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On the continuation of the limit distribution of intermediate order statistics under power normalization

H. M. Barakat , E. M. Nigm, and E.O.Abo-zaid

Department of Mathematics, faculty of science, Zagazig University
Email Address: s_nigm@yahoo.com

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Abstract: The property of the continuation of the convergence of the distribution function of intermediate order statistics under power normalizations is studied on an arbitrary nondegenerate interval to the whole real line.

Keywords: Weak convergence; intermediate order statistics; power normalization; continuation of the convergence.

1 Introduction

In the last two decades E. Pancheva and her collaborators (see, e.g., [13], [14] and the references therein) were investigating various limit theorems for extremes and extremal processes using a wider class of normalizing mapping than the linear ones to get a wider class of limit laws. This wider class of extreme limits can be used in solving approximation problems. Another, reason for using nonlinear normalization concerns the problem of refining the accuracy of approximation in the limit theorems. Actually, by using relatively non difficult monotone mapping in certain cases we may achieve a better rate of convergence, e.g., see [18] and [7]). Although, no one can claim that the employment of nonlinear normalization in general is preferable, but as in [14] (and other many authors) showed in some cases of practical interest it is not only better to use nonlinear transformation, but we have to use it. Pancheva in [13] considered the power normalization $G_n(x)=b_n|x|^{a_n}S(x), a_n, b_n>0$, where $S(x)=sign(x)$, and derived all the possible limit distributions of the maximum order statistics subjected to this normalization. These limit distributions are usually called the power max stable distribution functions. Mohan and Ravi in [12] showed that the power-max stable distributions, which are six types, attract more than linear stable df's. Therefore, using the power normalization, we get a wider class of limit distribution functions which can be used in solving approximation problems. The intermediate order statistics have many applications. For example intermediate order statistics can be used to estimate probabilities of future extreme observations and to estimate tail quantiles of the underlying distribution that are extremes relative to available sample size. To be more specific, let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables (rv's) with common df $F(x)=P(X_n \le x)$ and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote to the order statistics of $X_1, X_2, \cdots, X_n$. The sequence $\{X_{r:n}\}$ is refereed to a sequence of order statistics with rank r. If $r=r_n \to \infty$ as $n \to \infty$ and $\frac{r_n}{n} \to 0$ (i.e., lower intermediate) or $\frac{r_n}{n} \to 1$ (i.e., upper intermediate) then r is called the intermediate rank. When the intermediate rank sequence $\{r_n\}$ satisfies the limit relation

$$\lim_{n\to\infty}(\sqrt{r_{n+z_n(\nu)}}-\sqrt{r_n})=\frac{\alpha\nu\ell}{2}, \quad (1)$$

which is known as Chibisov's condition, for any sequence of integer values $\{z_n(\nu)\}$, for which $\frac{z_n(\nu)}{n^{1-\frac{\alpha}{2}}}\xrightarrow[n]{w}\nu$, where $0<\alpha<1, \ell>0$ and ν is any real number, i.e., $r_n\sim\ell^2n^{\alpha}(1+\circ(1))$, Chibisov in [8] and Wu in [19] have fully characterized the class of possible limit df's of the df. of the intermediate order statistics $\Phi_{r_n:n}(x)=P(X_{r_n:n}\le a_nx+b_n)$, where $a_n>0$ and b_n are suitable normalizing constants. Barakat and Omar in [3] have extended the work of Chibisov to the power normalization case by showing that all possible limit distributions of intermediate order statistics under power normalization can derived from corresponding results in the extremal case, and they have fully characterized the class of possible nondegenerate limits of the df

$$P\left(\left|\frac{X_{r_n:n}}{\alpha_n}\right|^{\frac{1}{\beta_n}}S(X_{r_n:n})\le x\right)=\Psi_{r_n}(\alpha_n|x|^{\beta_n}S(x)) \quad (2)$$

