)$ and $f \in FC(X,Y)$ such that $\lim_M(f_m^{-1}(K)) \subseteq f^{-1}(K)$, for every fuzzy closed subset K of Y. Let x_t be a fuzzy point in X and w be a fuzzy open nbd of $f(x_t)$ in Y. Let $K = w'$, then $x_t \not q f^{-1}(K)$. Then, we have that $x_t \not q \lim_M(f_m^{-1}(K))$. This means that there exists an element $m_0 \in M$ and a fuzzy open nbd V of x_t in X such that $f_m^{-1}(K) \not q V$, for every $m \in M$, $m \geq m_0$. Then, we have that $V \subseteq (f_m^{-1}(K))' = f_m^{-1}(K)' \subseteq f_m^{-1}(w)$. Therefore, $f_m(V) \subseteq w$, for every $m \in M$, $m \geq m_0$. Hence, the fuzzy net $\{f_m : m \in M\}$ fuzzy continuously converges to f.

Theorem 4. *If $\{\eta(n) : n \in D\}$ is a fuzzy net in $FC(X,Y)$ such that $\eta(n) = \eta$ for every $n \in D$, then $\eta(n)$ fuzzy continuously converges to $\eta \in FC(X,Y)$.*

Proof. Suppose that $\{\eta(n) : n \in D\}$ be a fuzzy net in $FC(X,Y)$ such that $\eta(n) = \eta$ for every $n \in D$. Let $\{S(e) : e \in E\}$ be a fuzzy net in X fuzzy converges to x_t. Since $\eta \in FC(X,Y)$, then the fuzzy net $\{\eta(S(e)) : e \in E\}$ fuzzy converges to $\eta(x_t)$ in Y. Therefore, $\eta_n(S_e) = \eta(S_e)$ fuzzy converges to $\eta(x_t)$. Hence, $\eta(n)$ fuzzy continuously converges to $\eta \in FC(X,Y)$.

Theorem 5. *If $\{\eta(n) : n \in D\}$ is a fuzzy net in $FC(X,Y)$ which fuzzy continuously converges to $\eta \in FC(X,Y)$ and $\{\xi(m) : m \in M\}$ is a subnet of $\{\eta(n) : n \in D\}$, then the fuzzy net $\{\xi(m) : m \in M\}$ is fuzzy continuously converges to η.*

Proof. Let x_t be a fuzzy point in X and V be a fuzzy open nbd of $\eta(x_t)$ in Y. Then, there is $n_0 \in D$ and a fuzzy open nbd U of x_t such that $\eta_n(U) \subseteq V$, for every $n \in D$, $n \geq n_0$. Since $\{\xi(m) : m \in M\}$ is a subnet of $\{\eta(n) : n \in D\}$, there is a map $f : M \to D$ such that:

(i) $\xi(m) = \eta_{f(m)}$;
(ii) for the element $n_0 \in D$, there is $m_0 \in M$ such that if $m \geq m_0$, $m \in M$, then $f(m) \geq n_0$.

Hence, we have $\xi_m(U) = \eta_{f(m)}(U) \subseteq V$, for every $m \geq m_0$, $m \in M$. Thus, the fuzzy net $\{\xi(m) : m \in M\}$ fuzzy continuously converges to η.

Theorem 6. *Let $\{f_m : m \in M\}$ be a fuzzy net in $FC(X,Y)$ which does not fuzzy continuously converges to f. Then, there is a subnet of $\{f_m : m \in M\}$ no subnet of which fuzzy continuously converges to $f \in FC(X,Y)$.*

