Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.12785/jsap/020210

Estimation for Rayleigh Distribution Using Progressive First-Failure Censored Data

Nasser A. Abou-Elheggag[1]

[1]Mathematics Department, Sohag University, Sohag 82524, Egypt

Received: 2 Mar. 2013, Revised: 7 Mar. 2013, Accepted: 13 Mar. 2013
Published online: 1 Jul. 2013

Abstract: In this paper, maximum likelihood and Bayes estimators of the scale parameter, survival function and hazard rate function are obtained for the Rayleigh failure time distribution, when the life test is progressively first-failure censored. Bayes estimators have been developed using the standard Bayes method under square error and LINEX loss functions, using inverted gamma prior for the parameter. Asymptotic confidence intervals and two bootstrap confidence intervals for the parameter are also proposed. We give an example to illustrate our proposed methods. Results from simulation studies assessing the performance of our proposed method are included. The Bayes estimates are found to be, generally, better than the maximum likelihood estimates against the proposed prior, in the sense of having smaller mean square errors.

Keywords: Rayleigh Distribution, Progressive First-Failure Censoring Scheme, Symmetric and Asymmetric Loss Functions, Maximum Likelihood Estimator, Bayes Estimator, Confidence Interval, Reliability function.

1 Introduction

Censoring is very common in life tests. There are survival types of censored tests. One of the most common censored test is type II censoring. It is noted that one can use type II censoring for saving time and money. However, when the lifetimes of products are very high, the experimental time of a type II censoring life test can be still too long. A generalization of type II censoring is the progressive type II censoring. Johnson [1] described a life test in which the experimenter might decide to group the test units into several sets, each as an assembly of test units and then run all the test units simultaneously until occurrence the first failure in each group. Such a censoring scheme is called first-failure censoring. Wu et al [2] and Wu and Yu [3] obtained MLE, exact confidence intervals and exact confidence regions for the parameters of the Gompertz and Burr Type XII distributions based on first failure-censored sampling, respectively. Also one can refer to Wu et al. [4] and Lee et al. [5]. Note that a first-failure-censoring scheme is terminated when the first failure in each set is observed. The first-failure censoring does not allow for sets to be removed from the test at the points other than the final termination point. however, this allowance will be desirable in practice. This leads us to the area of progressive censoring. Wu and Kuş. [6] combine the concepts of first-failure censoring and progressive censoring to develop a new life test plan called a progressive first-failure censoring scheme. Soliman et al. [7] studied the coefficient of variation of Gompertz distribution under progressive first-failure censoring. Soliman et al. [8,9] introduced MLE, Bayesian estimates, exact confidence intervals and exact confidence regions for the parameters of Gompertz and Burr Type XII distributions under progressive first failure-censored sampling.

Suppose that n independent groups with k items within each group are put in a life test. R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure $X^{\mathbf{R}}_{1;m,n,k}$ has occurred, R_2 groups and the group in which the second failure is observed are randomly removed from the test as soon as the second failure $X^{\mathbf{R}}_{2;m,n,k}$ has occurred and finally when the $m-th$ failure $X^{\mathbf{R}}_{m;m,n,k}$ is observed, the remaining groups R_m are removed from the test. Then the observed ordered lifetimes $X^{\mathbf{R}}_{1;m,n,k} < X^{\mathbf{R}}_{2;m,n,k} < \ldots < X^{\mathbf{R}}_{m;m,n,k}$ are called progressive first-

* Corresponding author e-mail:

failure censored order statistics with the progressive censored scheme $\mathbf{R} = (R_1, R_2, ..., R_m)$. It is clear that $n = m + \sum_{i=1}^{m} R_i$. If the failure times of the $n \times k$ items originally in the test are from a continuous population with distribution function $F(x)$ and probability density function $f(x)$, the joint probability density function for $X^{\mathbf{R}}_{1;m,n,k}, X^{\mathbf{R}}_{2;m,n,k}, ..., X^{\mathbf{R}}_{m;m,n,k}$ is given by

$$f_{1,2,...,m}(x^{\mathbf{R}}_{1;m,n,k}, x^{\mathbf{R}}_{2;m,n,k}, ..., x^{\mathbf{R}}_{m;m,n,k}) = Ck^m \prod_{i=1}^{m} f(x^{\mathbf{R}}_{i;m,n,k}) \times [1 - F(x^{\mathbf{R}}_{i;m,n,k})]^{k(Ri+1)-1} \tag{1}$$

$$0 < x^{\mathbf{R}}_{1;m,n,k} < x^{\mathbf{R}}_{2;m,n,k} < ... < x^{\mathbf{R}}_{m;m,n,k} < \infty,$$

where

$$C = n(n - R_1 - 1)(n - R_1 - R_2 - 2)...(n - R_1 - R_2 - ... - R_{m-1} - m + 1). \tag{2}$$

This censoring scheme has advantages in terms of reducing test time, in which more items are used but only m of $n \times k$ items are failures. Note that using the above notation, some censoring rules can be accommodated such as the first-failure censored order statistics when, $\mathbf{R} = (0, 0, ..., 0)$, a progressive type II censored order statistics when $k = 1$, a usual type II censored order statistics when $k = 1$ and $\mathbf{R} = (0, 0, ..., n - m)$ and complete sample case if $k = 1$ and $\mathbf{R} = (0, 0, ..., 0)$, with $n = m$. Also, it should be noted that the progressive first-failure censored sample $X^{\mathbf{R}}_{1;m,n,k}, X^{\mathbf{R}}_{2;m,n,k}, ..., X^{\mathbf{R}}_{m;m,n,k}$ with distribution function $F(x)$, can be viewed as a progressive type II censored sample from a population with distribution function $1 - (1 - F(x))^k$.