where $\alpha_n>0$ and $\beta_n>0$. Namely, [3] derived all the possible nondegenerate types of the limit df's $H(x)$ and $\Psi(x)$ of the df's of the upper and the lower intermediate order statistics $X_{k_n:n}$ and $X_{r_n:n}$, respectively, under the power normalization, where $k_n=n-r_n+1, \frac{r_n}{n}\to 0$, as $n\to\infty$. These types are given by $H_{i;\beta}(x)=N(V_{i;\beta}(x))$ and $\Psi_{i;\beta}(x)=N(W_{i;\beta}(x)), i=1,2,...,6,$ where $N(.)$ is the standard normal df and

$$\left.\begin{array}{l}(i)H_{1;\beta}(x)=N(V_{1;\beta}(x))=I_{[-1,\infty)}(x)+(1-N(\beta\log((\log|x|)))I_{(-\infty,-1)}(x);\\ (ii)H_{2;\beta}(x)=N(V_{2;\beta}(x))=I_{[0,\infty)}(x)+N(\beta\log((-\log|x|)))I_{[-1,0)}(x);\\ (iii)H_{3;\beta}(x)=N(V_{3;\beta}(x))=I_{[1,\infty)}(x)+(1-N(\beta\log(-\log x)))I_{[0,1)}(x);\\ (iv)H_{4;\beta}(x)=N(V_{4;\beta}(x))=N(\beta\log(\log x))I_{[1,\infty)}(x);\\ (v)H_{5;\beta}(x)=N(V_{5;\beta}(x))=I_{[0,\infty)}(x)+N(-\log|x|)I_{(-\infty,0)}(x);\\ (vi)H_{6;\beta}(x)=N(V_{6;\beta}(x))=N(\log x)I_{[0,\infty)}(x);\end{array}\right\} \quad (3)$$

and the corresponding types of the lower intermediate are

$$\left.\begin{array}{l}(i)\Psi_{1;\beta}(x)=N(W_{1;\beta}(x))=N(\beta\log\log x)I_{[1,\infty)}(x);\\ (ii)\Psi_{2;\beta}(x)=N(W_{2;\beta}(x))=I_{(-\infty,0)}(x)+N(-\beta\log(-\log x))I_{[0,1)}(x);\\ (iii)\Psi_{3;\beta}(x)=N(W_{3;\beta}(x))=N(\beta\log(-\log|x|))I_{[-1,0)}(x)+I_{[0,\infty)}(x);\\ (iv)\Psi_{4;\beta}(x)=N(W_{4;\beta}(x))=N(-\beta\log\log|x|)I_{(-\infty,-1)}(x)+I_{[-1,\infty)}(x);\\ (v)\Psi_{5;\beta}(x)=N(W_{5;\beta}(x))=N(\log x)I_{[0,\infty)}(x);\\ (vi)\Psi_{6;\beta}(x)=N(W_{6;\beta}(x))=N(-\log|x|)I_{(-\infty,0)}(x)+I_{[0,\infty)}(x);\end{array}\right\} \quad (4)$$

where $I_A(x)=0,1$ if $x\notin A, x\in A$, respectively.

The theory of the continuation of the convergence started with the work by [17], in which an elegant hypothesis is proved. This hypothesis states that if the distribution of the normalized sum of i.i.d. rv's converges weakly to the normal distribution, then this implies that convergence is achieved on the whole real line. More recently this result has been generalized in varies directions (e.g., [15] and [6]). However, some results of this problem concerning the asymptotic theory of order statistics have been obtained (see, e.g., [9], [10], [11], [1], [2] and [5]). Recently, [4] proved the continuation of the restricted convergence of

the power normalized extremes, on the half-line of real numbers to the whole real line. [6] proved that the restricted convergence of the power normalized extremes on an arbitrary nondegenerate interval implies the weak convergence.
In this paper an interesting stability property of the intermediate order statistics under power normalization is proved. This property is the continuation of the convergence. More precisely, let $\alpha_n > 0$ and $\beta_n > 0$ be some normalizing constants. Let $\Psi^*(x)$ be a monotone function which has more than two different values in a closed interval $[c,d]$. Finally, let

