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Approximations for some statistical moments of the solution process of stochastic Navier-Stokes equation using WHEP technique

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Abstract: Wiener-Hermite expansion linked with perturbation technique (WHEP) is used to solve the stochastic tow-dimensions non-linear Navier-Stokes equations. An approximate formula for the ensemble average, variance and some higher statistical moments of the stochastic solution process are obtained using WHEP technique and some cases study are considered to illustrate the method of analysis

Keywords: Stochastic Navier Stokes equations, perturbation method, WHEP Wiener–Hermite expansion (WHE).

1. Introduction

The Navier-Stokes equations are nothing more than a continuum formulation of Newton's laws of motion for material. They are a set of non linear partial differential equations which are thought to describe fluids motion for gasses and liquids, from laminar to turbulent flows, on scales ranging from below a millimeter to astronomical lengths.

The boundary value problems for PDEs with irregular coefficients, fluctuating source terms, or randomly excited boundary conditions are used in many fields of science and technology to describe uncertainty, probabilistic distribution of irregularities, or large ensembles of measurements under similar but randomly fluctuating conditions(e.g., see [1-7]).

Since Meecham and his co-workers [8] developed a theory of turbulence involving a truncated Wiener-Hermite expansion (WHE) of the velocity field, many authors studied problems concerning turbulence [9-14]. A number of general applications in fluid mechanics were also studied in [15-17]. Scattering problems attracted the WHE applications through many authors [18-22]. The non-linear oscillators were considered as an opened area for the applications of WHE as can be found in [23-29]. There are many applications in boundary value problems [30,31] and generally in different mathematical studies.

The application of the Wiener-Hermite expansion (WHE) aims at finding a truncated series solution to the solution process of differential equations. The truncated series are composed of two major parts; the first is the Gaussian part which consists of the first two terms, while the rest of the series constitute the non-Gaussian part. In non-linear cases, there exist always difficulties of solving the resultant set of deterministic integro-differential equations obtained from the applications of a set of comprehensive averages on the stochastic integro-differential equation resulted after the direct application of WHE. Many authors introduced different methods to face these obstacles. Among them, the WHEP technique was introduced in [28] using the perturbation method to solve non-linear problems. Many authors used the WHEP algorithm to find the approximations of the statistical moments of the solution process for a family of the stochastic differential equations which have an important in the applied sciences as diffusion equation, oscillatory, and Schrödinger equations (see, [32-43]).

This paper is organized as follows: in section 2, we give the description of the problem that includes the tow-dimensions Navier-Stokes equation under a stochastic excitation. In section 3, we introduce a survey on the Wiener-Hermite expansion and some formulas are obtained for the ensemble average, variance and some higher statistical moments of the stochastic process. In section 4, the equations which describe the deterministic kernels of the Wiener-Hermite expansion of the stochastic solution process of the problem are found using the statistical properties of Wiener-Hermite polynomials (WHPs). In section 5, we apply the perturbation theory to approximate the non-linear deterministic system. Finally in section 6, some case studies are presented to illustrate the mathematical analysis of the WHEP technique.

2. Description of the stochastic problem

Let us consider the following stochastic two-dimensions nonlinear perturbed Navier-Stokes equations

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\\
&\frac{\partial u}{\partial t}+\varepsilon\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\alpha\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]-\frac{\partial p}{\partial x},\\
&\frac{\partial v}{\partial t}+\varepsilon\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\alpha\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]-\frac{\partial p}{\partial y}+\sigma W(x;\omega),\\
&0\le x\le L,\quad 0\le y\le L,\quad t\ge 0,\\
&u(x,y,0)=\phi_1(x,y),\ v(x,y,0)=\phi_2(x,y),\\
&u(0,y,t)=u(L,y,t)=0, v(x,0,t)=v(x,L,t)=0,
\end{aligned}\right\} \tag{1}$$

where ε is a deterministic scale for the nonlinear term and the inhomogeneity term $\sigma W(x;\omega)$ is space Wiener process scaled by σ and ω is a random outcome of a triple probability space (Ω,P,B) where Ω is a sample space, B is a σ-algebra associated with Ω and P is a probability measure.

Eliminating p by differentiation the second and third equations from (1) with respect to y and x respectively, we get

$$\frac{\partial^2 v}{\partial x\partial t}-\frac{\partial^2 u}{\partial y\partial t}+\varepsilon\begin{bmatrix}\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+u\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}-\\ \frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}-v\frac{\partial^2 u}{\partial y^2}-u\frac{\partial^2 u}{\partial y\partial x}\end{bmatrix}$$
$$=\alpha\left[\frac{\partial^3 v}{\partial y^2\partial x}+\frac{\partial^3 v}{\partial x^3}-\frac{\partial^3 u}{\partial y\partial x^2}-\frac{\partial^3 u}{\partial y^3}\right]+\sigma n(x;\omega) \tag{2}$$

From the first equation of (1), we choose a random function $\psi(x,y,t;\omega)$ such that satisfies

$$v=\frac{\partial\psi}{\partial x},\quad u=-\frac{\partial\psi}{\partial y}\ . \tag{3}$$

Substituting from (3) into (2), hence the stochastic model (1) tends to the following form

$$\left.\begin{aligned}
&\frac{\partial\nabla^2\psi(x,y,t;\omega)}{\partial t}+\varepsilon\left[\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)}\right]=\alpha\nabla^4\psi+\sigma n(x;\omega)\\
&\psi(x,y,0)=\phi(x,y),\frac{\partial\psi(0,y,t)}{\partial y}=\frac{\partial\psi(L,y,t)}{\partial y}=0,\\
&\frac{\partial\psi(x,0,t)}{\partial x}=\frac{\partial\psi(x,L,t)}{\partial x}=0,
\end{aligned}\right\}, \tag{4}$$

where $\frac{\partial(\ldots,\ldots)}{\partial(x,y)}$ is the Jacobian determinate and

$$\nabla^4=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4},\quad \phi_1=-\frac{\partial\phi}{\partial y},\quad \phi_2=\frac{\partial\phi}{\partial x} \tag{5}$$

3. The Wiener Hermite expansion (WHE)

The Wiener-Hermite polynomials $H^{(i)}(x_1,x_2,..,x_i)$ (WHPs) are the elements of a complete set of statistically orthogonal random functions (See [14]) and satisfies the following recurrence relation,

$$H^{(i)}(x_1,x_2,\ldots\ldots,x_i)=H^{(i-1)}(x_1,x_2,\ldots\ldots,x_{i-1})H^{(1)}(x_i)$$
$$-\sum_{m=1}^{i-1}H^{(i-2)}(x_{i_1},x_{i_2},\ldots\ldots,x_{i_{i-2}})\delta(x_{i-m}-x_i),\ i\ge 2, \tag{6}$$

where

$$\left.\begin{aligned}
&H^{(0)}=1,\qquad H^{(1)}(x)=n(x),\\
&H^{(2)}(x_1,x_2)=H^{(1)}(x_1)H^{(1)}(x_2)-\delta(x_1-x_2),\\
&H^{(3)}(x_1,x_2,x_3)=H^{(2)}(x_1,x_2)H^{(1)}(x_3)-H^{(1)}(x_1)\delta(x_2-x_3)\\
&\qquad -H^{(1)}(x_2)\delta(x_1-x_3),\\
&H^{(4)}(x_1,x_2,x_3,x_4)=H^{(3)}(x_1,x_2,x_3)H^{(1)}(x_4)\\
&\qquad -H^{(2)}(x_1,x_2)\delta(x_3-x_4)\\
&\qquad -H^{(2)}(x_1,x_3)\delta(x_2-x_4)-H^{(2)}(x_2,x_3)\delta(x_1-x_4),
\end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}
&E\left[H^{(i)}(x_1,x_2,\ldots\ldots,x_i)\right]=0\quad\forall\, i\ge 1\\
&E\left[H^{(i)}(x_1,x_2,\ldots\ldots,x_i)H^{(j)}(x_1,x_2,\ldots\ldots,x_j)\right]=0\quad\forall\, i\ne j
\end{aligned}\right\} \tag{8}$$

where E denotes the ensemble average operator, $\delta(-)$ is the Dirac delta function and $n(x)$ is the stochastic white noise process which has the statistical properties,

$$E[n(x)]=0,\quad E[n(x_1)n(x_2)]=\delta(x_1-x_2) \tag{9}$$

Due to the completeness of the Wiener-Hermite set, any stochastic function $u(x;\omega)$ can be expanded as

$$\left.\begin{aligned}&u(x;\omega)=u^{(0)}(x)+\int_R u^{(1)}(x;x_1)H^{(1)}(x_1)dx_1+\\&\int_{R^2} u^{(2)}(x;x_1,x_2)H^{(1)}(x_1,x_2)dx_1dx_2+\\&\int_{R^3} u^{(3)}(x;x_1,x_2,x_3)H^{(3)}(x_1,x_2,x_3)dx_1dx_2dx_3+\ldots\ldots,\end{aligned}\right\} \tag{10}$$

