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On Some Properties of Spectral Radius for Brualdi-Li Matrix

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Abstract: Let B_{2m} denote the Brualdi-Li matrix,and let $\rho(B_{2m})$ denote the spectral radius of Brualdi-Li matrix. We obtain some properties of $\rho(B_{2m})$.

Keywords: Brualdi-Li Matrix,Spectral Radius,Reducible Matrix, Tournament Matrix

1 Introduction

A tournament matrix of order n is a $(0,1)$ matrix T satisfying the equation $T+T^t=J-I$, where J is the all ones matrix, I is the identity matrix, and T^t is the transpose of T. The tournament matrices are inspired in the round robin competitions.Tournament matrices(and their generalizations)appear in a variety of combinatorial applications (e.g., in biology, sociology, statistics, and networks).

Brualdi and Li matrix of order $2m$ is defined by

$$\mathscr{B}_{2m}=\begin{pmatrix} U_m & U_m^t \\ I+U_m^t & U_m \end{pmatrix},$$

where U_m is strictly lower triangular tournament matrix(all of whose entries below the main diagonal are equal to one). A matrix A of order n is said to be a reducible matrix if there exists a permutation matrix P such that

$$PAP^t=\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix},$$

where A_1 and A_2 are square (non-vacuous), or if $n=1$ and $A=O$. A matrix is called irreducible matrix if it is not reducible. The spectral radius of a matrix $A_{n\times n}$ defined as $\rho(A)=max\{|\lambda_1|,|\lambda_2|,\cdots,|\lambda_n|\}$, where $\lambda_1,\lambda_2,\cdots,\lambda_n$ are eigenvalues of $A_{n\times n}$. If an nonnegative matrix A is irreducible and it has exactly one eigenvalue of modulus $\rho(A)$,then the matrix is called a primitive matrix. Obviously,Brualdi and Li matrix $\mathscr{B}_{2m}$ $(m\geq 2)$is primitive matrix.

In 1983 Brualdi and Li conjectured that the maximal spectral radius for tournaments of order $2m$ is attained by the Brualdi-Li matrix [1]. This conjecture has recently be confirmed in [2]. The several interesting properties of Brualdi-Li matrix are studied. In this paper we investigate some properties of spectral radius for Brualdi-Li Matrix.

2 Preliminaries

The notation and terminology used in this paper will basically follow those in [3].

Let $\mathbf{1}_m=(1,1,\cdots,1)^t_{m\times 1}$,$\mathbf{0}_m=(0,0,\cdots,0)^t_{m\times 1}$, and

$$U_m=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{m\times m},$$

where $m\geq 2$ is an integer.

Lemma 2.1[3] Let n be a nonnegative integer,and A be a primitive matrix of order n. Then

$$\lim_{k\to\infty}(\frac{A}{\rho})^k\mathbf{1}_n=S,$$

where $\rho=\rho(A)>0$,$S>0$ is a eigenvector of A corresponding to the eigenvalue of $\rho(A)$.

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Let $b(2m,k)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^k\mathbf{1}_{2m}$,
$b_l(2m,k)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^k\binom{\mathbf{1}_m}{\mathbf{0}_m}$,and
$b_r(2m,k)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^k\binom{\mathbf{0}_m}{\mathbf{1}_m}$,then
$b(2m,k+1)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^{k+1}\mathbf{1}_{2m}=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^k\binom{(m-1)\mathbf{1}_m}{m\mathbf{1}_m}$
$=mb(2m,k)-b_l(2m,k)$
$=(m-1)b(2m,k)+b_r(2m,k)$.
It is easy to verify the following result.

Lemma 2.2 Let $k,m\geq 2$ be an integer,and $\rho=\rho(\mathscr{B}_{2m})$. Then
(1)$\lim_{k\to\infty}\sqrt[k]{b(2m,k)}=\rho$;
(2)$\lim_{k\to\infty}\frac{b(2m,k)}{b(2m,k-1)}=\rho$;
(3)$\lim_{k\to\infty}\frac{b_l(2m,k)}{b(2m,k)}=m-\rho$;
(4)$\lim_{k\to\infty}\frac{b_r(2m,k)}{b(2m,k)}=\rho-m+1$;
(5)$\lim_{k\to\infty}\frac{b_l(2m,k)}{b_r(2m,k)}=\frac{m-\rho}{\rho-m+1}$.