$$\Psi_{r_n}(\alpha_n |x|^{\beta_n} S(x)) \xrightarrow[n]{} \Psi^*(x), \forall x \in [c,d].$$ Then

$$\Psi_{r_n}(\alpha_n |x|^{\beta_n} S(x)) \xrightarrow[n]{w} \Psi(x) = N(W_{i;\beta}(x)), i = 1,2,...,6,$$

where $\Psi(x) = \Psi^*(x), \forall x \in [c,d]$, and $\xrightarrow[n]{w}$ stands for weak convergence, as $n \to \infty$ (everywhere in what follows the symbol ($\xrightarrow[n]{}$) stands for convergence, as $n \to \infty$).

We end this introductory section with a definition and two lemmas, which help us in establishing the aimed results.

Definition 1.1. Let $\{F_n\}_n$ be a sequence of df's. Then, the restricted convergence $F_n(x) \xrightarrow[n]{S} F(x)$, where S is a set of real numbers and F is a nondecreasing function, means that the convergence of $\{F_n\}_n$ to the limit F is restricted on S, for all continuity points of F. Moreover, a function $F(x)$ is said to be a nondegenerate on S, if it has at least two growth points on S.

Lemma 1.1. (see, [3]). Let $\{u_n, n \geq 1\}$ be a sequence of real numbers and $0 \leq \tau \leq \infty$. Then

$$\Psi_n(u_n) \xrightarrow[n]{} N(\tau) \quad \text{if and only if} \quad \frac{n F(u_n) - r_n}{\sqrt{r_n}} \xrightarrow[n]{} \tau.$$

Lemma 1.2. ([4]). Let $F(x)$ be a nondecreasing and nondegenerate right continuous function on $[c,d]$. If, for a sequence $\{F_n\}_n$ of df's and some constants $a_n, b_n, \alpha_n, \beta_n > 0$, we have

$$F_n\left(a_n |x|^{b_n} S(x)\right) \xrightarrow[n]{[c,d]} F(x), \quad F_n\left(\alpha_n |x|^{\beta_n} S(x)\right) \xrightarrow[n]{[c,d]} F(x), \tag{5}$$

then,

$$\left(\frac{\alpha_n}{a_n}\right)^{\frac{1}{b_n}} \xrightarrow[n]{} 1 \quad \text{and} \quad \frac{\beta_n}{b_n} \xrightarrow[n]{} 1. \tag{6}$$

Conversely, if (1.6) holds, then each of the relationships in (1.5) implies the other.

2 Main Results

Theorem 2.1. Let $F(x)$ be a df for which there exist real constants $\alpha_n > 0$ and $\beta_n > 0$ such that

$$\Psi_{r_n}(\alpha_n |x|^{\beta_n} S(x)) \xrightarrow[n]{[c,d]} \Psi^*(x), \tag{7}$$

where $\Psi^*(x)$ is any nondecreasing (right continuous) function which has at least two growth points in the interval (c,d) and the real rank sequence $\{r_n\}$ satisfies the condition (1), i.e., Chibisov's condition. Then $\Psi_{r_n}(\alpha_n |x|^{\beta_n} S(x)) \xrightarrow[n]{w} \Psi(x) = N(W_{i;\beta}(x)), i \in \{1,2,...,6\}$, where $N(W_{i;\beta}(x))$ is defined in (4). Moreover, $\Psi^*(x) = \Psi(x), \forall x \in [c,d]$.

Before giving the details of the proof of Theorem 2.1, we first establish the following lemma.

Lemma 2.1. Let the rank sequence satisfy (1). Furthermore, let (7) be satisfied. Then $\{\Psi_{r_n}(\alpha_n |x|^{\beta_n} S(x))\}_n$ is the sequence of stochastically bounded df's (for the definition of stochastically bounded, see [1]).