Proof. Let $\{f_m : m \in M\}$ be a fuzzy net in $FC(X,Y)$, $f \in FC(X,Y)$ and let $\{f_m : m \in M\}$ does not fuzzy continuously converges to f. This means that $\lim_M(f_m^{-1}(K)) \not\subseteq f^{-1}(K)$, for some fuzzy closed subset K of Y. Let $x_t q \lim_M(f_m^{-1}(K))$. Let N_{x_t} be the set of all fuzzy open nbds of x_t in X directed by inclusion and let $H = M \times N_{x_t}$. If $\nu = (m,\mu) \in M \times N_{x_t}$, then we denote by $\tilde{m}$ the element of M such that $\tilde{m} \geq m$ and $f_{\tilde{m}}^{-1}(K) q \mu$ where $\tilde{m} = \phi(\nu)$. $\phi : M \times N_{x_t} \to M$. Obviously, the fuzzy net $\{g_\nu = f_{\tilde{m}} : \nu \in H\}$ is a subnet of $\{f_m : m \in M\}$. Let $\{\ell_s : s \in S\}$ be a subnet of $\{g_\nu : \nu \in H\}$ and ξ be the corresponding map of S into H. Let $s_0 \in S$ and V be an arbitrary fuzzy open nbd of x_t in X. If $\xi(s_0) = \nu_0 = (m_0, \mu_0)$, then if we take $V_0 = \mu_0 \cap \mu$, we have that there exists an element $s_1 \in S$ such that $s_1 \geq s_0$ and for every $s \geq s_1$ we have $\xi(s) \geq \nu_0$. Let $s \geq s_1$ and $\xi(s) = (m,V)$. Then $\ell_s^{-1}(K) \cap \mu = f_{\phi(\xi(s))}^{-1}(K) \cap \mu \geq f_{\phi(\xi(s))}^{-1}(K) \cap V_0 \geq f_{\phi(\xi(s))}^{-1}(K) \cap V$. Therefore, $x_t q \lim_S(\ell_s^{-1}(K))$. Hence $\lim_S(\ell_s^{-1}(K)) \not\subseteq f^{-1}(K)$. That is, $\{\ell_s : s \in S\}$ does not fuzzy continuously converges to f.

Theorem 7. *Let $FC(Y,Z)$ be a fuzzy topological space, let D be a directed set and $\{E_n : n \in D\}$ a family of directed sets, If $\{f_n : n \in D\}$ be a fuzzy net in $FC(Y,Z)$ continuously converges to f and $\{f_n(m) : m \in E_n\}$ be a fuzzy net in $FC(Y,Z)$ continuously converges to $f(n)$. Then, the induced fuzzy net $\{f_{(n,g)} : (n,g) \in D \times \prod_{n\in D} E_n\}$ in $FC(Y,Z)$ continuously converges to f.*

Proof. Let $FC(Y,Z)$ be a fuzzy topological space, let $\{f_n : n \in D\}$ be a fuzzy net in $FC(Y,Z)$ continuously converges to f, then there exists $\{\eta_\tau : \tau \in T\}$ be a fuzzy net in Y converges to y_r in Y, the fuzzy net $\{f_n(\eta_\tau) : (n,\tau) \in D \times T\}$ which fuzzy converges to $f(y_r)$. Thus there exists fuzzy open nbd v of $f(y_r)$ and there exists $n_0 \in D$ such that $f_n(\eta_\tau) \in v$ for every $n \geq n_0$. Since $f_n(\tau)(\eta_\tau) \in v$ for all $(n,\tau) \geq (n_0, h(n))$. Now, for $(n,g) \geq (n_0,h)$, we have $n \geq n_0, g(n) \geq h(n)$ and hence $f_{(n,g)}(\eta_\tau) \in v$. Hence, $\{f_{(n,g)} : (n,g) \in D \times \prod_{n\in D} E_n\}$ the induced fuzzy net in $FC(Y,Z)$ continuously converges to f.

Hence, the class C of all pairs (S,f) where S is a fuzzy net in $FC(X,Y)$ and SC-converges to $f \in FC(X,Y)$ is a continuously convergence class.

3 Fuzzy Function Spaces

In this section, we introduce fuzzy splitting topology and fuzzy jointly continuous topology on the set $FC(Y,Z)$. Also, we give a necessary and sufficient condition for the existence of the splitting and jointly continuous topology on the set $FC(Y,Z)$.

Notation: By FC^* we denote the class of all pairs $(\{f_n : n \in D\}, f)$ where $\{f_n : n \in D\}$ is a fuzzy net in $FC(Y,Z)$ which fuzzy continuously converges to f. If $\Im$ is a fuzzy topology on $FC(Y,Z)$, then by $FC(\Im)$ we denote the class of all pairs $(\{f_n : n \in D\}, f)$ where $\{f_n : n \in D\}$ is a fuzzy net in $FC(Y,Z)$ which fuzzy converges to $f \in FC(Y,Z)$ in the fuzzy topology $\Im$.