The RD provides a population model which is useful in several areas of statistics, Rayleigh [10]. References on this model may be found, among many others in Sinha and Howlader [11] and Anand [12]. Arturo [13] studied a Bayesian inference from Type-II doubly censored Rayleigh data. Statistical inference on Rayleigh distribution based on record values can be found in Soliman and AL-Aboud [14].

The probability density function (pdf), cumulative distribution function (cdf), failure rate $H(t)$, and reliability function $S(t)$ of the Rayleigh distribution with parameter $\lambda > 0$ respectively, given by

$$f(x;\lambda) = \frac{2x}{\lambda} \exp(-\frac{x^2}{\lambda}), \quad x > 0, \quad \lambda > 0, \tag{3}$$

$$F(x;\lambda) = 1 - \exp(-\frac{x^2}{\lambda}), \tag{4}$$

$$H(t) = \frac{2t}{\lambda}, \tag{5}$$

$$S(t) = \exp(-\frac{x^2}{\lambda}). \tag{6}$$

The rest of the paper is organized as follows. Section 2 introduces the MLE estimators and the asymptotic approximate confidence interval of the parameter. Two bootstrap confidence interval are discussed in Section 3. Section 4 describes Bayes method to estimate parameter as well as reliability and hazard rate functions. Section 5 contains the analysis of a simulate life data set to illustrate our proposed method. Results from simulation studies are given in Section 6. Finally we conclude with some comments in Section 7.

2 Maximum Likelihood Estimation

Let $X_i = X^{\mathbf{R}}_{i;m,n,k}$, $i = 1, 2, ..., m$, be a progressive first-failure censored order statistics from RD, with censored scheme $\mathbf{R} = (R_1, R_2, ..., R_m)$. From (1), (3) and (4), the likelihood function is given by

$$L\left(\underline{x};\lambda\right) = Ck^m 2^m \lambda^{-m} \left(\prod_{i=1}^{m} x_i\right) \exp\left(\frac{-1}{\lambda} \sum_{i=1}^{m} x_i^2 - \frac{1}{\lambda} \sum_{i=1}^{m} (k(R_i + 1) - 1) x_i^2\right), \tag{7}$$

where C is given by (2). The logarithm of the likelihood function may then be written as

$$l\left(\underline{x};\lambda\right) = Log(Ck^m 2^m) - mLog\lambda + Log(\prod_{i=1}^{m} x_i) - \frac{1}{\lambda} \sum_{i=1}^{m} k(R_i + 1) x_i^2. \tag{8}$$

Calculating the first partial derivatives of (8) with respect to λ and equating to zero, we obtain the likelihood equation

$$\frac{\partial \ell(\underline{x};\lambda)}{\partial \lambda} = -\frac{m}{\lambda} + \frac{k}{\lambda^2}\sum_{i=1}^{m}(R_i+1)x_i^2 = 0,$$

hence, the MLE of λ is

$$\hat{\lambda}_{\text{MLE}} = \frac{k\sum_{i=1}^{m}(R_i+1)x_i^2}{m} \tag{9}$$

By the invariance property of the MLE, we can obtain the MLEs of $H(t)$ and $S(t)$ by replacing λ by $\hat{\lambda}_{\text{MLE}}$ in (5) and (6), respectively.

2.1 Approximate Confidence Interval

From the log-likelihood function in (8), we have

$$\frac{\partial^2 \ell(\underline{x};\lambda)}{\partial \lambda^2} = \frac{m}{\lambda^2} - \frac{2k}{\lambda^3}\sum_{i=1}^{m}(R_i+1)x_i^2 \tag{10}$$

where $X_i \equiv X^R_{i;m,n,k}$. The Fisher information $I(\lambda)$ is then obtained by taking expectation of minus eqs. (10). In practice , we usually estimate $I\ (\lambda)$ by $I_0\ (\widehat{\lambda})$ where

$$I\ (\lambda)\ = -E\left[\frac{\partial^2 \ell(\underline{x};\lambda)}{\partial \lambda^2}\right] \tag{11}$$

$$I_{\ 0}(\widehat{\lambda})\ = -\left[\frac{m}{\widehat{\lambda}^2} - \frac{2k}{\widehat{\lambda}^2}\sum_{i=1}^{m}(R_i+1)\right] \tag{12}$$

Under some mild regularity conditions, $\widehat{\lambda}$ is approximately normally distributed with mean (λ) and variance $I(\lambda)$, i.e.

$$(\widehat{\lambda}) \sim N\left(\lambda, I_0(\widehat{\lambda})\right),$$

Thus, the 100$(1$-$\gamma)$% approximate confidence interval for λ is

$$\left(\widehat{\lambda} - z_{\frac{\gamma}{2}}\sqrt{I_0(\widehat{\lambda})}, \widehat{\lambda} + z_{\frac{\gamma}{2}}\sqrt{I_0(\widehat{\lambda})}\right) \tag{13}$$

where $z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\gamma}{2}$.

