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A Characterization of Distributions based on Conditional Expectations of Generalized Order Statistics

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Abstract: A general form of continuous distributions has been characterized by considering two conditional expectations of generalized order statistics (*gos*) conditioned on a non-adjacent *gos*. Further, some of its deductions are also discussed.

Keywords: Characterization of distributions, conditional expectation, probability distribution, generalized order statistics, order statistics and record statistics.

1 Introduction

The concept of generalized order statistics (*gos*) was introduced and extensively studied by [1].

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j > 0 \quad \text{for } 1 \leq i \leq n-1.$$

Then $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k), \ldots, X(n,n,\tilde{m},k)$ are called generalized order statistics from a continuous population with the distribution function (*df*) $F(x)$ and the probability density function (*pdf*) $f(x)$ if their joint *pdf* has the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)[1-F(x_n)]^{k-1} f(x_n) \tag{1.1}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$ of $\Re^n$.

The model of generalized order statistics contains special cases such as ordinary order statistics $(\gamma_i = n-i+1; i = 1,2,\ldots,n$ i.e. $m_1 = m_2 = \ldots = m_{n-1} = 0, k = 1)$, $k^{th}-$ record values $(\gamma_i = k$ i.e. $m_1 = m_2 = \ldots = m_{n-1} = -1, k \in N)$, sequential order statistics $(\gamma_i = (n-i+1)\alpha_i; \alpha_1, \alpha_2, \ldots, \alpha_n > 0)$, order statistics with non-integral sample size

$(\gamma_i = \alpha - i + 1; \alpha > 0)$, Pfeifer's record values $(\gamma_i = \beta_i;\ \beta_1, \beta_2, ..., \beta_n > 0)$ and progressive type II censored order statistics $(m_i \in N,\ k \in N)$ [1, 2].

Here we consider two cases:

Case I: $m_1 = m_2 = \cdots = m_{n-1} = m\ \ (m - gos)$.

Case II: $\gamma_i \neq \gamma_j\ \ , i \neq j$ for all $i, j \in (1, \cdots, n)$.

For Case I, the *pdf* of $X(r,n,m,k)$, the r^{th} $m - gos$ is given by [1]

$$f_{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x)\, [g_m(F(x))]^{r-1}, \tag{1.2}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is given by

$$f_{X(r,n,m,k), X(s,n,m,k)}(x, y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}\ [\bar{F}(x)]^m g_m^{r-1}[F(x)]$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(x)\, f(y) \tag{1.3}$$

where

$$\bar{F}(x) = 1 - F(x),$$

$$c_{s-1} = \prod_{i=1}^{s} \gamma_i\ ,$$

$$h_m(x) = \begin{cases} -\dfrac{(1-x)^{m+1}}{m+1}, & m \neq -1 \\ \log \dfrac{1}{(1-x)}, & m = -1 \end{cases} \qquad , x \in (0, 1),$$

and $$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m\, dt.$$

For Case II, the *pdf* of $X(r,n,\tilde{m},k)$ is given by [2]

$$f_{X(r,n,m,k)}(x) = c_{r-1}\ f(x) \sum_{i=1}^{r} a_i(r)\, [\bar{F}(x)]^{\gamma_i - 1} \tag{1.4}$$

and the joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$ is given by [2]

$$f_{X(r,n,\tilde{m},k), X(s,n,\tilde{m},k)}(x, y) = c_{s-1} \left(\sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{1-F(y)}{1-F(x)} \right]^{\gamma_i} \right)$$

$$\times \left(\sum_{i=1}^{r} a_i\,(r)\, [1 - F(x)]^{\gamma_i} \right) \frac{f(x)}{[1-F(x)]}\ \frac{f(y)}{[1-F(y)]} \tag{1.5}$$

where

$$a_i = a_i(r) = \prod_{\substack{j=1 \\ j\neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)} \quad , \quad \gamma_j \neq \gamma_i , \quad 1 \le i \le r \le n \tag{1.6}$$

and

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j\neq i}}^{s} \frac{1}{(\gamma_j - \gamma_i)} \quad , \quad \gamma_j \neq \gamma_i , \quad r+1 \le i \le s \le n \tag{1.7}$$