4. Application of the WHE to approximate the stochastic solution process

In this section, the first order series of Wiener-Hermite expansion of the stochastic solution process $\psi(x,y,t;\omega)$ of the problem (4) is considered and takes the following form,

$$\psi(x,y,t;\omega)=\psi^{(0)}(x,y,t)+\int_0^L \psi^{(1)}(x,y,t;x_1)H^{(1)}(x_1)dx_1 \tag{11}$$

where the mean and the variance of the stochastic solution processes $u(x,y,t;\omega)$ and $v(x,y,t;\omega)$ are obtained from the following relations,

$$\left.\begin{aligned}&E[u(x,y,t;\omega)]=-\frac{\partial\psi^{(0)}}{\partial y},\\&E[v(x,y,t;\omega)]=\frac{\partial\psi^{(0)}}{\partial x}\\&Var[u(x,y,t;\omega)]=\int_0^L\left(\frac{\partial\psi^{(1)}}{\partial y}\right)^2dx_1\\&Var[v(x,y,t;\omega)]=\int_0^L\left(\frac{\partial\psi^{(1)}}{\partial x}\right)^2dx_1\end{aligned}\right\} \tag{12}$$

Hence, substituting from (11) into (4), we get the following stochastic equation

$$\left.\begin{aligned}&\left(\frac{\partial\nabla^2}{\partial t}-\alpha\nabla^4\right)\left(\psi^{(0)}(x,y,t)+\int_0^L\psi^{(1)}H^{(1)}(x_1)dx_1\right)+\\&\int_0^L\left(\frac{\partial(\psi^{(1)},\nabla^2\psi^{(0)})}{\partial(x,y)}+\frac{\partial(\psi^{(0)},\nabla^2\psi^{(1)})}{\partial(x,y)}\right)H^{(1)}(x_1)dx_1+\\&\int_0^L\int_0^L\frac{\partial(\psi^{(1)}(x_1),\nabla^2\psi^{(1)}(x_2))}{\partial(x,y)}H^{(1)}(x_1)H^{(1)}(x_2)dx_1dx_2\\&+\frac{\partial(\psi^{(0)},\nabla^2\psi^{(0)})}{\partial(x,y)}-\sigma\, n(x;\omega)=0\end{aligned}\right\}. \tag{13}$$

The deterministic kernels $\psi^{(0)}$ and $\psi^{(1)}$ are obtained from non-linear deterministic system which reduces after taking some statistical averages of Eq. (13) using the statistical properties of WHPs (See Appendix 1) and it is given by the following form.

$$\left.\begin{aligned}&\frac{\partial\nabla^2\psi^{(0)}}{\partial t}+\frac{\partial(\psi^{(0)},\nabla^2\psi^{(0)})}{\partial(x,y)}+\int_0^L\frac{\partial(\psi^{(1)},\nabla^2\psi^{(1)})}{\partial(x,y)}dx_1=\nabla^4\psi^{(0)},\\&\frac{\partial\nabla^2\psi^{(1)}}{\partial t}+\frac{\partial(\psi^{(1)},\nabla^2\psi^{(0)})}{\partial(x,y)}+\frac{\partial(\psi^{(0)},\nabla^2\psi^{(1)})}{\partial(x,y)}\\&\qquad=\nabla^4\psi^{(1)}+\sigma\delta(x-x_1),\\&\psi^{(0)}(x,y,0)=\phi(x,y),\ \psi^{(1)}(x,y,0,x_1)=0,\\&\frac{\partial\psi^{(0)}(0,y,t)}{\partial y}=\frac{\partial\psi^{(0)}(L,y,t)}{\partial y}=0\\&\frac{\partial\psi^{(0)}(x,0,t)}{\partial x}=\frac{\partial\psi^{(0)}(x,L,t)}{\partial y}=\frac{\partial\psi^{(1)}(x,0,t,x_1)}{\partial x}=\\&\frac{\partial\psi^{(1)}(x,L,t,x_1)}{\partial x}=\frac{\partial\psi^{(1)}(0,y,t,x_1)}{\partial y}=\frac{\partial\psi^{(1)}(0,L,t,x_1)}{\partial y}=0\end{aligned}\right\} \tag{14}$$

5. The application of perturbation method

In this section, we consider the perturbation theory to find some approximations for the deterministic non-linear system (14). Let us consider the kernels $\psi^{(0)}$ and $\psi^{(1)}$ in second correction of ε (See Appendix 2, [33]) as follow,

$$\begin{aligned}&\psi^{(0)}=\psi_0^{(0)}(x,y,t)+\varepsilon\psi_1^{(0)}(x,y,t)+\varepsilon^2\psi_2^{(0)}(x,y,t),\\&\psi^{(1)}=\psi_0^{(1)}(x,y,t;x_1)+\varepsilon\psi_1^{(1)}(x,y,t;x_1)+\varepsilon^2\psi_2^{(1)}(x,y,t;x_1).\end{aligned} \tag{15}$$

Substituting from (15) into (14), we get the following system of iterative linear equations

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\nabla^2\psi_0^{(0)}\right]=\alpha\nabla^4\psi_0^{(0)},\\&\psi_0^{(0)}(x,y,0)=\phi(x,y),\quad \frac{\partial\psi_0^{(0)}(0,y,t)}{\partial y}=\frac{\partial\psi_0^{(0)}(L,y,t)}{\partial y}\\&\qquad=\frac{\partial\psi_0^{(0)}(x,0,t)}{\partial x}=\frac{\partial\psi_0^{(0)}(x,L,t,)}{\partial x}=0,\end{aligned} \tag{16}$$

$$\frac{\partial\nabla^2\psi_0^{(1)}}{\partial t}=\alpha\nabla^4\psi_0^{(1)}+\sigma\delta(x-x_1),$$

$$\psi_0^{(1)}(x,y,0,x_1)=\frac{\partial\psi_0^{(1)}(0,y,t,x_1)}{\partial y}=\frac{\partial\psi_0^{(1)}(L,y,t,x_1)}{\partial y}$$

$$=\frac{\partial\psi_0^{(1)}(x,0,t,x_1)}{\partial x}=\frac{\partial\psi_0^{(1)}(x,L,t,x_1)}{\partial x}=0, \tag{17}$$

$$\frac{\partial\nabla^2\psi_1^{(0)}}{\partial t}+\left[\frac{\partial(\psi_0^{(0)},\nabla^2\psi_0^{(0)})}{\partial(x,y)}+\int_0^L\frac{\partial(\psi_0^{(1)},\nabla^2\psi_0^{(1)})}{\partial(x,y)}dx_1\right]=\alpha\nabla^4\psi_1^{(0)}$$

$$\psi_1^{(0)}(x,y,0)=\frac{\partial\psi_1^{(0)}(0,y,t)}{\partial y}=\frac{\partial\psi_1^{(0)}(L,y,t)}{\partial y}$$

$$=\frac{\partial\psi_1^{(0)}(x,0,t)}{\partial x}=\frac{\partial\psi_1^{(0)}(x,L,t,)}{\partial x}=0, \tag{18}$$

$$\frac{\partial\nabla^2\psi_1^{(1)}}{\partial t}+\left[\frac{\partial(\psi_1^{(0)},\nabla^2\psi_0^{(0)})}{\partial(x,y)}+\frac{\partial(\psi_0^{(0)},\nabla^2\psi_0^{(1)})}{\partial(x,y)}\right]=\alpha\nabla^4\psi_1^{(1)},$$

$$\psi_1^{(1)}(x,y,0;x_1)=\frac{\partial\psi_1^{(1)}(0,y,t;x_1)}{\partial y}=\frac{\partial\psi_1^{(1)}(L,y,t;x_1)}{\partial y}$$

$$=\frac{\partial\psi_1^{(1)}(x,0,t;x_1)}{\partial x}=\frac{\partial\psi_1^{(1)}(x,L,t;x_1)}{\partial x}=0, \tag{19}$$

$$\frac{\partial\nabla^2\psi_2^{(0)}}{\partial t}+\left[\begin{array}{l}\frac{\partial(\psi_1^{(0)},\nabla^2\psi_0^{(0)})}{\partial(x,y)}+\frac{\partial(\psi_0^{(0)},\nabla^2\psi_1^{(0)})}{\partial(x,y)}\\ \int_0^L\left[\frac{\partial(\psi_0^{(1)},\nabla^2\psi_1^{(1)})}{\partial(x,y)}+\frac{\partial(\psi_1^{(1)},\nabla^2\psi_0^{(1)})}{\partial(x,y)}\right]dx_1\end{array}\right]$$