3 Bootstrap Confidence Intervals

The bootstrap is a resampling method for statistical inference. It is commonly used to estimate confidence intervals. More survey of the nonparametric and parametric bootstrap methods can be found in Davison and Hinkley [15], Efron and Tibshirani [16]. In this section, we use the parametric bootstrap method to construct Confidence Intervals (CI) for the unknown parameter. Two parametric bootstrap methods are used: (i) Studentized-t bootstrap (Boot-t) CI suggested by Hall (1988). (ii) Percentile bootstrap (Boot-p) CI suggested by Efron (1982).

The following steps are followed to obtain bootstrap sample from RD with parameter λ and based on simulated progressive first-failure-censoring order statistics.

1. From an original data set $\underline{x} \equiv x^R_{1;m,n,k}, x^R_{2;m,n,k}, \ldots, x^R_{m;m,n,k}$, compute the MLE of parameters λ say $\widehat{\lambda}$ from equations (9) .
2. Use $\widehat{\lambda}$ to generate a bootstrap sample $\underline{x}^*$with the same values of R_i, $(i = 1,2,\ldots,m)$ using the algorithm of Balakrishnan and Sandhu [17].
3. As in step 1 based on $\underline{x}^*$ compute the bootstrap sample estimates of λ say $\widehat{\lambda}^*$.
4. Repeat steps 2-3 N times representing N bootstrap MLE's of λ based on N different bootstrap samples.

5.Arrange all $\widehat{\lambda}^{*i}$ in an ascending order to obtain bootstrap sample $\left(\varphi^{[1]}, \varphi^{[2]}, ..., \varphi^{[N]}\right)$, where $(\varphi = \widehat{\lambda}^{*})$.

I- Percentile bootstrap method (Boot-p)

Let $G(x) = P(\hat{\varphi}^{*} \leq x)$ be the cumulative distribution function of $\hat{\varphi}_l^{*}$. Define $\varphi_{lboot-p} = G^{-1}(x)$ for given x. The approximate bootstrap $100(1-\gamma)\%$ confidence interval of φ is given by

$$\left[\varphi_{Boot-p}(\frac{\gamma}{2}), \varphi_{Boot-p}(1-\frac{\gamma}{2})\right]. \tag{14}$$

II- Bootstrap-t method (Boot-t)

Compute the following statistic:

$$T^{*} = \frac{\sqrt{m}(\hat{\varphi}^{*} - \hat{\varphi})}{\sqrt{Var(\hat{\varphi}^{*})}},$$

where $Var(\hat{\varphi}^{*})$ are obtained using the observed Fisher information matrix obtained in (12) . Using T^{*} values, determine the upper and lower bounds of the $100(1-\gamma)\%$ confidence interval of φ as follows: let $H(x) = P(T^{*} \leq x)$ be the cumulative distribution function of T^{*}. For a given x, define

$$\hat{\varphi}_{Boot-t}(x) = \hat{\varphi} + m^{-1/2}\sqrt{Var(\hat{\varphi})}H^{-1}(x).$$

Here also, $Var(\hat{\varphi})$ can be computed as same as computing the $Var(\hat{\varphi}^{*})$. The approximate $100(1-\gamma)\%$ confidence interval of φ are given by

$$\left(\hat{\varphi}_{Boot-t}(\frac{\gamma}{2}), \hat{\varphi}_{Boot-t}(1-\frac{\gamma}{2})\right). \tag{15}$$

4 Bayes Estimation

The Bayesian approach to reliability analysis allows prior subjective knowledge on lifetime parameters and technical information on the failure mechanism, as well as experimental data, to be incorporated into the inferential procedure. Hence Bayesian methods usually require less sample data to achieve the same quality of inferences than methods based on sampling theory, which becomes extremely important in case of expensive testing procedures. In this section, we discuss the Bayesian estimation of the RD based on progressive first-failure censored data.

4.1 Loss Functions

A wide variety of loss functions have been developed in literature to describe various types of loss structures. The symmetric squared error loss (SE) is one of the most popular loss functions. It is widely employed in inference, but its application is motivated by its good mathematical properties, not by its applicability to representing a true loss structure. A loss function should represent the consequences of different errors. There are situations where over- and under-estimation can lead to different consequences. For example, when we estimate the average reliable working life of the components of a spaceship or an aircraft, over-estimation is usually more serious than under-estimation. The SE loss equally penalizes over- and under-estimation of the same magnitude. The SE depend on the scalar error $(\widetilde{u} - u)$, which represents the distance between an unknown parameter u and the decision $\widetilde{u}$. Another way to measure this error is the expression $\widetilde{u}/u$. A value of this quotient close to 1 corresponds to a situation where the estimator is close to the unknown parameter. This expression is less than 1 in the case of over-estimation and greater than 1 for under-estimation. A useful asymmetric loss function of this type is known as the general entropy (GE) loss with the following form

$$L_1(\widetilde{u}, u) \propto \left(\frac{\widetilde{u}}{u}\right)^{q} - q\log\left(\frac{\widetilde{u}}{u}\right) - 1, \tag{16}$$

whose minimum occurs at $\widetilde{u} = u$. This loss function is a generalization of the Entropy-loss used in several papers where $q = 1$, see for example, Day et al. [18] and Day and Liu [19]. When $q > 0$, a positive error $(\widetilde{u} > u)$ causes more serious consequences than a negative error. The Bayes estimate $\widetilde{u}_{\text{BG}}$ of u under GE loss (16) is

$$\widetilde{u}_{\text{BG}} = (E_u[u^{-q}])^{-1/q}, \tag{17}$$

provided that $E_u[u^{-q}]$ exists, and is finite, where E_u is the posterior expectation.