2 Characterizations of distributions when $m_1 = m_2 = \cdots = m_{n-1} = m$

The conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$ is

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y \mid x)$$

$$= \frac{c_{s-1}}{(s-r-1)!\, c_{r-1}} \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[1-F(y)]^{\gamma_s - 1} f(y)}{[1-F(x)]^{\gamma_{r+1}}} \tag{2.1}$$

Theorem 2.1: Let X be an absolutely continuous random variable with the *df* $F(x)$ and the *pdf* $f(x)$ in the interval (α, β), where α and β may be finite or infinite, then for $1 \le r < s < t \le n$,

$$E[h\{X(t,n,m,k)\} \mid X(r,n,m,k) = x]$$

$$= a_{t|s}^{*}\, E[h\{X(s,n,m,k)\} \mid X(r,n,m,k) = x] + b_{t|s}^{*} \tag{2.2}$$

if and only if

$$F(x) = 1 - [a\,h(x) + b]^{c} \tag{2.3}$$

Where $h(x)$ is monotonic and differentiable function and a, b, c are constant such that (2.3) is a *df*,

$$a_{t|s}^{*} = \prod_{j=s+1}^{t} \frac{c\gamma_j}{(1 + c\gamma_j)} \text{ and } b_{t|s}^{*} = -\frac{b}{a}(1 - a_{t|s}^{*})$$

Proof: In view of Khan and Alzaid (2004), it is easy to prove the necessary part.

For the sufficiency part, we have

$$\frac{c_{t-1}}{(t-r-1)!\, c_{r-1}(m+1)^{t-r-1}} \int_x^{\beta} h(y)[(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{t-r-1}$$

$$\times [1-F(y)]^{\gamma_t - 1} f(y)\, dy = a_{t|s}^{*} \frac{c_{s-1}}{(s-r-1)!\, c_{r-1}(m+1)^{s-r-1}}$$

$$\times \int_x^\beta h(y)[(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-1}$$

$$\times [1-F(y)]^{\gamma_s - 1} f(y)\,dy + b^*_{t|s}\ [1-F(x)]^{\gamma_{r+1}} \qquad (2.4)$$

Differentiating $(s-r)$ times both the sides of (2.4), *w.r.t.* x, we get

$$\frac{c_{t-1}}{(t-s-1)!\, c_{s-1}\,(m+1)^{t-s-1}} \int_x^\beta h(y) \frac{[(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{t-s-1}}{[1-F(x)]^{\gamma_{s+1}}}$$

$$\times [1-F(y)]^{\gamma_t - 1} f(y)\,dy = a^*_{t|s}\, h(x) + b^*_{t|s} \qquad (2.5)$$

or,

$$g_{t|s}(x) = E[h\{X(t,n,m,k)\} \mid X(s,n,m,k) = x] = a^*_{t|s}\, h(x) + b^*_{t|s}$$

Using the result of Khan *et al.* (2006), we have

$$\frac{f(x)}{\bar{F}(x)} = -\frac{a\,c\,h'(x)}{[a\ h(x) + b]}$$

Thus

$$\bar{F}(x) = [a\,h(x) + b]^c$$

and hence the theorem.

3 Characterizations of distributions when $\gamma_i \neq \gamma_j$, $i \neq j$

The conditional *pdf* of $X(s,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k) = x$, $1 \le r < s \le n$ is

$$f_{X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)}(y \mid x)$$

$$= \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_i} \frac{f(y)}{[1-F(y)]},\ x \le y \qquad (3.1)$$

Theorem 3.1: Under the conditions as given in the Theorem 2.1 and for $1 \le r < s < t \le n$,

$$E[h\{X(t,n,\tilde{m},k)\} \mid X(r,n,\tilde{m},k) = x]$$

$$= a^*_{t|s}\ E[h\{X(s,n,\tilde{m},k)\} \mid X(r,n,\tilde{m},k) = x] + b^*_{t|s} \qquad (3.2)$$

if and only if

$$F(x) = 1 - [a\,h(x) + b]^c \qquad (3.3)$$

where

$$a^*_{t|s} = \prod_{j=s+1}^{t} \frac{c\gamma_j}{1 + c\gamma_j} \text{ and } b^*_{t|s} = -\frac{b}{a}(1 - a^*_{t|s})$$

Proof: It is easy to prove the necessary part in view of Khan and Alzaid (2004) and the Theorem 2.1.