Let $\mathscr{B}_{2m}^k=\begin{pmatrix}B_{11}^{(k)} & B_{12}^{(k)}\\ B_{21}^{(k)} & B_{22}^{(k)}\end{pmatrix}$,and
$b_{ij}(2m,k)=\mathbf{1}_m^tB_{ij}^{(k)}\mathbf{1}_m, i,j=1,2,$
where $B_{11}^{(k)},B_{12}^{(k)},B_{21}^{(k)},B_{22}^{(k)}$ are matrices of order m.

Now that $b(2m,k+1)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^{k+1}\mathbf{1}_{2m}$
$=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^k\binom{(m-1)\mathbf{1}_m}{m\mathbf{1}_m}$
$=(m-1)(b_{11}(2m,k)+b_{21}(2m,k))$
$+m(b_{12}(2m,k)+b_{22}(2m,k))$,
$b(2m,k+1)=\mathbf{1}_{2m}^t(\mathscr{B}_{2m}^t)^{k+1}\mathbf{1}_{2m}$
$=\mathbf{1}_{2m}^t(\mathscr{B}_{2m}^t)^k\binom{m\mathbf{1}_m}{(m-1)\mathbf{1}_m}$
$=m(b_{11}(2m,k)+b_{12}(2m,k))$
$+(m-1)(b_{21}(2m,k)+b_{22}(2m,k))$,
we have
$b_{11}(2m,k)=b_{22}(2m,k)$.
$b(2m,k+2)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^{k+2}\mathbf{1}_{2m}$
$=(m\mathbf{1}_m^t,(m-1)\mathbf{1}_m^t)\mathscr{B}_{2m}^k\binom{(m-1)\mathbf{1}_m}{m\mathbf{1}_m}$
$=2(m-1)mb_{11}(2m,k)+m^2b_{12}(2m,k)$
$+(m-1)^2b_{21}(2m,k)$.
Leading to the following result.
Lemma 2.3 Let $k,m\geq 2$ be an integer. Then

$$(1)\begin{pmatrix}b(2m,k)\\ b(2m,k+1)\\ b(2m,k+2)\end{pmatrix}=\begin{pmatrix}2 & 1 & 1\\ 2m-1 & m & m-1\\ 2(m-1)m & m^2 & (m-1)^2\end{pmatrix}\begin{pmatrix}b_{11}(2m,k)\\ b_{12}(2m,k)\\ b_{21}(2m,k)\end{pmatrix};$$

$$(2)\begin{pmatrix}b_{11}(2m,k)\\ b_{12}(2m,k)\\ b_{21}(2m,k)\end{pmatrix}=\begin{pmatrix}-(m-1)m & 2m-1 & -1\\ (m-1)^2 & -2m+2 & 1\\ m^2 & -2m & 1\end{pmatrix}\begin{pmatrix}b(2m,k)\\ b(2m,k+1)\\ b(2m,k+2)\end{pmatrix}.$$

Lemma 2.4 ([4]) Let $m\geq 2$ be an integer, $\rho=\rho(\mathscr{B}_{2m})$,and $(v_1,v_2,\ldots,v_m,w_1,w_2,\ldots,w_m)^t$ be a eigenvector of $\mathscr{B}_{2m}$ corresponding to the eigenvalue of $\rho(\mathscr{B}_{2m})$,where $\Sigma_{i=1}^m v_i+\Sigma_{i=1}^m w_i=1$. Then
(1) $\rho=m-\Sigma_{i=1}^m w_i=m-1+\Sigma_{i=1}^m v_i$;
(2)$v_m=\frac{\rho+1-m}{\rho+1}$;
(3) $w_k-v_k=\frac{1-v_k(2\rho+1)}{\rho+1}$ and
$v_k=\frac{1}{2\rho+1}-\frac{2(\rho+\frac{1-m}{2})^2+(1-m)(\frac{1+m}{2})}{\rho(2\rho+1)}\cdot(\frac{\rho+1)}{\rho})^{2k+1}$,
$k=1,2,\cdots,m$.