4.2 Prior Distribution and Posterior Analysis

In this subsection we first describe the prior information needed for the Bayesian analysis of the unknown parameter. When the parameters λ is assumed to be unknown, we can use the conjugate inverted gamma (α, β) prior given by

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{-(\alpha+1)} \exp\left(\frac{-\beta}{\lambda}\right), \lambda > 0,\ \alpha > 0,\ \beta > 0. \tag{18}$$

By using the Bayes theorem, the conditional posterior density function of the parameter λ take the form

$$\pi(\lambda \mid \underline{x}) = \frac{L(\underline{x};\lambda)\, g(\lambda)}{\int_{\lambda} L(\underline{x};\lambda)\, g(\lambda)\, d\lambda}, \tag{19}$$

using (7) and (18), the conditional posterior density function of the parameter λ is the inverted gamma given by

$$\pi(\lambda \mid \underline{x}) = \frac{B^{A}}{\Gamma(A)} \lambda^{-(A+1)} \exp\left(-\frac{B}{\lambda}\right), \tag{20}$$

where

$$A = \alpha + m \text{ and B=k} \sum_{i=1}^{m} (R_i + 1) x_i^2 + \beta \tag{21}$$

4.2.1 **Bayes Estimation Under a Squared Error Loss function**

Based on progressively first-failure censored data, the Bayes estimator for the parameter λ under SE loss function, can be derived as

$$\widehat{\lambda}_{\mathrm{BS}} = E(\lambda \mid \underline{x}) = \int_{\lambda} \lambda . \pi(\lambda \mid \underline{x})\, d\lambda = \frac{B}{A-1}. \tag{22}$$

Similarly, the Bayes estimators $\widehat{S}_{\mathrm{BS}}(t)$ and $\widehat{H}_{\mathrm{BS}}(t)$, at mission time t of the reliability function $S(t)$ and hazard function $H(t)$ are given, respectively, by

$$\hat{S}_{\mathrm{BS}}(t) = \left[\frac{B}{B+t^2}\right]^{A}, \tag{23}$$

and

$$\widehat{H}_{\mathrm{BS}}(t) = \frac{2At}{B} \tag{24}$$

where A and B are as given by (21).

4.2.2 **Bayes Estimation Under General Entropy loss Function**

Under GE loss function, the Bayes estimate of the parameter λ is

$$\widehat{\lambda}_{\mathrm{BG}} = \left(E(\lambda^{-q} \mid \underline{x})\right)^{\frac{-1}{q}} = B\left[\frac{\Gamma(A+q)}{\Gamma(A)}\right]^{\frac{-1}{q}} \tag{25}$$

Similarly, the Bayes estimators $\widehat{S}_{\mathrm{BG}}(t)$ and $\widehat{H}_{\mathrm{BG}}(t)$ of $S(t)$ and $H(t)$ are given respectively, by

$$\widehat{S}_{\mathrm{BG}}(t) = \left[\frac{B}{B-q\,t^2}\right]^{\frac{-A}{q}}, \tag{26}$$

and

$$\widehat{H}_{\mathrm{BG}}(t) = \left[\frac{\Gamma(A-q)}{\Gamma(A)}\right]^{\frac{-1}{q}} \left[\frac{2t}{B}\right], \tag{27}$$

4.3 Bayesian Two-Sided Equitailed Probability Intervals

To construct $100(1-\gamma)\%$ Bayesian two-sided equitailed probability intervals, we need to solve the following two equations

$$\int_0^L \pi(\lambda \mid \underline{x})\, d\lambda = \frac{\gamma}{2} \text{ and } \int_U^\infty \pi(\lambda \mid \underline{x})\, d\lambda = \frac{\gamma}{2}, \tag{28}$$

hence

$$\frac{\Gamma(A, \frac{B}{L})}{\Gamma(A)} = \frac{\gamma}{2}, \tag{29}$$

$$\left(\frac{B}{U}\right)^A \frac{\Gamma(A, \frac{B}{U})}{\Gamma(A)} = 1 - \frac{\gamma}{2}, \tag{30}$$

where for given value of γ, the lower L and the upper U bounds are obtained from (29) and (30) numerically.

5 Illustrative Example

To illustrate the use of the estimation methods proposed in this article, a progressively first-failure-censored sample is generated from a RD using the algorithm of Balakrishnan and Sandhu [17]. We use: $\lambda = 2$, $k = 3$, $n = 30$ and $m = 15$. Table 1 lists the generated data. Concerning the hyperparameters of the prior, we consider two cases: prior1 (noninfarmative prior with $\alpha = \beta = 0$) and proir2: (informative prior with $\alpha = 4, \beta = 3$). The point estimates of the parameter, reliability and failure rate functions using the ML, bootostrap and Bayes methods are presented in Table 2. The 90% and 95% approximate confidence intervals (CIs), using maximum likelihood, bootstrap (Boot-p and Boot-t), as well as Bayes probability interval of paramter λ are presented in Table 3.