$$=\alpha\nabla^4\psi_2^{(0)},$$

$$\psi_2^{(0)}(x,y,0)=\frac{\partial\psi_2^{(0)}(0,y,t)}{\partial y}=\frac{\partial\psi_2^{(0)}(L,y,t)}{\partial y}$$

$$=\frac{\partial\psi_2^{(0)}(x,0,t)}{\partial x}=\frac{\partial\psi_2^{(0)}(x,L,t,)}{\partial x}=0, \tag{20}$$

$$\frac{\partial\nabla^2\psi_2^{(1)}}{\partial t}+\left[\begin{array}{l}\frac{\partial(\psi_0^{(1)},\nabla^2\psi_1^{(0)})}{\partial(x,y)}+\frac{\partial(\psi_1^{(1)},\nabla^2\psi_0^{(0)})}{\partial(x,y)}+\\ \frac{\partial(\psi_0^{(0)},\nabla^2\psi_1^{(1)})}{\partial(x,y)}+\frac{\partial(\psi_1^{(0)},\nabla^2\psi_0^{(1)})}{\partial(x,y)}\end{array}\right]=\alpha\nabla^4\psi_2^{(1)},$$

$$\psi_2^{(1)}(x,y,0;x_1)=\frac{\partial\psi_2^{(1)}(0,y,t;x_1)}{\partial y}=\frac{\partial\psi_2^{(1)}(L,y,t;x_1)}{\partial y}$$

$$=\frac{\partial\psi_2^{(1)}(x,0,t;x_1)}{\partial x}=\frac{\partial\psi_2^{(1)}(x,L,t;x_1)}{\partial x}=0, \tag{19}$$

6. The application of eigenfunctions expansion

The system of iterative linear partial differential equations (16-21) of the pervious section has the following general form

$$\frac{\partial}{\partial t}\left[\nabla^2\Psi(x,y,t)\right]+f(x,y,t)=\alpha\nabla^4\Psi(x,y,t),$$

$$t\ge 0,\quad 0\le x\le L,\quad 0\le y\le L,\quad \Psi(x,y,0)=\Phi(x,y),$$

$$\frac{\partial\Psi(0,y,t)}{\partial y}=\frac{\partial\Psi(L,y,t)}{\partial y}=\frac{\partial\Psi(x,0,t)}{\partial x}=\frac{\partial\Psi(x,L,t)}{\partial x}=0, \tag{22}$$

which has a general exact solution which is discussed in appendix 1. Finally, from the pervious results, the exact solutions of the components of the perturbation series take the following form,

$$\left.\begin{array}{l}\psi_0^{(0)}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}R_1(n,m,t)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}y\right),\\ R_1(n,m,t)=T_{n,m}\,e^{-\alpha\lambda_{n,m}t},\\ T_{n.m}=\frac{4}{L^2}\int_0^L\int_0^L\phi(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)dxdy\end{array}\right\} \tag{23}$$

$$\psi_0^{(1)}=$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}R_2(n,m,t)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{(2m-1)\pi}{L}y\right)\sin\left(\frac{n\pi}{L}x_1\right),$$

$$R_2(n,m,t)=-\frac{8\sigma}{L\pi(2m-1)\lambda_{n,2m-1}}\int_0^t e^{-\alpha\lambda_{n,2m-1}(t-\tau)}d\tau \tag{24}$$

$$\left.\begin{array}{l}\psi_1^{(0)}=\left(\sum_{n=1}^{4}Q_n(t)\sin\left(\frac{n\pi}{L}y\right)\right)\sin\left(\frac{2\pi}{L}x\right),\\ Q_1(t)=\frac{3L^2}{16}R_3(2,1,t),\quad Q_2(t)=\frac{L^2}{16}R_4(2,2,t),\\ Q_3(t)=-\frac{L^2}{16}R_3(2,3,t),\quad Q_4(t)=-\frac{L^2}{2}R_4(2,4,t),\\ R_3(n,m,t)=\frac{4}{L^2\lambda_{n,m}}\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}q_1(\tau)d\tau,\\ R_4(n,m,t)=\frac{4}{L^2\lambda_{n,m}}\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}q_2(\tau)d\tau,\\ q_1(t)=3\left(\frac{\pi}{L}\right)^4\left[R_1(1,1,t)R_1(1,2,t)\right],\\ q_2(t)=\frac{16\pi^4}{L^3}\left[R_2(1,1,t)R_2(1,2,t)\right]\end{array}\right\} \tag{25}$$

$$
\left.
\begin{aligned}
&\psi_1^{(1)}=\left(\sum_{n=1}^{5}V_n(t)\sin\left(\frac{n\pi}{L}y\right)\right)\sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{\pi}{L}x_1\right)\\
&V_1(t)=\frac{L^2}{8}\left[2R_9(2,1,t)-R_6(2,1,t)\right]\\
&V_2(t)=\frac{L^2}{8}\left[R_5(2,2,t)-2R_7(2,2,t)\right],\\
&V_3(t)=\frac{L^2}{8}\left[R_6(2,3,t)-R_8(2,3,t)\right],\\
&V_4(t)=\frac{L^2}{8}R_7(2,4,t),\qquad V_5(t)=\frac{L^2}{8}R_8(2,5,t),\\
&R_5(n,m,t)=\frac{2}{L^2\lambda_{n,m}}\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}q_3(\tau)d\tau,\\
&R_6(n,m,t)=\frac{4}{L^2\lambda_{n,m}}\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}q_4(\tau)d\tau,\\
&R_7(n,m,t)=\frac{4}{L^2\lambda_{n,m}}\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}q_5(\tau)d\tau,\\
&R_8(n,m,t)=\frac{4}{L^2\lambda_{n,m}}\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}q_6(\tau)d\tau,\\
&R_9(n,m,t)=\frac{4}{L^2\lambda_{n,m}}\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}q_7(\tau)d\tau,
\end{aligned}
\right\}\qquad (26)
$$

$$
\left.
\begin{aligned}
&q_3(t)=-2\left(\frac{\pi}{L}\right)^4\left[R_1(1,1,t)R_2(1,2,t)\right],\\
&q_4(t)=-\frac{1}{2}\left(\frac{\pi}{L}\right)^4R_1(1,2,t)\left[7R_2(1,1,t)+25R_2(1,2,t)\right]\\
&q_5(t)=\frac{13}{2}q_3(t),\quad q_6(t)=-\frac{25}{2}\left(\frac{\pi}{L}\right)^4\left[R_1(1,2,t)R_2(1,2,t)\right],\\
&q_7(t)=\frac{1}{2}\left(\frac{\pi}{L}\right)^4R_1(1,2,t)\left[3R_2(1,1,t)-10R_2(1,2,t)\right].
\end{aligned}
\right\}\qquad (27)
$$

$$
\left.
\begin{aligned}
&\psi_2^{(0)}=\left(\sum_{n=1}^{2}a_n(t)\sin\left(\frac{(2n-1)\pi}{L}x\right)\right)\sin\left(\frac{2\pi}{L}y\right)\\
&a_1(t)=\frac{3L^2}{16}\left[R_{10}(2,1,t)\right],\quad a_2(t)=-\frac{L^2}{16}\left[R_{10}(2,3,t)\right],\\
&R_{10}(n,m,t)=\frac{4}{L^2\lambda_{n,m}}\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}q_8(\tau)d\tau,\\
&q_8(\tau)=-3\left(\frac{\pi}{L}\right)^4\left[R_1(1,1,t)Q_1(t)+L\,R_2(1,1,t)V_1(t)\right]
\end{aligned}
\right\}\qquad (28)
$$

$$
\left.
\begin{aligned}
&\psi_2^{(1)}=\left(\sum_{n=1}^{2}b_n(t)\sin\left(\frac{(2n-1)\pi}{L}x\right)\right)\sin\left(\frac{2\pi}{L}y\right)\sin\left(\frac{\pi}{L}x_1\right),\\
&b_1(t)=\frac{L^2}{16}\left[4R_{11}(1,2,t)-2R_{12}(1,2,t)+3R_{13}(1,2,t)\right],\\
&b_2(t)=\frac{L^2}{8}\left[R_{12}(3,2,t)+R_{13}(2,3,t)\right],\\
&R_{11}(n,m,t)=\frac{4}{L^2\lambda_{n,m}}\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}q_9(\tau)d\tau,\\
&R_{12}(n,m,t)=\frac{4}{L^2\lambda_{n,m}}\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}q_{10}(\tau)d\tau,\\
&R_{13}(n,m,t)=\frac{4}{L^2\lambda_{n,m}}\int_0^t e^{-\alpha\lambda_{n,m}(t-\tau)}q_{11}(\tau)d\tau,\\
&q_9(t)=-\frac{3}{2}\left(\frac{\pi}{L}\right)^4\left[R_1(1,1,t)V_1(t)\right],\quad q_{10}(t)=\frac{7}{3}q_9(t),\\
&q_{11}(t)=-3\left(\frac{\pi}{L}\right)^4\left[R_2(1,1,t)Q_1(t)\right]
\end{aligned}
\right\}\qquad (29)
$$