For the sufficiency part, we have

$$\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t} a_i^{(r)}(t)\int_x^\beta h(y)\left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_i}\frac{f(y)}{[1-F(y)]}\,dy$$
$$=a_{t|s}^{*}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s} a_i^{(r)}(s)\int_x^\beta h(y)\left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_i}\frac{f(y)}{[1-F(y)]}\,dy+b_{t|s}^{*} \tag{3.4}$$

Differentiating (3.4) *w.r.t.* x, we have

$$\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t} a_i^{(r)}(t)\left[-\frac{h(x)[\bar F(x)]^{\gamma_i-1}f(x)}{[\bar F(x)]^{\gamma_i}}+\gamma_i\int_x^\beta\frac{h(y)[\bar F(y)]^{\gamma_i-1}}{[\bar F(x)]^{\gamma_i+1}}f(x)\,f(y)\,dy\right]$$
$$=a_{t|s}^{*}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left[-\frac{h(x)[\bar F(x)]^{\gamma_i-1}f(x)}{[\bar F(x)]^{\gamma_i}}\right.$$
$$\left.+\gamma_i\int_x^\beta\frac{h(y)[\bar F(y)]^{\gamma_i-1}}{[\bar F(x)]^{\gamma_i+1}}f(x)\,f(y)\,dy\right]$$

or,

$$-\frac{f(x)}{[1-F(x)]}\frac{c_{t-1}}{c_{r-1}}\sum_{i=r+1}^{t} a_i^{(r)}(t)\,h(x)+\frac{f(x)}{[1-F(x)]}\frac{c_{t-1}}{c_{r-1}}$$
$$\times\sum_{i=r+1}^{t}\gamma_i a_i^{(r)}(t)\int_x^\beta\frac{h(y)[1-F(y)]^{\gamma_i-1}f(y)}{[1-F(x)]^{\gamma_i}}\,dy$$
$$=-a_{t|s}^{*}\frac{f(x)}{[1-F(x)]}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s} a_i^{(r)}(s)\,h(x)$$
$$+a_{t|s}^{*}\frac{f(x)}{[1-F(x)]}\frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}\gamma_i a_i^{(r)}(s)\int_x^\beta\frac{h(y)[1-F(y)]^{\gamma_i-1}f(y)}{[1-F(x)]^{\gamma_i}}\,dy$$

After noting that $\sum_{i=r+1}^{s} a_i^{(r)}(s)=0$, $c_r=\gamma_{r+1}c_{r-1}$ and $a_i^{(r+1)}(t)=(\gamma_{r+1}-\gamma_i)\,a_i^{(r)}(t)$, we get

$$\frac{f(x)}{[1-F(x)]}\gamma_{r+1}[g_{t|r}(x)-g_{t|r+1}(x)]=a_{t|s}^{*}\frac{f(x)}{[1-F(x)]}\gamma_{r+1}[g_{s|r}(x)-g_{s|r+1}(x)]$$

where

$$g_{s|r}(x)=E[h\{X(s,n,\tilde m,k)\}\,|\,X(r,n,\tilde m,k)=x]$$

or,

$$g_{t|r}(x)-a_{t|s}^{*}\,g_{s|r}(x)=g_{t|r+1}(x)-a_{t|s}^{*}\,g_{s|r+1}(x)$$
$$=\cdots=g_{t|s}(x)-a_{t|s}^{*}\,g_{s|s}(x)=b_{t|s}^{*} \tag{3.5}$$

Noting that $g_{s|s}(x) = h(x)$, we have

$$g_{t|s}(x) = a^*_{t|s}\, h(x) + b^*_{t|s}$$

i.e.

$$E[h\{X(t,n,\tilde{m},k)\} \mid X(s,n,\tilde{m},k) = x] = a^*_{t|s}\, h(x) + b^*_{t|s} \tag{3.6}$$

Using the result (Khan *et al.*, 2006)

$$E[h\{X(t,n,\tilde{m},k)\} \mid X(s,n,\tilde{m},k) = x] = g_{t|s}(x),$$

we get,

$$\bar{F}(x) = [a\, h(x) + b]^c$$

and hence the theorem.