Table1: Simulated progressively first-failure-censored sample.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
R_i	3	0	2	0	0	2	0	0	0	1	0	0	3	0	4
$X_{i:15:30:3}$	0.092	0.098	0.231	0.325	0.388	0.389	0.458	0.464	0.487	0.571	0.685	0.774	0.854	0.863	894

Table2: The ML, Boot-p and Bayes estimates of λ, $S(t)$ and $H(t)$, $t = 0.8$

	$(.)_{ML}$	$(.)_{Boot}$	$(.)_{BS}$	$(.)_{BG}$	$(.)_{BG}$	$(.)_{BS}$	$(.)_{BG}$	$(.)_{BG}$
			Prior1			Prior2		
				q=-1	q=2		q=-1	q=2
λ	1.9347	1.9072	2.11228	2.03594	1.84597	1.66892	1.77895	1.64265
$S(t)$	0.71835	0.71419	0.6866	0.717172	0.70926	0.6701	0.6866	0.6787
$h(t)$	0.82698	0.83893	0.94937	0.840077	0.758295	0.89013	0.94937	0.87407

Table 3:Two-sided 90% and 95% confidence/probability intervals of λ

Method	95% C.I.	Length	90% C.I.	Length
MLE	(1.6959, 2.1939)	0.498	(1.7603, 2.3941)	0.6092
Boot-p	(1.2112, 2.8891)	1.6778	(1.1113, 3.0221)	1.9107
Boot-t	(0.8859, 3.5052)	2.6193	(0.9209, 3.5245)	2.6036
Bayes	(1.7098, 2.1446)	0.4348	(1.7944, 2.3065)	0.5121

6 Simulation Study

In this section we report some numerical experiments performed to evaluate the behavior of the proposed methods for different effective sample sizes, different sampling schemes, different parameter values and different priors. Using the fact that the progressive first-failure censored sample with distribution function $F(x)$, can be viewed as a progressive type II censored sample from a population with distribution function $1-(1-F(x))^k$, we generate a progressively first-falure censored samples from the CRD using the algorithm described in Balakrishnan and Sandhu [17], with $\lambda = 0.5$, $\lambda = 2$. We used the sample size $(n = 30)$, with effective sample sizes $(m = 15 \text{ and } 20)$, different k $(k = 1 \text{ and } 5)$, different values of the hyper parameters α and β, and different sampling schemes (i.e., different R_i values). For computing Bayes estimates, we used two informative priors, the first one is the prior 1, with ($\alpha = 0.5$, $\beta = 0.5$), the second is the prior 2, with ($\alpha = 2$, $\beta = 2$). In each case, we compute the MLE and Bayes estimates of λ, $S(t)$ and $H(t)$.We replicate the process 1000 times and compute the average values of the estimates and mean squared error (MSE). The results, up to four decimal places, are reported in Tables (4-7). It should be noted that in tables (4-7), Corresponding to each scheme, the first figure represents the average estimates, with the corresponding MSEs reported below.

7 Conclusions

Censoring is a common phenomenon in life-testing, and reliability studies. The subject of progressive censoring has received considerable attention in the past few years, due in part to the availability of high speed computing resources, which make it both a feasible topic for simulation studies for researchers and a feasible method of gathering lifetime data for practitioners. It has been illustrated by Viveros and Balakrishnan [20] that the inference is feasible and practical when the sample data are gathered according to a Type-II progressively censored experimental scheme. Combining the concept of first-failure censoring and the concept of progressive censoring, a progressive first-failure censoring scheme has been introduced by Wu and Kuş [6]. This censoring scheme has advantages in terms of reducing test cost and test time, in which more items are used but only m of $n \times k$ items are failures. Based on this new censoring scheme, the present paper shows how the things can be routinely managed for the Rayleigh model in a Bayesian and classical frameworks. We have considered the ML and Bayes estimates for survival time parameter, reliability and hazard functions, as well as the parameter of the Rayleigh model using progressively first-failure censored data. The Bayes estimators are discussed under symmetric and asymmetric loss functions. A simulation study was conducted to examine the performance of the different estimators. From the results, we observe the following:

1. Tables 4 and 6 shows that the Bayes estimates relative to the general entropy loss function has the smallest (MSE) if compared with both quadratic Bayes estimates or the MLEs for different choices of k, n, m and censoring scheme **R**.
2. If we adopt Bayesian approach in estimating reliability and hazard functions for different prior under symmetric and asymmetric loss functions, one would expect that estimators to be better (in the sense of MSE's) than the MLEs. In general, this can be seen in the results in Tables 5 and 6 for censoring scheme **R** . Also, the MSE's of the asymmetric Bayes estimates of reliability and hazard functions are smaller than MSE's of the symmetric Bayes estimates.
3. When the effective sample proportion m/n increases, the MSE of different Bayes estimators and MLEs are reduced. The censoring scheme $\mathbf{R} = (n-m, \ldots, 0)$ is most efficient for all choices, it seems to usually provide the smallest MSE for all estimators.
4. The results establish that for optimum decision making, important should be given on the choice of loss function and not just the choice of prior distribution only.