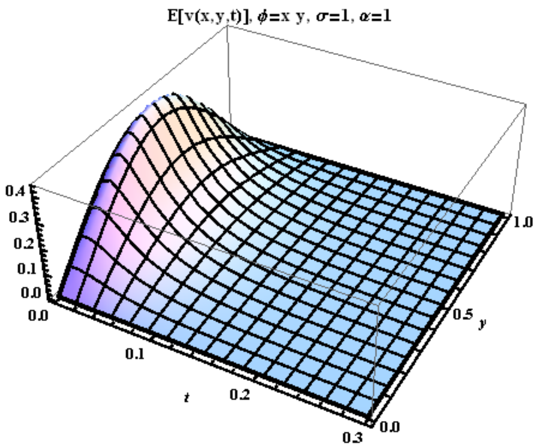
6. Cases study

In section, we consider some cases study illustrate graphically some approximations of the statistical moments of the stochastic solution process which were obtained from the application of the WHEP technique.

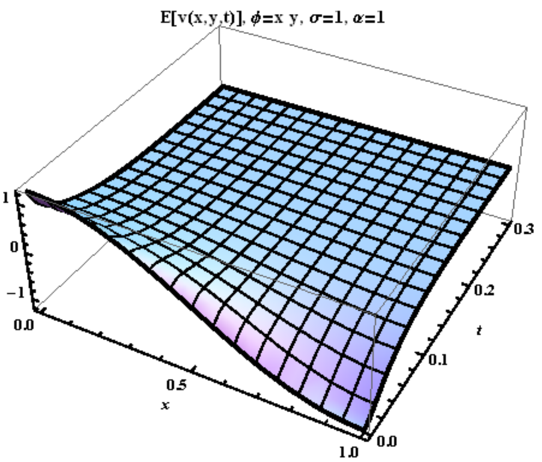
Case study 1:

In this case, the mean and variance functions approximations are presented for a given initial condition over a given domain of the coordinates of the problem and the graphical behavior of this case are shown in the figures (1-4).

$(1.a)$



(1.*b*)



(1.*c*)

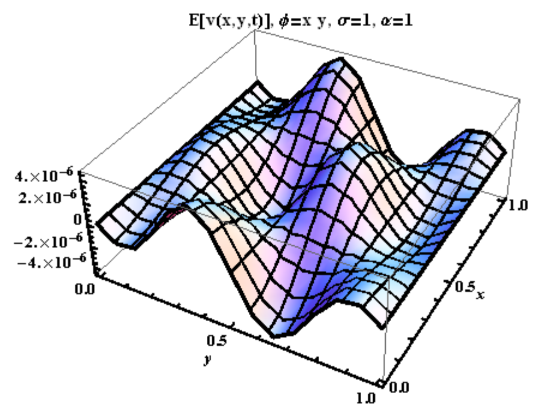
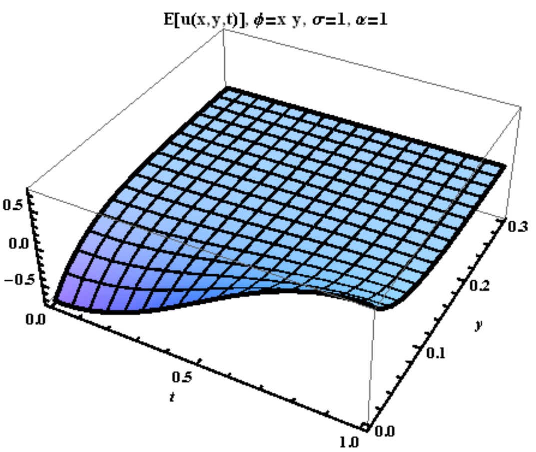
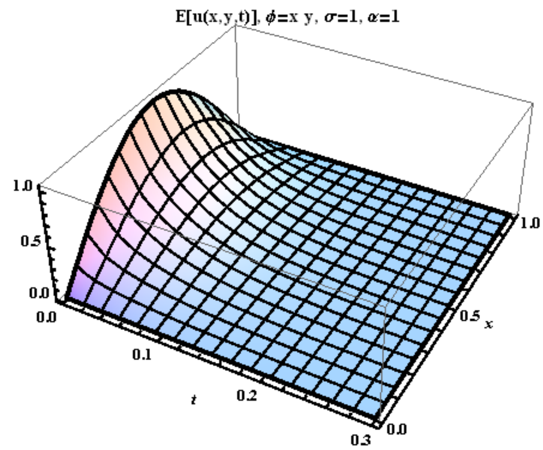


Fig.1 The second correction of $E[v(x,y,t;\omega)]$ for $\varepsilon = 10,\ L = 1$ in (a) $x = 0.8$, (b) $y = 0.8$ and (c) $t = 1$

(2.*a*)



(2.*b*)



(2.*c*)

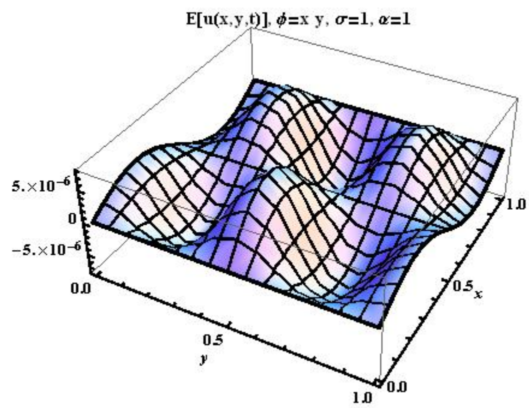
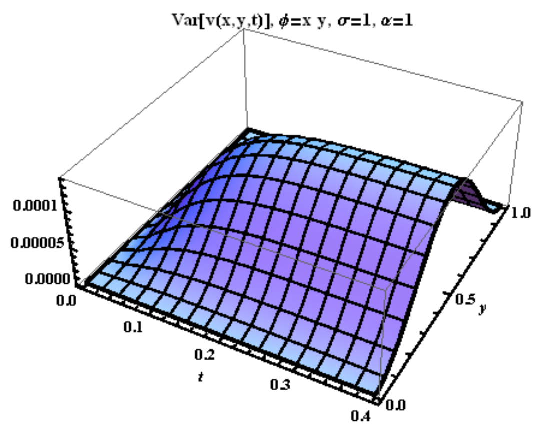
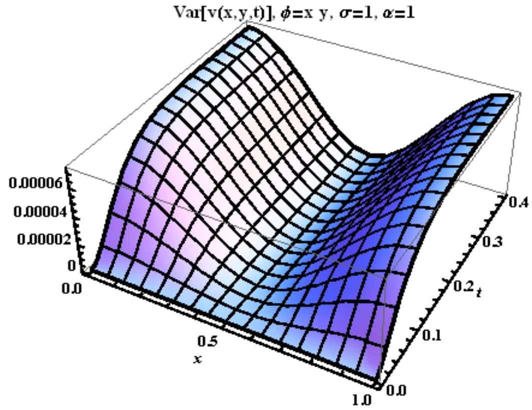


Fig.2 The second correction of $E[u(x,y,t;\omega)]$ for $\varepsilon = 10,\ L = 1$ in (a) $x = 0.8$, (b) $y = 0.8$ and (c) $t = 1$

(3.*a*)