3 Some properties for spectral radius of Brualdi-Li matrix

Theorem 3.1 Let $m\geq 2$ be an integer,and $\rho=\rho(\mathscr{B}_{2m})$.Then
(1)$\lim_{k\to\infty}\frac{b_{11}(2m,k)}{b(2m,k)}=-\rho^2+(2m-1)\rho-m(m-1)$;
(2)$\lim_{k\to\infty}\frac{b_{12}(2m,k)}{b(2m,k)}=\rho^2-2(m-1)\rho+(m-1)^2$;
(3)$\lim_{k\to\infty}\frac{b_{21}(2m,k)}{b(2m,k)}=\rho^2-2m\rho+m^2$.
Proof By Lemma 2.3(1),
$b_{11}(2m,k)=-m(m-1)b(2m,k)+(2m-1)b(2m,k+1)-b(2m,k+2)$, then
$\frac{b_{11}(2m,k)}{b(2m,k)}=-m(m-1)\frac{b(2m,k)}{b(2m,k)}$
$+(2m-1)\frac{b(2m,k+1)}{b(2m,k)}-\frac{b(2m,k+2)}{b(2m,k)}$. By Lemma 2.2(2),we have
$\lim_{k\to\infty}\frac{b_{11}(2m,k)}{b(2m,k)}=-m(m-1)\lim_{k\to\infty}\frac{b(2m,k)}{b(2m,k)}$
$+(2m-1)\lim_{k\to\infty}\frac{b(2m,k+1)}{b(2m,k)}-\lim_{k\to\infty}\frac{b(2m,k+2)}{b(2m,k)}$
$=-m(m-1)+(2m-1)\rho-\lim_{k\to\infty}(\frac{b(2m,k+2)}{b(2m,k+1)}\cdot\frac{b(2m,k+1)}{b(2m,k)})$
$=-\rho^2+(2m-1)\rho-m(m-1)$.
Using a similar approach,we have obtained (2)and (3).

Theorem 3.2 Let $m\geq 2$ be an integer,and $\rho=\rho(\mathscr{B}_{2m})$.Then
(1)$\lim_{k\to\infty}\frac{b_l(2m,k)}{b_l(2m,k-1)}=\lim_{k\to\infty}\frac{b_r(2m,k)}{b_r(2m,k-1)}=\rho$;
(2)$\lim_{k\to\infty}\frac{b_{ij}(2m,k)}{b_{ij}(2m,k-1)}=\rho$, for $i,j=1,2$.

Proof By Lemma 2.3(2),(3),
$\lim_{k\to\infty}\frac{b_l(2m,k)}{b_l(2m,k-1)}$
$=\lim_{k\to\infty}(\frac{b_l(2m,k)}{b(2m,k)}\cdot(\frac{b_l(2m,k-1)}{b(2m,k-1)})^{-1}\cdot\frac{b(2m,k)}{b(2m,k-1)})$
$=(m-\rho)(m-\rho)^{-1}\rho$
$=\rho$.
Using a similar approach,we have obtained
$\lim_{k\to\infty}\frac{b_r(2m,k)}{b_r(2m,k-1)}=\rho$ and (2).

Theorem 3.3 Let $m\geq 2$ be an integer, $\rho=\rho(\mathscr{B}_{2m})$,and $(v_1,v_2,\ldots,v_m,w_1,w_2,\ldots,w_m)^t$ be a eigenvector of $\mathscr{B}_{2m}$ corresponding to the eigenvalue of $\rho(\mathscr{B}_{2m})$,where $\Sigma_{i=1}^m v_i+\Sigma_{i=1}^m w_i=1$. Then

$$\Sigma_{i=1}^m i(w_i-v_i)=(m-\rho)^2.$$

Proof By Lemma 2.4,

$$\Sigma_{i=1}^{m} i(w_i - v_i)$$
$$= \Sigma_{i=1}^{m} i(\frac{1-v_i(2\rho+1)}{\rho+1})$$
$$= \Sigma_{i=1}^{m} \frac{i}{\rho+1} - \frac{2\rho+1}{\rho+1}\Sigma_{i=1}^{m} iv_i$$
$$= \Sigma_{i=1}^{m} \frac{i}{\rho+1} - \frac{2\rho+1}{\rho+1}\Sigma_{i=1}^{m} i(\frac{1}{2\rho+1}$$
$$-(\frac{2(\rho+\frac{1-m}{2})^2+\frac{(1-m)(1+m)}{2}}{\rho(2\rho+1)})(\frac{\rho+1}{\rho})^{2i+1})$$
$$= \frac{(2\rho^2+2(1-m)\rho+1-m)}{\rho(\rho+1)}\Sigma_{i=1}^{m} i(\frac{\rho+1}{\rho})^{2i-1}.$$