Definition 14. *A fuzzy topology $\Im$ on $FC(Y,Z)$ is called fuzzy splitting iff for every fts X, the fuzzy continuity of the map $\tilde{F} : X \times Y \to Z$ implies that of the map $\hat{F} : X \to FC_\Im(Y,Z)$, for which $\tilde{F}(x_t, y_m) = \hat{F}(x_t)(y_m)$.*

Theorem 8. *There exists the greatest splitting topology on the set $FC(Y,Z)$.*

Proof. Suppose that $\{\tau_i\}_{i\in\Lambda}$ be a family of fuzzy splitting topologies on $FC(Y,Z)$ and let $\Im = \sup_i\{\tau_i\}$. For any fuzzy topological space X, let $\tilde{F} : X \times Y \to Z$ be a fuzzy continuous map. Consider the map $\hat{F} : X \to FC_{\Im}(Y,Z)$. Let x_t in X and let U be a fuzzy open nbd of $\hat{F}(x_t)$ in $FC(Y,Z)$. Since $\Im = \sup_i \tau_i$, we have that $U \in \tau_i$ for some i. Also, since $\hat{F} : X \to FC_{\tau_i}(Y,Z)$ is fuzzy continuous, there exists a fuzzy open nbd V of x_t such that $\hat{F}(V) \subseteq U$. Thus, the map $\tilde{F}$ is fuzzy continuous and the fuzzy topology $\Im$ is fuzzy splitting.

Theorem 9. *A fuzzy topology $\Im$ on $FC(Y,Z)$ is fuzzy splitting topology iff $FC^* \subseteq FC(\Im)$.*

*Proof.*Let $\Im$ be a fuzzy splitting topology on $FC(Y,Z)$ and let $(\{f_\lambda : \lambda \in \wedge\}, f) \in FC^*$. Consider the set $X = \wedge \cup \{z\}$, where $z \notin \wedge$ is a symbol such that $z \geq \lambda$, for every $\lambda \in \wedge$. Then, we define a fuzzy topology on X by defining any singleton $\{x_\alpha\}$ where $x \in \wedge$ to be fuzzy open and a fuzzy nbds of z_t are the fuzzy sets $\{\chi_\lambda : \lambda \in X$ and $\lambda \geq \lambda_0\}$, for some $\lambda_0 \in \wedge\}$. Let $\tilde{F} : X \times Y \to Z$ be a map, for which $\tilde{F}(\lambda, y) = f_\lambda(y), \lambda \neq z$, and $\tilde{F}(z,y) = f(y)$, for every $y \in Y$. The map $\tilde{F}$ is fuzzy continuous. Also, $\hat{F}(\lambda) = f_\lambda$ and $\hat{F}(z) = f$. Since, the fuzzy topology $\Im$ is fuzzy splitting, the map $\hat{F} : X \to FC_{\Im}(Y,Z)$ is fuzzy continuous. Then, for every fuzzy open nbd μ of f in $FC_{\Im}(Y,Z)$, there exists a fuzzy open nbd v of z in X such that $\hat{F}(v) \subseteq \mu$. Hence, there exists $\lambda_0 \in \wedge$ such that $\lambda \in v$, for every $\lambda \in \wedge, \lambda \geq \lambda_0$. Therefore $\hat{F}(\lambda) = f_\lambda \in \mu$ for every $\lambda \in \wedge, \lambda \geq \lambda_0$ which means that the fuzzy net $\{f_\lambda : \lambda \in \wedge\}$ fuzzy converges to f in the fuzzy topology $\Im$. Thus $FC^* \subseteq FC(\Im)$.