$(3.b)$



$(3.c)$

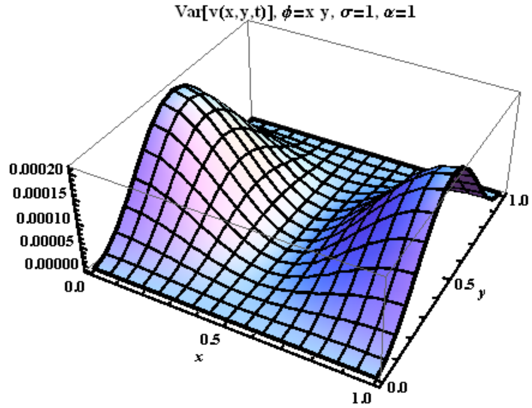
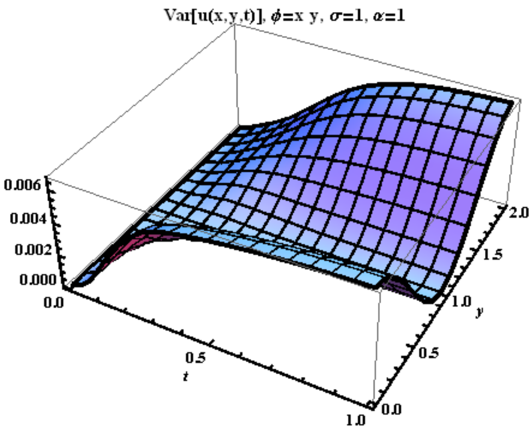
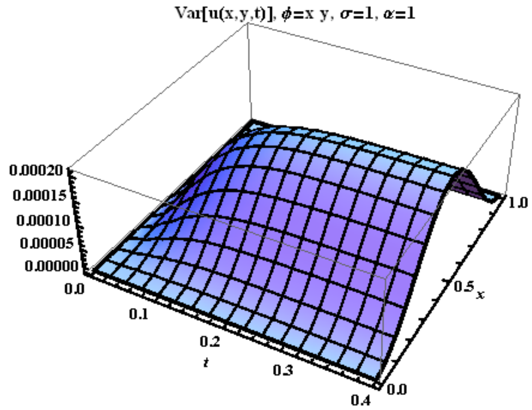


Fig.3 The second correction of $Var[v(x,y,t;\omega)]$ for $\varepsilon=10,\ L=1$ in (a) $x=0.2$, (b) $y=0.2$ and (c) $t=1$

$(4.a)$



$(4.b)$



$(4.c)$

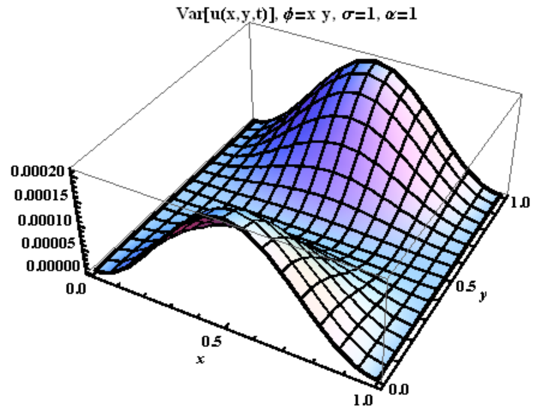
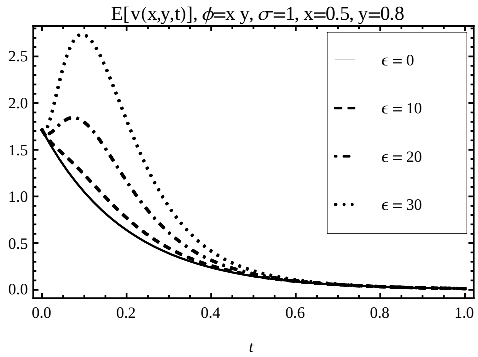


Fig.4 The second correction of $Var[u(x,y,t;\omega)]$ for $\varepsilon=10,\ L=1$ in (a) $x=0.2$, (b) $y=0.2$ and (c) $t=1$

Case study 2:

The variation of the statistical moments functions with the time at fixed point is present in this case for different values of ε and for given domain and some prompters, the figures (6-8) are displayed this case.

$(5.a)$



(5.*b*)

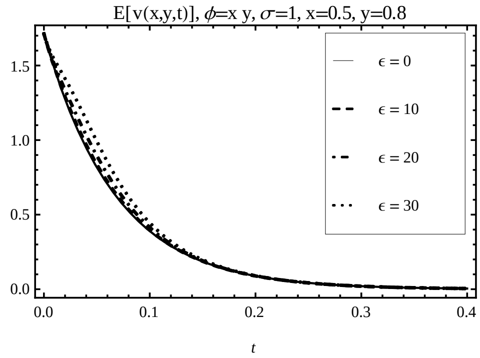
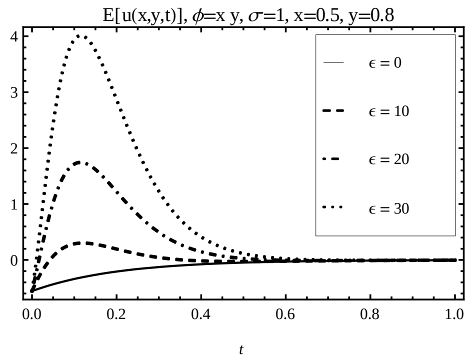


Fig. 5 The second correction of $E[v(x,y,t;\omega)]$ at fixed point for different values of ε, for $L=2$ in **(a)** at $\alpha=1$ and **(b)** at $\alpha=3$

(6.*a*)



(6.*b*)

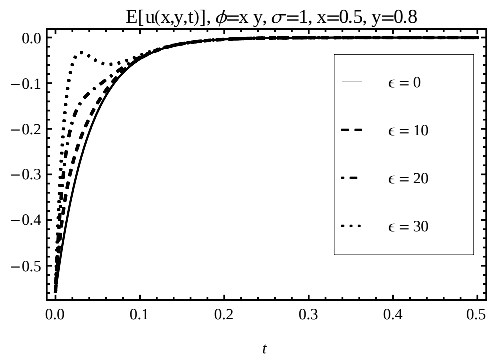
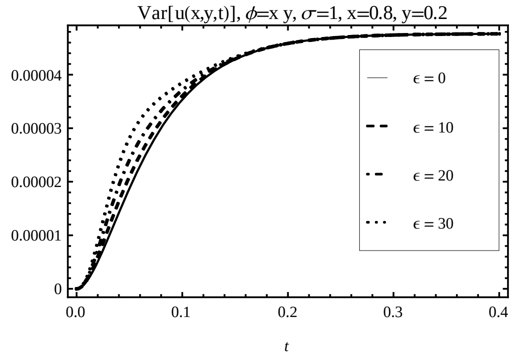


Fig. 6 The second correction of $E[u(x,y,t;\omega)]$ at fixed point for different values of ε, for $L=2$ in **(a)** at $\alpha=1$ and **(b)** at $\alpha=5$

(7.*a*)



(7.*b*)

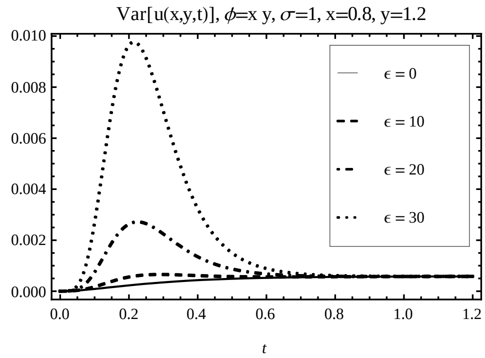
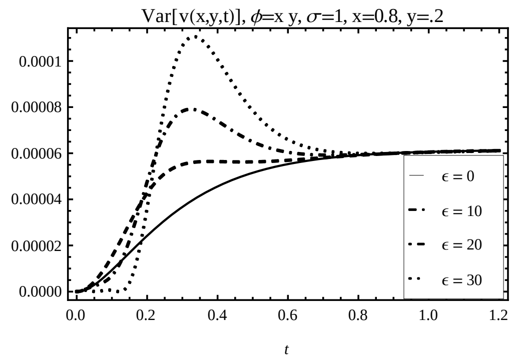


Fig. 7 The second correction of $Var[u(x,y,t;\omega)]$ at fixed point for different values of ε, for $\alpha=1$ in **(a)** at $L=1$ and **(b)** at $L=2$

(8.*a*)



$(8.b)$

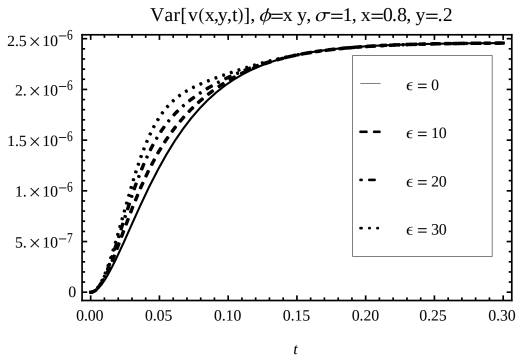
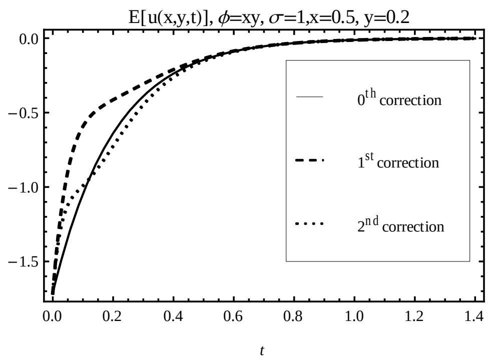


Fig. 8 The second correction of $Var[v(x,y,t;\omega)]$ at fixed point for different values of ε, for $L=2$ in **(a)** at $\alpha=1$ and **(b)** at $\alpha=6$