Now that $\Sigma_{k=1}^{m} kx^{k-1} = \frac{mx^{m+1}+1-(m+1)x^m}{(x-1)^2}$,thence

$$\Sigma_{i=1}^{m} i(\frac{\rho+1}{\rho})^{2i-1} = \frac{m(1+\frac{1}{\rho})^{2m+3}-(m+1)(1+\frac{1}{\rho})^{2m+1}+1+\frac{1}{\rho})}{((1+\frac{1}{\rho})^2-1)^2}$$
$$= \frac{\rho^4}{(2\rho+1)^2}(m(1+\frac{1}{\rho})^{2m+3}$$
$$-(m+1)(1+\frac{1}{\rho})^{2m+1}+1+\frac{1}{\rho}).$$

We have

$$\Sigma_{i=1}^{m} i(w_i - v_i)$$
$$= \frac{(2\rho^2+2(1-m)\rho+1-m)}{\rho(\rho+1)}(\frac{\rho^4}{(2\rho+1)^2})\cdot$$
$$(m(1+\frac{1}{\rho})^{2m+3}-(m+1)(1+\frac{1}{\rho})^{2m+1}+1+\frac{1}{\rho})$$
$$= \frac{\rho^3}{2\rho+1}(\frac{(2\rho^2+2(1-m)\rho+1-m)}{\rho(2\rho+1)})\cdot$$
$$(m(1+\frac{1}{\rho})^{2m+2}-(m+1)(1+\frac{1}{\rho})^{2m}+1)$$
$$= \frac{\rho^3}{2\rho+1}(\frac{(2\rho^2+2(1-m)\rho+1-m)}{\rho(2\rho+1)})(1+\frac{1}{\rho})^{2m-1}\cdot$$
$$(m(1+\frac{1}{\rho})^3-(m+1)(1+\frac{1}{\rho}))+\frac{\rho^2(2\rho^2+2(1-m)\rho+1-m)}{(2\rho+1)^2}$$
$$= \frac{\rho^3}{(2\rho+1)}(\frac{1}{2\rho+1}-\frac{\rho+1-m}{\rho+1})\cdot$$
$$(m(1+\frac{1}{\rho})^3-(m+1)(1+\frac{1}{\rho}))+\frac{\rho^2(2\rho^2+2(1-m)\rho+1-m)}{(2\rho+1)^2}$$
$$= \frac{1}{(2\rho+1)^2}(m+2m\rho-\rho^2)(m+2m\rho-2\rho^2-2\rho)$$
$$+\frac{\rho^2(2\rho^2+2(1-m)\rho+1-m)}{(2\rho+1)^2}$$
$$= \frac{1}{(2\rho+1)^2}(4\rho^4-4(2m-1)\rho^3+(4m^2-8m+1)\rho^2$$
$$+2m(2m-1)\rho+m^2)$$
$$= \frac{1}{(2\rho+1)^2}(4\rho^4+4\rho^3+(4m^2+1)\rho^2+4m^2\rho+m^2$$
$$-(8m\rho^3+8m\rho^2+2m\rho))$$
$$= \frac{1}{(2\rho+1)^2}((4\rho^2+4\rho+1)\rho^2+4m^2\rho^2+4m^2\rho+m^2$$
$$-(4\rho^2+4\rho+1)2m\rho)$$
$$= (m-\rho)^2.$$

References

[1] R.A. Brualdi and Q. Li, *Problem 31*, Discrete Math., **43**(1983) 329–330.

[2] S.W. Drury, *Solution of the Conjecture of Brualdi and Li*, Linear Algebra Appl., **436**(2012) 3392–3399.

[3] Liu Bolian, *Combinatorial matrix theory*, Bejing: Science Press,2006.

[4] J.Burk and M.Tsatsomeros, *On the Brualdi-Li Matrix and its Perron Eigenspace*, Electron J. Linear Algebra, **23** (2012)212–230.

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