Conversely, let $\Im$ be a fuzzy topology on $FC(Y,Z)$ such that $FC^* \subseteq FC(\Im)$. We aim to prove that the fuzzy topology $\Im$ is fuzzy splitting. Let X be any fts and , let $\tilde{F} : X \times Y \to Z$ be a fuzzy continuous map. Consider the map $\hat{F} : X \to FC_{\Im}(Y,Z)$. Let $\{S(n) : n \in D\}$ be a fuzzy net in X which fuzzy converges to x_t in X. We prove that the fuzzy net $\{\hat{F}(S(n)) : n \in D\}$ fuzzy converges to $\hat{F}(x_t)$. Let $\{\eta(m) : m \in M\}$ be a fuzzy net in Y which fuzzy converges to y_r in Y. Since the map $\tilde{F}$ is fuzzy continuous and the fuzzy net $\{(S(n), \eta(m)) : (n,m) \in D \times M\}$ in $X \times Y$ fuzzy converges to (x_t, y_r) in $X \times Y$, we have that $\{\tilde{F}(S(n), \eta(m)) : (n,m) \in D \times M\}$ fuzzy converges to $\tilde{F}(x_t, y_r)$ which means that $\{\hat{F}_{S(n)}\eta(m) : (n,m) \in D \times M\}$ fuzzy converges to $\hat{F}_{x_t}(y_r)$. Therefore, the fuzzy net $\{\hat{F}(S(n)) : n \in D\}$ fuzzy continuously converges to $\hat{F}(x_t)$. Since $FC^* \subseteq FC(\Im)$, then the fuzzy net $\{\hat{F}(S(n)) : n \in D\}$ fuzzy converges to $\hat{F}(x_t)$. Hence, the map $\hat{F}$ is fuzzy continuous and the fuzzy topology $\Im$ is fuzzy splitting.

Theorem 10. *A subset U of $FC(Y,Z)$ is fuzzy open in the finest splitting topology iff for every $f \in U$ and for every fuzzy net $\{f_n : n \in D\}$ in $FC(Y,Z)$ such that $\underline{\lim}_D(f_n^{-1}(K)) \subseteq f^{-1}(K)$ for each fuzzy closed subset K of Z, there exists $n_0 \in D$ such that $f_n \in U$ for every $n \geq n_0$.$(*)$*

Proof. It is clear that the set $\Im$ of all subsets U of $FC(Y,Z)$ satisfy the condition $(*)$ is a fuzzy topology on $FC(Y,Z)$. Also, we prove that this fuzzy topology is splitting. For any fuzzy topological space X, let $\tilde{F} : X \times Y \to Z$ be a fuzzy continuous map. Consider the map $\hat{F} : X \to FC_{\Im}(Y,Z)$, let $\{S(n) : n \in D\}$ be a fuzzy net in X which fuzzy converges to x_t in X. We prove that the fuzzy net $\{\hat{F}(S(n)) : n \in D\}$ in $FC(Y,Z)$ fuzzy converges to $\hat{F}(x_t)$. Let $\{\eta(m) : m \in M\}$ be a fuzzy net in Y fuzzy converges to y_r in Y. Since the map $\tilde{F}$ is fuzzy continuous and the fuzzy net $\{(S(n), \eta(m)) : (n,m) \in D \times M\}$ in $X \times Y$ fuzzy converges to (x_t, y_r) in $X \times Y$, we have that $\{\tilde{F}(S(n), \eta(m)) : (n,m) \in D \times M\}$ fuzzy converges to $\tilde{F}(x_t, y_r)$ which means that $\{\hat{F}_{S(n)}(\eta(m)) : (n,m) \in D \times M\}$ fuzzy converges to $\hat{F}_{x_t}(y_r)$. Therefore, the fuzzy net $\{\hat{F}(S(n)) : n \in D\}$ fuzzy converges to $\hat{F}(x_t)$. Hence, the map $\hat{F}$ is fuzzy continuous and the fuzzy topology $\Im$ is fuzzy splitting. Now, we prove that $\Im$ is the finest splitting topology on $FC(Y,Z)$. Let $\Im'$ be a fuzzy splitting topology on $FC(Y,Z)$ and let $V \in \Im'$. Suppose that $f \in V$ and $\{f_\lambda : \lambda \in D\}$ be a fuzzy net in $FC(Y,Z)$ such that the condition $(*)$ is satisfied, for every fuzzy closed subset K of Z. Then, $(\{f_\lambda : \lambda \in D\}, f) \in FC^*$. Since, $\Im'$ is a fuzzy splitting topology, $FC^* \subseteq FC(\Im')$ and so, $(\{f_\lambda : \lambda \in D\}, f) \in FC(\Im')$. Therefore, there exists $\lambda_0 \in D$ such that $f_\lambda \in V$, for each $\lambda \geq \lambda_0$. Thus, $V \in \Im$. Hence $\Im$ is the finest splitting topology.