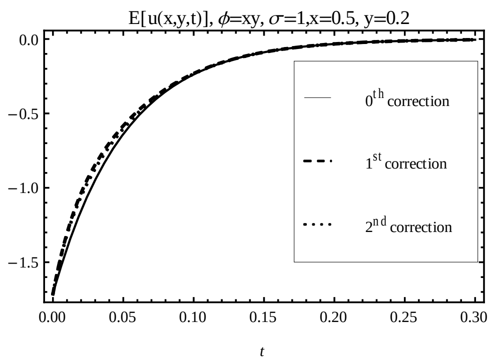
Case study 3:

The comparison between different corrections related to the approximation of the statistical moments of the stochastic solutions process are considered in this case and the figures (9-14) illustrate this comparison for some given data related to the problem

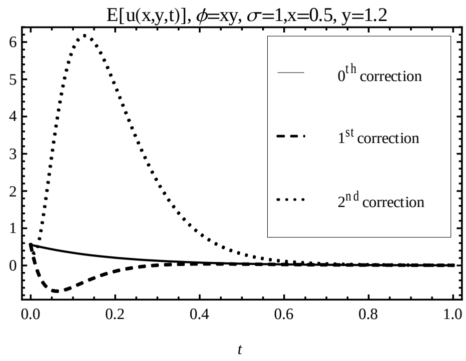
$(9.a)$



$(9.b)$



$(9.c)$



$(9.d)$

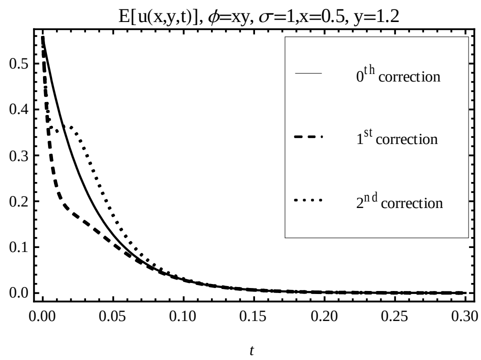
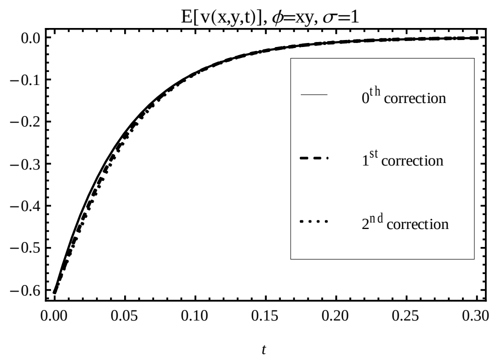


Fig. 9 A comparison between different corrections of $E[u(x,y,t;\omega)]$ for $L=2$ in **(a)** $\alpha=1, \varepsilon=10$, **(b)** $\alpha=4, \varepsilon=10$, (c) $\alpha=1,\ \varepsilon=40$ and (d) $\alpha=1,\ \varepsilon=40$

$(10.a)$



(10.*b*)

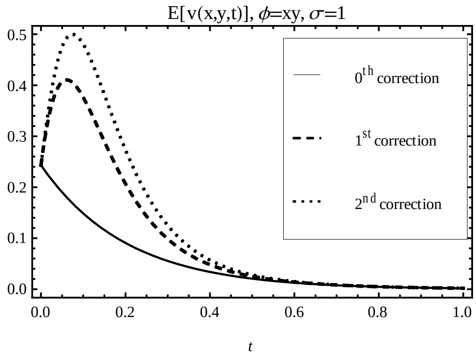
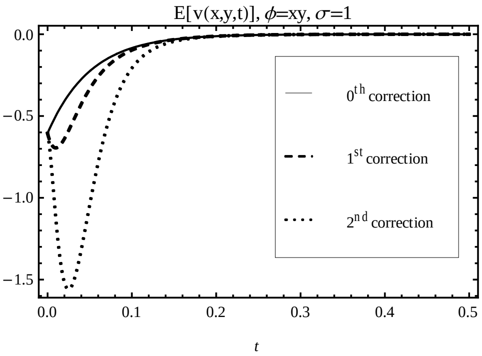


Fig. 10 A comparison between different corrections of $E[v(x,y,t;\omega)]$ for $\alpha=1, \varepsilon=10,\ x=0.8,\ \ y=0.2$ in **(a)** $L=1$ and **(b)** $L=2$,

(11.*a*)



(11.*b*)

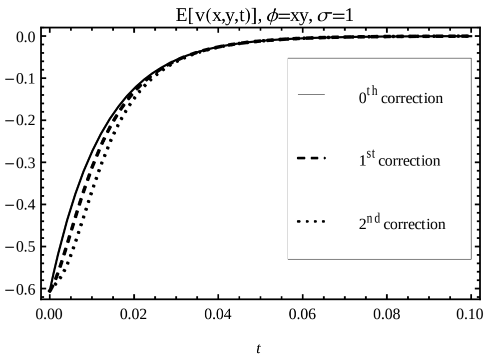
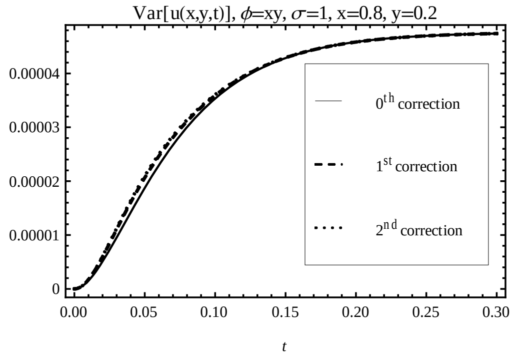


Fig. 11 A comparison between different corrections of $E[v(x,y,t;\omega)]$ for $L=2, \varepsilon=100,\ x=0.8,\ \ y=0.2$ in **(a)** $\alpha=1$ and **(b)** $\alpha=5$,

(12.*a*)



(12.*b*)

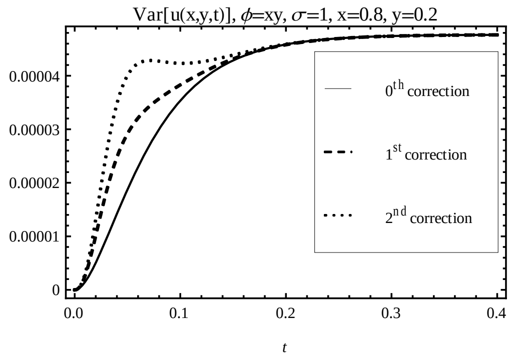
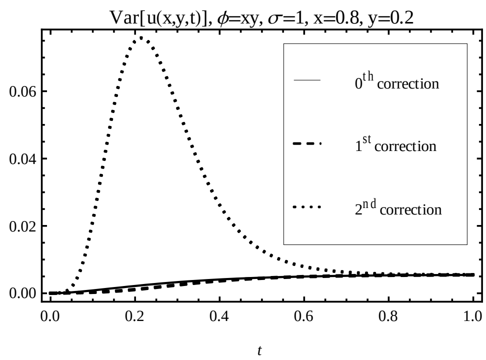


Fig. 12 A comparison between different corrections of $Var[u(x,y,t;\omega)]$ for $L=1, \alpha=1$ in **(a)** $\varepsilon=10$ and **(b)** $\varepsilon=50$,

(13.*a*)



$(13.b)$

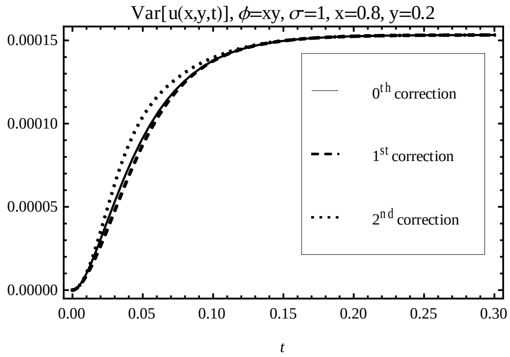
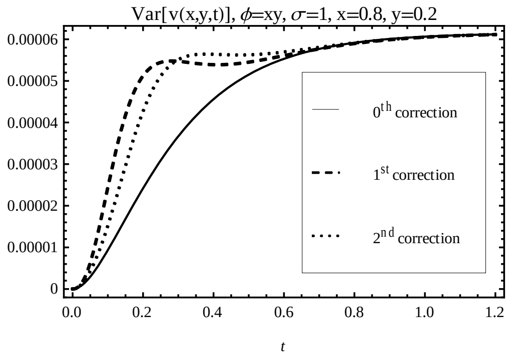


Fig. 13 A comparison between different corrections of $Var[u(x,y,t;\omega)]$ for $L=2, \varepsilon=50$ in **(a)** $\alpha=1$ and **(b)** $\alpha=6$,