Table 4: Average values of the different estimates and the corresponding MSE when λ =0.5

k	m	(scheme)	MLE	Bayes Prior1			Bayes Prior2		
					q =0.5	q =2		q =0.5	q =2
			λ_{ML}	λ_{BS}	λ_{BG}	λ_{BG}	λ_{BS}	λ_{BG}	λ_{BG}
1	15	$(15,14^0)$	0.4995	0.5512	0.5241	0.5036	0.5893	0.5628	0.5467
			0.0177	0.0215	0.0176	0.0145	0.0225	0.0172	0.0139
		(15^1)	0.5025	0.5544	0.5173	0.5026	0.5799	0.5678	0.5439
			0.0161	0.0202	0.0160	0.0142	0.0196	0.0177	0.0146
		$(14^0,15)$	0.5007	0.5524	0.5252	0.4976	0.5979	0.5687	0.5469
			0.0164	0.0165	0.0165	0.0147	0.0246	0.0183	0.0148
	20	$(10,19^0)$	0.4986	0.537	0.5171	0.5026	0.5751	0.5552	0.5329
			0.012	0.014	0.012	0.0108	0.0173	0.0139	0.0104
		$(1,0,\cdots,1,0)$	0.5033	0.5418	0.5217	0.5032	0.5753	0.5493	0.5371
			0.0141	0.0165	0.0140	0.0114	0.0104	0.0125	0.0104
		$(19^0,10)$	0.5058	0.5444	0.5242	0.5031	0.5694	0.5490	0.5316
			0.0124	0.0149	0.0123	0.0118	0.0155	0.0123	0.0103
5	15	$(15,14^0)$	0.5034	0.6931	0.5278	0.5036	0.5633	0.5711	0.5445
			0.0174	0.0559	0.0176	0.0153	0.0194	0.0176	0.0144
		(15^1)	0.5012	0.6909	0.5257	0.4961	0.5761	0.5646	0.5393
			0.0158	0.0533	0.0158	0.0148	0.0211	0.0171	0.0145
		$(14^0,15)$	0.5008	0.6905	0.5253	0.5006	0.5691	0.5645	0.5448
			0.0171	0.0546	0.0171	0.0143	0.0196	0.0168	0.0138
	20	$(10,19^0)$	0.5044	0.6456	0.5214	0.5044	0.5500	0.5522	0.5313
			0.0114	0.0332	0.0124	0.0104	0.0131	0.0137	0.0114
		$(1,0,\cdots,1,0)$	0.4923	0.6332	0.5139	0.4928	0.5549	0.5536	0.5347
			0.0122	0.0305	0.0124	0.0111	0.0177	0.0135	0.0117
		$(19^0,10)$	0.4934	0.6342	0.5171	0.4938	0.5504	0.5497	0.5306
			0.0111	0.0297	0.0127	0.0101	0.0141	0.0127	0.0105

Table 5. Average values of the different estimates and the corresponding MSE when $\lambda = 0.5, S(0.8) = 0.278037, H(0.8). = 3.2$

k	m	scheme	MLE		Bayes Prior1					
							q = 0.5		q = 2	
			S_{ML}	H_{ML}	S_{BS}	H_{BS}	S_{BG}	H_{BG}	S_{BG}	H_{BG}
1	15	$(15,14^0)$	0.2696	3.4426	0.2953	3.3033	0.2727	3.1452	0.2544	2.9454
			0.0082	0.9897	0.0073	0.7394	0.0068	0.6206	0.0081	0.6079
		(15^1)	0.2724	3.408	0.2978	3.2726	0.2700	3.1795	0.2539	2.9540
			0.0076	0.9474	0.0082	0.7083	0.0072	0.6866	0.0082	0.6333
		$(14^0,15)$	0.2710	3.4179	0.2966	3.2819	0.2754	3.1235	0.2499	2.9938
			0.0076	0.9333	0.0086	0.6936	0.0070	0.6281	0.0087	0.6756
	20	$(10,19^0)$	0.2713	3.3687	0.2909	3.2723	0.2747	3.1529	0.2603	3.0072
			0.0057	0.6135	0.0052	0.4969	0.0054	0.4586	0.0059	0.4494
		$(1,0,\cdots,1,0)$	0.2737	3.3608	0.2931	3.264	0.2770	3.1448	0.2605	3.0100
			0.0065	0.6824	0.0060	0.5567	0.0062	0.5161	0.0062	0.4739
		$(19^0,10)$	0.2763	3.3258	0.2955	3.2322	0.2795	3.1142	0.2602	3.0177
			0.0058	0.6269	0.0055	0.5124	0.0055	0.4821	0.0065	0.4966
5	15	$(15,14^0)$	0.2724	3.4078	0.2979	3.2727	0.2768	3.1148	0.2541	2.9536
			0.0079	0.9286	0.0071	0.6981	0.0073	0.6349	0.0085	0.6285
		(15^1)	0.2715	3.4014	0.2971	3.2687	0.2759	3.1110	0.2486	2.9924
			0.0073	0.8244	0.0066	0.6239	0.0068	0.5688	0.0086	0.5891
		$(14^0,15)$	0.2706	3.4268	0.2963	3.2897	0.2751	3.1309	0.2523	2.9647
			0.0080	0.9462	0.0071	0.7095	0.0074	0.6402	0.0083	0.6192
	20	$(10,19^0)$	0.2757	3.3215	0.295	2.9917	0.2776	3.1334	0.2618	2.9917
			0.0054	0.5681	0.0051	0.4418	0.0055	0.4977	0.0057	0.4418
		$(1,0,\cdots,1,0)$	0.2668	3.4208	0.2866	3.3196	0.2722	3.1804	0.2529	3.0756
			0.006	0.7001	0.0053	0.5576	0.0056	0.4933	0.0065	0.4818
		$(19^0,10)$	0.268	3.3993	0.2878	3.3006	0.2744	3.1625	0.2541	3.0580
			0.0056	0.6255	0.0049	0.501	0.0057	0.5011	0.0060	0.4415