Definition 15. *A fuzzy topology $\Im$ on $FC(Y,Z)$ is called fuzzy jointly continuous iff for any fts X, the fuzzy continuity of the map $\hat{G} : X \to FC_{\Im}(Y,Z)$ implies the fuzzy continuity of the map $\tilde{G} : X \times Y \to Z$, for which $\tilde{G}(x_t, y_m) = \hat{G}(x_t)(y_m)$.*

Theorem 11. *A fuzzy topology $\Im$ on $FC(Y,Z)$ is fuzzy jointly continuous iff the fuzzy evaluation map $e : FC_{\Im}(Y,Z) \times Y \to Z$ defined by $e(f, y_r) = f(y_r)$ is fuzzy continuous.*

Proof. Obviously, the identity map

$$\hat{G} = 1 : FC_{\Im}(Y,Z) \to FC_{\Im}(Y,Z)$$

is fuzzy continuous. Since, the fuzzy topology $\Im$ is fuzzy jointly continuous. Then, the map

$$\tilde{G} = e : FC_{\Im}(Y,Z) \times Y \to Z$$

is fuzzy continuous.

Conversely, let X be an fts, $\hat{G} : X \to FC_{\Im}(Y,Z)$ be a fuzzy continuous map and $1 : Y \to Y$ be the identity map. The map $\hat{G} \times 1 : X \times Y \to FC_{\Im}(Y,Z) \times Y$ is fuzzy continuous. Hence, the map $e \circ (\hat{G} \times 1) : X \times Y \to Z$ is fuzzy continuous.

Theorem 12. *A fuzzy topology $\Im$ on $FC(Y,Z)$ is fuzzy jointly continuous iff $FC(\Im) \subseteq FC^*$.*

Proof. Let $\Im$ be a fuzzy jointly continuous topology on $FC(Y,Z)$, X be the space which was defined in the proof of theorem 9 and $(\{f_\lambda : \lambda \in \wedge\}, f) \in FC(\Im)$. The map $\hat{G} : X \to FC_\Im(Y,Z)$, where $\hat{G}(\lambda) = f_\lambda$ and $\hat{G}(z) = f$ is fuzzy continuous. Thus, the map $\tilde{G} : X \times Y \to Z$ is fuzzy continuous. Let $\{S(n) : n \in D\}$ be a fuzzy net in Y fuzzy converges to y_r in Y. So, the fuzzy net $\{\chi_\lambda : \lambda \in \wedge\}$ in X fuzzy converges to z. Hence, the fuzzy net $\{(\chi_\lambda, S(n)) : (\lambda, n) \in \wedge \times D\}$ fuzzy converges to (z, y_r). Since the map $\tilde{G}$ is fuzzy continuous, the fuzzy net $\{\tilde{G}(\chi_\lambda, S(n)) = \hat{G}(\chi_\lambda(S(n))) = f_\lambda(S(n)) : (\lambda, n) \in \wedge \times D\}$ fuzzy converges to $\tilde{G}(z, y_r) = f(y_r)$ in Y.