$(14.a)$



$(14.b)$

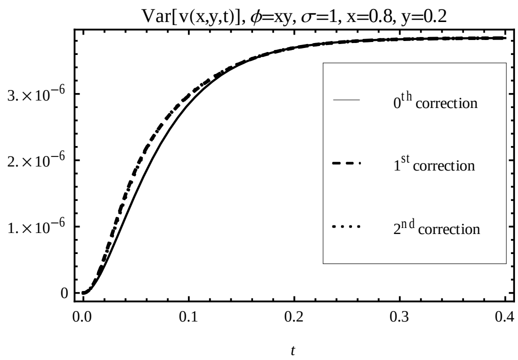
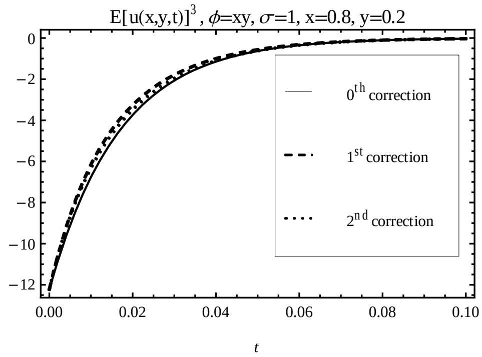


Fig. 14 A comparison between different corrections of $Var[v(x,y,t;\omega)]$ for $L=2, \ \varepsilon=10$ in **(a)** $\alpha=1$ and **(b)** $\alpha=4$,

Case study 4:

Some higher statistical moments are considered in this case for some a given data. The figures (15-20) display some comparison between different corrections of the approximation functions of the third, fourth and fifth statistical moments of the solution process.

$(15.a)$



$(15.b)$

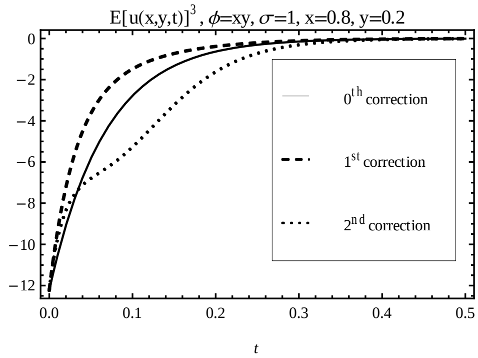
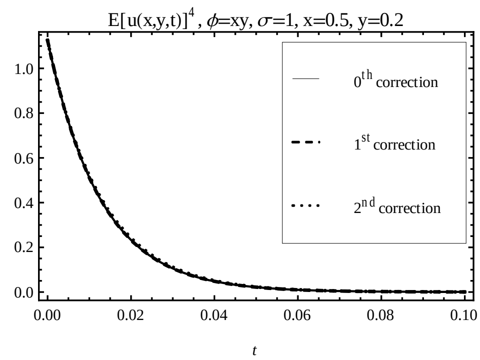


Fig. 15 A comparison between different corrections of $E[u(x,y,t;\omega)]^3$ for $L=2, \ \varepsilon=10$ in **(a)** $\alpha=1$ and **(b)** $\alpha=4$,

$(16.a)$



(16.*b*)

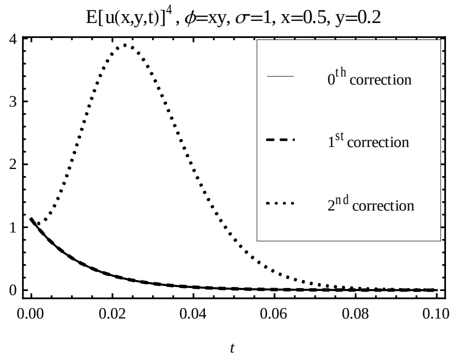
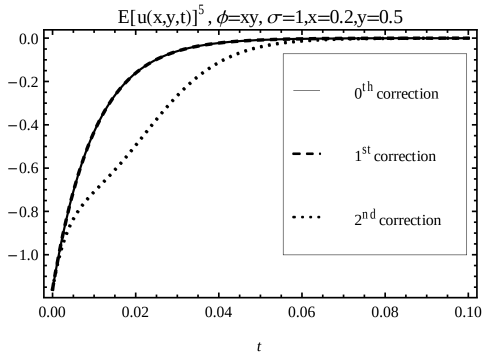


Fig. 16 A comparison between different corrections of $E[u(x,y,t;\omega)]^4$ for $L=1,\ \alpha=1$ in **(a)** $\varepsilon=10$ and **(b)** $\varepsilon=100$

(17.*a*)



(17.*b*)

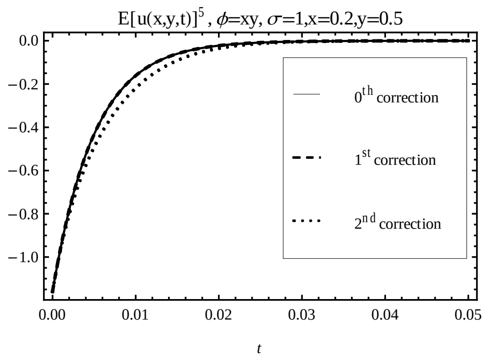
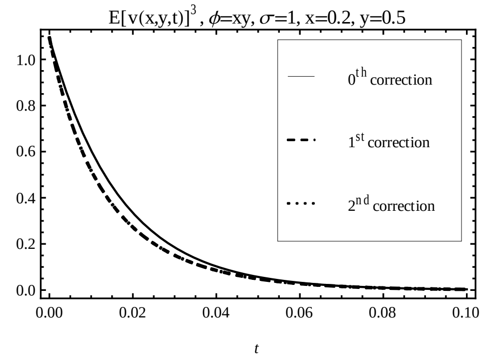


Fig. 17 A comparison between different corrections of $E[u(x,y,t;\omega)]^5$ for $L=1,\ \varepsilon=50$ in **(a)** $\alpha=1$ and **(b)** $\alpha=2$

(18.*a*)



(18.*b*)

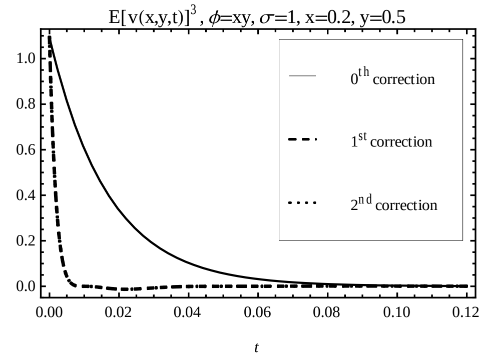
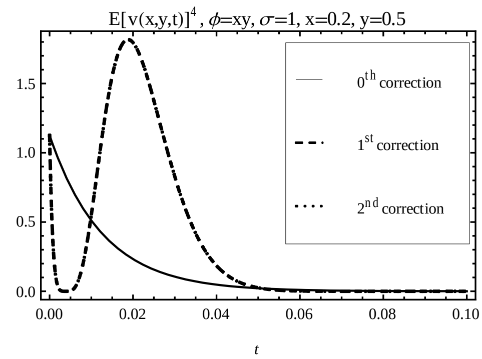


Fig. 18 A comparison between different corrections of $E[v(x,y,t;\omega)]^3$ for $L=1,\ \alpha=1$ in **(a)** $\varepsilon=10$ and **(b)** $\varepsilon=200$

(19.*a*)



(19.b)

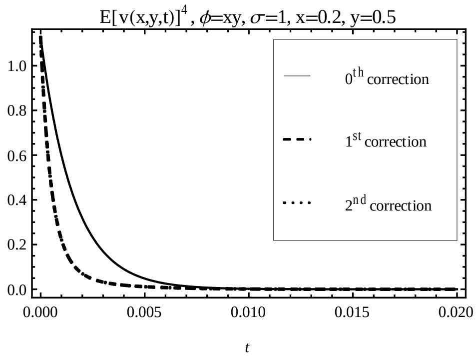
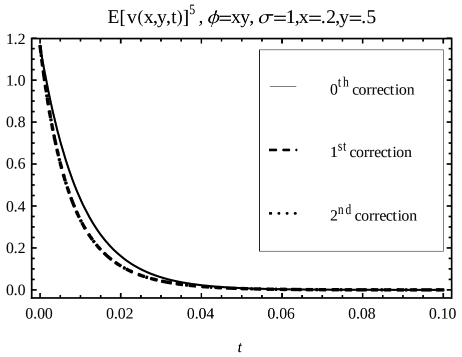


Fig. 19 A comparison between different corrections of $E[v(x,y,t;\omega)]^4$ for $L=1,\ \varepsilon=400$ in **(a)** $\alpha=1$ and **(b)** $\alpha=8$