Table 6: Average values of the different estimates and the corresponding MSE when $\lambda = 2$

k	m	(scheme)	MLE	Bayes Prior1			Bayes Prior2		
					q =0.5	q =2		q =0.5	q =2
			λ_{ML}	λ_{BS}	λ_{BG}	λ_{BG}	λ_{BS}	λ_{BG}	λ_{BG}
1	15	$(15,14^0)$	1.9900	2.0931	1.9899	1.9053	2.0141	1.9237	1.8265
			0.2641	0.2912	0.2554	0.2490	0.2395	0.2242	0.2218
		(15^1)	1.9899	2.0930	1.9898	1.9253	2.0056	1.9156	1.8211
			0.2736	0.3013	0.2646	0.2425	0.2239	0.2113	0.2065
		$(14^0,15)$	1.9997	2.1032	1.9995	1.9457	1.9672	1.9030	1.7993
			0.2714	0.3011	0.2625	0.2507	0.2253	0.2105	0.2278
	20	$(10,19^0)$	1.9842	2.0607	1.9842	1.9171	2.0005	1.9055	1.8676
			0.2008	0.2147	0.1959	0.1988	0.1807	0.1753	0.1750
		$(1,0,\cdots,1,0)$	1.9697	2.0458	1.9699	1.9242	2.0044	1.9351	1.8689
			0.1953	0.2066	0.1905	0.1896	0.1747	0.1670	0.1827
		$(19^0,10)$	1.985	2.0615	1.9850	1.9451	2.0008	1.9317	1.8726
			0.1935	0.2071	0.1888	0.1776	0.1677	0.1610	0.1802
5	15	$(15,14^0)$	1.9954	2.2366	1.9952	1.9093	2.5162	1.9257	1.8501
			0.2569	0.3309	0.2485	0.2331	0.5168	0.2339	0.2210
		(15^1)	1.9959	2.2372	1.9957	1.8847	2.4938	1.9043	1.8315
			0.2711	0.3464	0.2622	0.2473	0.4675	0.2132	0.2169
		$(14^0,15)$	1.9821	2.2228	1.9821	1.8758	2.5224	1.9316	1.8134
			0.2748	0.3434	0.2658	0.2562	0.5108	0.2217	0.2357
	20	$(10,19^0)$	2.0007	2.1802	2.0006	1.9352	2.374	1.9246	1.8906
			0.1869	0.2291	0.1823	0.1823	0.3174	0.1708	0.1694
		$(1,0,\cdots,1,0)$	1.9813	2.1603	1.9814	1.9353	2.3812	1.9083	1.8674
			0.1956	0.2311	0.1908	0.1978	0.3235	0.1871	0.1729
		$(19^0,10)$	1.9855	2.1646	1.9855	1.9056	2.3621	1.9127	1.8538
			0.1890	0.2257	0.1843	0.1721	0.2995	0.1646	0.1779

Table 7. Average values of the different estimates and the corresponding MSE
when $\lambda = 2, S(0.8) = 0.726149$, H(0.8) = 0.8

k	m	scheme	MLE				Bayes Prior1			
							q = 0.5		q = 2	
			S_{ML}	H_{MLE}	S_{BS}	H_{BS}	S_{BG}	H_{BG}	S_{BG}	H_{BG}
1	15	$(15,14^0)$	0.7114	0.8628	0.7108	0.8746	0.7066	0.8324	0.7032	0.7860
			0.0045	0.0635	0.0043	0.0641	0.0047	0.0541	0.0050	0.0466
		(15^1)	0.7110	0.8654	0.7104	0.8771	0.7061	0.8347	0.7012	0.7916
			0.0048	0.0724	0.0045	0.0719	0.0049	0.8347	0.0050	0.0445
		$(14^0,15)$	0.7122	0.8601	0.7116	0.8719	0.7074	0.8298	0.7083	0.7703
			0.0046	0.0647	0.0044	0.0652	0.0048	0.0553	0.0048	0.0468
	20	$(10,19^0)$	0.7141	0.8497	0.7135	0.8589	0.7104	0.8276	0.7117	0.7808
			0.0032	0.0431	0.0031	0.0437	0.0033	0.0381	0.0032	0.0308
		$(1,0,\cdots,1,0)$	0.7123	0.8563	0.7118	0.8655	0.7086	0.8339	0.7088	0.7905
			0.0034	0.0455	0.0033	0.0462	0.0035	0.0401	0.0034	0.0326
		$(19^0,10)$	0.7147	0.8474	0.7140	0.8567	0.7110	0.8254	0.7119	0.7802
			0.0031	0.0410	0.0030	0.0416	0.0032	0.0363	0.0031	0.0314
5	15	$(15,14^0)$	0.7126	0.8578	0.7119	0.8697	0.7077	0.8277	0.7045	0.7814
			0.0043	0.0586	0.0040	0.0593	0.0044	0.0500	0.0047	0.0438
		(15^1)	0.7120	0.8605	0.7114	0.8723	0.7072	0.8302	0.7000	0.7963
			0.0045	0.0629	0.0043	0.0634	0.0047	0.0536	0.0054	0.0498
		$(14^0,15)$	0.7101	0.8678	0.7095	0.8796	0.7053	0.8371	0.699	0.7989
			0.0047	0.0673	0.0044	0.0677	0.0049	0.057	0.0053	0.0478
	20	$(10,19^0)$	0.7169	0.839	0.7163	0.8483	0.7132	0.8173	0.7105	0.7850
			0.0029	0.0374	0.0028	0.0379	0.0030	0.0334	0.0032	0.0317
		$(1,0,\cdots,1,0)$	0.7138	0.8513	0.7132	0.8605	0.7101	0.8290	0.7095	0.7890
			0.0034	0.0459	0.0032	0.0465	0.0035	0.0406	0.0037	0.0362
		$(19^0,10)$	0.7146	0.8482	0.7140	0.8574	0.7109	0.8261	0.7070	0.7959
			0.0033	0.0443	0.0031	0.0448	0.0033	0.0392	0.0034	0.0321