Conversely, let $\Im$ be a fuzzy topology on $FC(Y,Z)$ such that $FC(\Im) \subseteq FC^*$. Our aim is to show that the fuzzy topology $\Im$ is fuzzy jointly continuous. Let X be arbitrary fts and let $\hat{G} : X \to FC_\Im(Y,Z)$ be a fuzzy continuous map. We shall prove that the map $\tilde{G} : X \times Y \to Z$ is fuzzy continuous. Let $\{(S(n), \eta(m)) : (n,m) \in D \times M\}$ be a fuzzy net in $X \times Y$ fuzzy converges to (x_t, y_r). Since the fuzzy net $\{S(n) : n \in D\}$ fuzzy converges to x_t in X and the map $\hat{G}$ is fuzzy continuous. The fuzzy net $\{\hat{G}(S(n)) : n \in D\}$ fuzzy converges to $\hat{G}(x_t)$. By the hypothesis the fuzzy net $\{\hat{G}(S(n)) : n \in D\}$ fuzzy continuously converges to $\hat{G}(x_t)$. Since $FC(\Im) \subseteq FC^*$. Therefore, the fuzzy net $\{\hat{G}(S(n))(\eta(m)) = \tilde{G}(S(n), \eta(m)) : (n,m) \in D \times M\}$ fuzzy converges to $\hat{G}(x_t)(y_r) = \tilde{G}(x_t, y_r)$. Hence, the fuzzy topology $\Im$ is fuzzy jointly continuous.

Acknowledgements

The authors are grateful to the referees for their fruitful comments for the improvement of this paper.

References

[1] R. Arens and J. Dugundji, "Topologies for Function Spaces", Pacific J. Math., **1** (1951), pp. 5-31.
[2] M. E. El-Shafei and A. I. Aggour, "Some Weaker Forms of Fuzzy Topologies on Fuzzy Function Spaces", J. Egypt. Math. Soc., **16**(1) (2008), pp. 27-35.
[3] Ying-Ming liu, "On Fuzzy Convergence Classes", Fuzzy Sets and Systems, **30** (1989), pp. 47-51.
[4] R. Lowen, "Fuzzy Topological Spaces and Fuzzy Compactness", J. Math. Anal. Appl., **56** (1976), pp. 621-633.
[5] R. Lowen, "Initial and Final Fuzzy Topologies and the Fuzzy Tychonoff Theorem", J. Math. Anal. Appl., **58** (1977), pp. 11-21.
[6] P. M. Pu and Y. M. Liu, "Fuzzy Topology I, Neighborhood Structure of a Fuzzy Point and Moore-Smith Convergence", J. Math. Anal. Appl. **76** (1980), pp. 571-599.
[7] P. M. Pu and Y. M. Liu, "Fuzzy Topology II, Product and quotient spaces", J. Math. Anal. Appl. **77** (1980), pp. 20-37.
[8] M. Macho Stadler and M. A. de Prada Vicente, "On N-convergence of fuzzy nets", Fuzzy Sets and Systems **51** (1992), pp. 203-217.
[9] M. N. Mukherjee and S. P. Sinha, "On Some Near-Fuzzy Continuous Functions Between Fuzzy Topological Spaces", Fuzzy Stes and Systems, **34** (1990), pp. 245-254.
[10] C. K. Wong, "Fuzzy Topology: Product and Quotient Theorems", J. Math. Anal. Appl., **45** (1974), pp. 512-521.
[11] C. K. Wong, "Fuzzy Points and Local Properties of Fuzzy Topologies", J. Math. Anal. Appl., **46** (1974), pp. 316-328.
[12] L. A. Zadeh, "Fuzzy sets", Inform. and Control **8** (1965), pp. 338-353.

A. I. Aggour is a leading world-known figure in mathematics and is presently employed as Assist. Prof. at Mathematics Department, Faculty of Sciences, Al-Azhar University. He obtained Ph D from Al-Azhar University. He introduced a new types of convergence on fuzzy function spaces by using fuzzy nets he is an active researcher, teaching experience in various countries of the arabic world. He has published more than 15 papers.

F. E. Attounsi is studied mathematics for five in a secondary school, she obtained her B.SC. (Mathematics) Faculty of Science, 7^th October University, Libya-Misurata, she studied pre-Masters courses of Ani Shams University (Egypt): Functional Analysis, Abstract algebra, Topology, Partial differential equations, Mathematical logic, specialized this manuscript in fuzzy topology. The notion of convergence is one of the basic notion in analysis, we introduce and study the fuzzy continuously converges theory of fuzzy nets on the set $FC(X,Y)$, we introduce and study of the important and properties of fuzzy nets on the set $FC(X,Y)$. Also, we introduce new fuzzy topologies on fuzzy function spaces and then introduce some results properties of the above concepts.