(20.a)



(20.b)

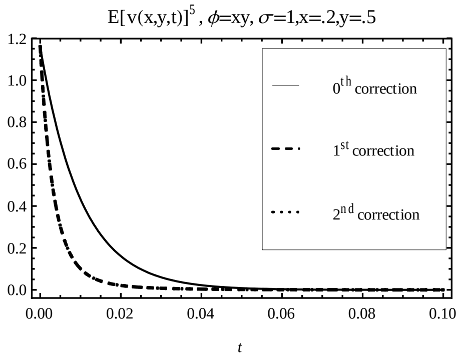


Fig. 20 A comparison between different corrections of $E[v(x,y,t;\omega)]^5$ for $L=1,\ \alpha=1$ in **(a)** $\varepsilon=10$ and **(b)** $\varepsilon=50$

7. Conclusion

The WHEP technique proved to be successful in introducing an approximation of the stochastic solution process for the non-linear perturbed tow-dimensions Navier-Stokes equation exited randomly by the space white noise process. The application of WHEP algorithm includes tow steps; the first step shows the application of Wiener-Hermite expansion to approximate the stochastic solution process of the problem in terms of deterministic kernels and use the statistical properties of WHPs to obtain a set of deterministic equations in these deterministic kernels. The second step of WHEP technique uses the perturbation technique to approximate the deterministic kernels in the first step. Some statistical moments of the solution process were obtained to illustrate the statistical behavior of the solution process. From the results of WHEP technique and the use of mathematical software (Mathematica 7), some case studies were presented to illustrate many corrections for the statistical moments of the solution process of the problem.

Appendix 1

The statistical properties of Wiener-Hermite polynomials (WHPs) (See [45]) which were used in this paper are

$$E[H^{(i)}(x_1,....x_i)H^{(j)}(x_1,....x_j)]=0 \quad \forall i\neq j$$

$$E\left[\prod_{m=1}^{k}H^{(1)}(x_m)\right]=0 \quad \forall\ k\in \ \textit{odd numbers}$$

$$E[H^{(1)}(x_1)H^{(2)}(x_2,x_3)H^{(2)}(x_4,x_5)]=0$$

$$E[H^{(2)}(x_1,x_2)H^{(2)}(x_3,x_4)]=$$

$$\delta(x_1-x_4)\delta(x_2-x_3)+\delta(x_1-x_3)\delta(x_2-x_4)$$

$$E[H^{(1)}(x_1)H^{(1)}(x_2)H^{(2)}(x_3,x_4)]=$$

$$\delta(x_1-x_3)\delta(x_2-x_4)+\delta(x_1-x_4)\delta(x_2-x_3)$$

$$E[H^{(1)}(x_1)H^{(1)}(x_2)H^{(1)}(x_3)H^{(1)}(x_4)]=$$

$$\delta(x_1-x_2)\delta(x_3-x_4)+\delta(x_1-x_3)\delta(x_2-x_4)$$

$$+\delta(x_1-x_4)\delta(x_2-x_3)$$

(30)

Appendix 2

Proof 1. From the application of eigenfunction expansion (See [46]), we put the general solution of Eq. (22) in the following form

$$\Psi(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}T_{n,m}(t)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)$$

. (31)

Subsisting from (31) into (22), we get

$$\left.\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\lambda_{n,m}\left[\dot{T}_{n,m}(t)+\lambda_{n,m}T_n(t)\right]\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)\\&\qquad= f(x,y,t),\quad \lambda_{n,m}=\alpha\left(\frac{\pi}{L}\right)^2(n^2+m^2),\\&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}T_{n,m}(0)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)=\Phi(x,y)\end{aligned}\right\} \tag{32}$$

Multiplying Eqs. (32) by $\sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{l\pi}{L}y\right)$, and integrate both sides from them respect to x and y, we get

$$\left.\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\lambda_{n,m}\left[\dot{T}_{n,m}(t)+\lambda_{n,m}T_n(t)\right]I_{n,k}I_{m,l}\\&=\int_0^L\int_0^L f(x,y,t)\sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{l\pi}{L}y\right)dxdy,\\&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}T_{n,m}(0)\,I_{n,k}I_{m,l}=\int_0^L\int_0^L\Phi(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right),\end{aligned}\right\} \tag{33}$$

where

$$I_{n,k}=\int_0^L\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx=\begin{cases}0 & n\neq k\\ \dfrac{L}{2} & n=k\end{cases} \tag{34}$$

Then from the property (34), the final expansions of Eq. (33) tends to the following recurrence ordinary differential equation,

$$\left.\begin{aligned}&\left[\dot{T}_{k,l}(t)+\lambda_{k,l}T_{k,l}(t)\right]=F_{k,l}(t),\\&F_{k,l}(t)=\frac{4}{L^2\lambda_{n,m}}\int_0^L\int_0^L f(x,y,t)\sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{l\pi}{L}y\right)dxdy,\\&T_{n,m}(0)=\frac{4}{L^2}\int_0^L\int_0^L\Phi(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right),\end{aligned}\right\} \tag{35}$$

which has the general solution under the initial condition

$$T_{k,l}(t)=T_{k,l}(0)\,e^{-\lambda_{k,l}t}+\int_0^t e^{-\lambda_{k,l}(t-\tau)}F_{k,l}(\tau)d\tau \tag{36}$$

Proof 2. The solution of equation (4), if exists, is a series power of ε.

Rewriting Eq. (4), it can take the following form

$$\frac{\partial\nabla^2\psi}{\partial t}-\alpha\nabla^4\psi=\sigma\,n(x;\omega)-\varepsilon\left[\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\right] \tag{37}$$

Following Pickard approximation, the equation can be rewritten as

$$\frac{\partial\nabla^2\psi_{n+1}}{\partial t}-\alpha\nabla^4\psi_{n+1}=\sigma\,n(x;\omega)-\varepsilon\left[\frac{\partial(\psi_n,\nabla^2\psi_n)}{\partial(x,y)}\right] \tag{38}$$

where the solution at $n=0$, ψ_0, is corresponding for the simple linear case at $\varepsilon=0$.

At $n=1$, the iteration takes the form:

$$\frac{\partial\nabla^2\psi_1}{\partial t}-\alpha\nabla^4\psi_1=\sigma\,n(x;\omega)-\varepsilon\left[\frac{\partial(\psi_0,\nabla^2\psi_0)}{\partial(x,y)}\right] \tag{39}$$

which has the following general solution (See the result of proof 1)

$$\begin{aligned}&\psi_1(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\varepsilon\,I_n(t)+\varphi(t)\right]\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right),\\&I_n(t)=\frac{4}{L^2\lambda_{n,m}}\int_0^t\int_0^L\int_0^L G(x,y,t)\,\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)d\tau\,dxdy,\\&G_1(x,y,t)=e^{-\lambda_{n,m}(t-\tau)}\left[\frac{\partial(\psi_0,\nabla^2\psi_0)}{\partial(x,y)}\right]\\&\varphi(t)=T_{n,m}\,e^{-\lambda_{mn}t}-\\&\qquad\frac{4\sigma}{L^2\lambda_{n,m}}\int_0^t\int_0^L\int_0^L G_2(x,y,t)\,\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)d\tau\,dxdy,\\&G_2(x,y,t)=e^{-\lambda_{n,m}(t-\tau)}n(x;\omega)\end{aligned} \tag{40}$$

or

$$\psi_1(x,y,t)=\psi_1^{(0)}+\varepsilon\,\psi_1^{(1)}. \tag{41}$$

At $n=2$, the iteration takes the form

$$\frac{\partial\nabla^2\psi_2}{\partial t}-\alpha\nabla^4\psi_2=\sigma\,n(x;\omega)-\varepsilon\left[\frac{\partial(\psi_1,\nabla^2\psi_1)}{\partial(x,y)}\right] \tag{42}$$

which has the following general solution by the same method

$$\psi_2(x,y,t)=\psi_2^{(0)}+\varepsilon\,\psi_2^{(1)}+\varepsilon^2\,\psi_2^{(2)} \tag{43}$$

Proceeding like this, one can get the following

$$u_n(x,y,t)=\psi_n^{(0)}+\varepsilon\,\psi_n^{(1)}+\varepsilon^2\,\psi_n^{(2)}+\varepsilon^3\,\psi_n^{(3)}+........+\varepsilon^n\,\psi_n^{(n)} \tag{44}$$

Assuming the solution exists, it will be

$$\psi(x,y,t)=\lim_{n\to\infty}\psi_n(x,y,t)=\sum_{i=0}^{\infty}\varepsilon^i\psi_i \tag{45}$$

and therefore that the kernels of the stochastic Wiener-Hermite of $\psi(x,y,t;\omega)$ are a power series of ε.

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