References

[1] L. G. Johnson, "Theory and Technique of Variation Research," *Elsevier, Amsterdam,* 1964

[2] I. W. Wu, W .L. Hung and C. H. Tsai, "Estimation of the Parameters of the Gompertz Distribution Under the First Failure-Censored Sampling Plan," *Statistics,* **37**, 2003, pp 517-525.

[3] J. W. Wu and H. Y. Yu, "Statistical Inference about the Shape Parameter of the Burr Type XII Distribution Under the Failure-Censored Sampling Plan," *Applied Mathematics and and Computation,* **163**, 2005, pp. 443–482.

[4] J. W. Wu, T. R. Tsai and L. Y. Ouyang, "Limited Failure-Censored Life Test for the Weibull Distribution," *IEEE Transactions on Reliability,* **50**, 2001, pp. 107-111.

[5] W. C. Lee, J. W. Wu and H. Y. Yu, "Statistical Inference about the Shape Parameter of the Bathtub-Shaped Distribution under the Failure-Censored Sampling Plan, " *International Journal of Information and Management Sciences,* **18**, 2007, pp. 157-172.

[6] S. J. Wu and C. Kuş, "On Estimation Based on Progressive First-Failure-Censored Sampling," *Computational Statistics and Data Analysis,* **53**, 2009, pp. 3659-3670.

[7] A. A. Soliman, A. H. Abd Ellah, N. A. Abo-Elheggag and G.A. Abd-Elmougod, "A simulation-Based Approach to the Study of Coefficient of Variation of Gompertz Distribution," *Indian Journal of Pure and Applied Mathematics,* **42**, No. 5, 2011, pp. 335-356.

[8] A. A. Soliman, A. H. Abd Ellah, N. A. Abo-Elheggag and G.A. Abd-Elmougod, "Estimation of the Parameters of Life for Gompertz Distribution Using Progressive First-Failure Censored Data," *Computational Statistics and Data Analysis,***51**, 2012, pp. 2065-2077.

[9] A. A. Soliman, A. H. Abd Ellah, N. A. Abo-Elheggag and, A. A. Modhesh, "Bayesian Inference and Prediction of Burr Type XII Distribution for Progressive First Failure-Censored Sampling," *Intelligent Information Management,***3**, 2011, pp. 175-185.

[10] L. Rayleigh, "On Waves Propagated Along the Plane Surface of an Elastic Solid," Proc. Lond. Math. Soc., **17**, 1985, pp. 4.

[11] S. K. Sinha and H. A. Howlader, "Credible and HPD Intervals of the Parameter and Reliability of Rayleigh Distribution," *IEEE Trans.Reliability,* Vol. **32**, 1983, pp. 217-220.

[12] S. Lalitha and M. Anand, "Modified maximum likelihood estimation for Rayleigh distribution, " Commun Statist. Theory Meth . **25** (2), (1996), 389-401.

[13] Arturo J. Fernandez, "Bayesian inference from type II doubly censored Rayleigh data," Statistics & Probability Letters **48** (2000) 393-399

[14] A. A. Soliman and AL-Aboud, "Bayesian Inference Using Recored Values from Rayleigh Model with Application," *European Journal of Operational Research,***185**, 2008, pp. 659-672.

[15] A. C. Davison and D. V. Hinkley, “Bootstrap Methods and their Applications,” *2nd, Cambridge University Press, Cambridge United Kingdom*, 1997.
[16] B. Efron and R. J. Tibshirani, “An Introduction to the Bootstrap,” *New York Chapman and Hall*, 1993.
[17] N. Balakrishnan and R. A. Sandhu, “A simple Simulation Algorithm for Generating Progressively Type-II Censored Samples,” *The American Statistician,* **49**, 1995, pp. 229-230.
[18] D.K. Dey, M. Ghosh and C. Srinivasan, “Simultaneous Estimation of Parameters Under Entropy Loss,” *Journal of Statistical Planning and Inference,* **25**, 1987, pp. 347–363.
[19] D.K. Dey and P.L. Liu, “On Comparison of Estimators in a Generalized Life Model,” *Microelectronics Reliability*, **32**, 1992, pp. 207–221.
[20] R. Viveros and N. Balakrishnan, “Interval Estimation of Parameters of Life from Progressively Censored Data,” *Technometrics,* **36**, 1994, pp. 84-91.