Remark 3.1: It may be seen that when $\gamma_i \neq \gamma_j$ but at $m_i = m_j = m$, $i, j = 1, \cdots, n-1$, then [Khan *et al.* 2006]

$$a_i^{(r)}(s) = \frac{1}{(m+1)^{s-r-1}} (-1)^{s-i} \frac{1}{(i-r-1)!(s-i)!}$$

$$a_i(r) = \frac{1}{(m+1)^{r-1}} (-1)^{r-i} \frac{1}{(i-1)!\,(r-i)!}$$

and consequently the conditional *pdf* given in (3.1) will reduce to the conditional *pdf* given in (2.1). Thus the Theorem 3.1 will reduce to the Theorem 2.1 with $\gamma_j = k + (n-j)(m+1)$.

Remark 3.2: At $s = r$, result reduces to as obtained by Khan and Alzaid (2004).

Table 3.1: Examples based on the *df* $F(x) = 1 - [a\, h(x) + b]^c$

Distribution	$F(x)$	a	b	c	$h(x)$
Power function	$a^{-p}x^p$, $0 < x \le a$	$-a^{-p}$	1	1	x^p
Pareto	$1 - a^p x^{-p}$, $a \le x < \infty$	a^{-q}	0	$-p/q$	x^q, $q \neq 0$
Beta of the first kind	$1-(1-x)^p$, $0 \le x \le 1$	1 -1	0 1	p/q p	$(1-x)^q$, $q \neq 0$ x
Weibull	$1 - e^{-\theta x^p}$, $0 \le x < \infty$	1 $-\theta/c$	0 1	θ/q c	$e^{-qx^p}, q \neq 0$ $x^p, c \to 0$
Inverse Weibull	$e^{-\theta x^{-p}}$, $0 \le x < \infty$	-1	1	1	$e^{-\theta x^{-p}}$
Burr type II	$[1+e^{-x}]^{-k}$, $-\infty < x < \infty$	-1	1	1	$(1+e^{-x})^{-k}$

Burr type III	$(1+x^{-c})^{-k}$, $0 \le x < \infty$	-1	1	1	$(1+x^{-c})^{-k}$
Burr type IV	$\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}$, $0 \le x \le c$	-1	1	1	$\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}$
Burr type V	$[1+ce^{-\tan x}]^{-k}$, $-\pi/2 \le x \le \pi/2$	-1	1	1	$[1+ce^{-\tan x}]^{-k}$
Burr type VI	$[1+ce^{-k\sinh x}]^{-k}$, $-\infty < x < \infty$	-1	1	1	$[1+ce^{-k\sinh x}]^{-k}$
Burr type VII	$2^{-k}(1+\tanh x)^{k}$, $-\infty < x < \infty$	-2^{-k}	1	1	$[1+\tanh x]^{k}$
Burr type VIII	$\left(\frac{2}{\pi}\tan^{-1}e^{x}\right)^{k}$, $-\infty < x < \infty$	$-\left(\frac{2}{\pi}\right)^{k}$	1	1	$(\tan^{-1}e^{x})^{k}$
Burr type IX	$1-\frac{2}{c[(1+e^{x})^{k}-1]+2}$, $-\infty < x < \infty$	$\frac{c}{2}$	$1-\frac{c}{2}$	-1	$(1+e^{x})^{-k}$
Burr type X	$(1-e^{-x^{2}})^{k}$, $0 < x < \infty$	-1	1	1	$(1-e^{-x^{2}})^{k}$
Burr type XI	$\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^{k}$, $0 \le x \le 1$	-1	1	1	$(x-\frac{1}{2\pi}\sin 2\pi x)^{k}$
Burr type XII	$1-(1+\theta x^{p})^{-m}$, $0 \le x < \infty$	θ	1	$-m$	x^{p}
Cauchy	$\frac{1}{2}+\frac{1}{\pi}\tan^{-1}x$, $-\infty < x < \infty$	$-\frac{1}{\pi}$	$\frac{1}{2}$	1	$\tan^{-1}x$

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