Proof. To prove this lemma, it suffices to show that , for any subsequence n_k such that

$$\Psi_{r_{n_k}}(\alpha_{n_k} |x|^{\beta_{n_k}} S(x)) \xrightarrow[k]{R} \tilde{\Psi}(x), \quad (8)$$

where $\tilde{\Psi}(x)$ is a nondecreasing right continuous function, we must have $\tilde{\Psi}(-\infty)=0$ and $\tilde{\Psi}(\infty)=1$. On the other hand, in view of Lemma 1.1, (7) is equivalent to

$$W_{r_n:n}(\alpha_n |x|^{\beta_n} S(x)) = \frac{nF(\alpha_n |x|^{\beta_n} S(x)) - r_n}{\sqrt{r_n}} \xrightarrow[n]{[c,d]} W^*(x).$$

Moreover, (8) is equivalent to

$$W_{r_{nk}:n_k}(\alpha_{n_k} |x|^{\beta_{n_k}} S(x)) = \frac{n_k F(\alpha_{n_k} |x|^{\beta_{n_k}} S(x)) - r_{n_k}}{\sqrt{r_{n_k}}} \xrightarrow[k]{R} \tilde{W}(x),$$

where $N(W^*(x)) = \Psi^*(x)$, for all $x \in [c,d]$, $N(\tilde{W}(x)) = \tilde{\Psi}(x)$, for all x , $W^*(x) = \tilde{W}(x)$, for all $x \in [c,d]$, and

$$W_{r_{n1}:n_2}(x) = \frac{n_2 F(x) - r_{n_1}}{\sqrt{r_{n_1}}}.$$

Hence, to prove this lemma it suffices to show that $\tilde{W}(-\infty) = -\infty$ and $\tilde{W}(\infty) = \infty$. Now, for any finite real number t, (7) may be written as

$$\Psi_{r_{n(t)}:n(t)}(\alpha_{n(t)} |x|^{\beta_{n(t)}} S(x)) \xrightarrow[n]{[c,d} \Psi^*(x),$$

where $n(t) = n + [n^{1-\alpha/2} t]$ and $[\theta]$ denotes the integer part of θ. The last limit relation, in view of Lemma 1.1, is equivalent to $W_{r_{n(t)}:n(t)}(a_{n(t)} |x|^{b_{n(t)}} S(x)) \xrightarrow[n]{[c,d} W^*(x)$. On the other hand, we can write

$$W_{r_{n(t)}:n(t)}(\alpha_{n(t)} |x|^{\beta_{n(t)}} S(x)) = \frac{n(t)}{n}\sqrt{\frac{r_n}{r_{n(t)}}} W_{r_n:n}(\alpha_{n(t)} |x|^{\beta_{n(t)}} S(x)) + \frac{n(t) r_n - n r_{n(t)}}{n\sqrt{r_{n(t)}}}.$$

It is clear that $\{z_n\} = \{n(t) - n\} = \{[n^{1-\alpha/2} t]\}$ are sequences of integer values for which $\frac{z_n}{n^{1-\alpha/2}} \xrightarrow[n]{} t$.

Therefore, we can deduce that r_n satisfies the limit relation

$$\lim_{n\to\infty}(\sqrt{r_{n(t)}} - \sqrt{r_n}) = \lim_{n\to\infty}(\sqrt{r_{n+z_n}} - \sqrt{r_n}) = \frac{\alpha \ell t}{2}, \ell > 0.$$

Moreover, the following limit relations can easily be verified:

$$\frac{n(t)}{n}\sqrt{\frac{r_n}{r_{n(t)}}} \xrightarrow[n]{} 1$$

and

$$\frac{n(t) r_n - n r_{n(t)}}{n\sqrt{r_{n(t)}}} \xrightarrow[n]{} \ell(1-\alpha)t.$$

Hence, we get $W_{r_n:n}(\alpha_{n(t)}|x|^{\beta_{n(t)}}S(x))+t(1-\alpha)\ell \xrightarrow[n]{[c,d]} W^*(x)$, which in view of Lemma 1.1 yields

$$\Psi_{r_n:n}(\alpha_{n(t)}|x|^{\beta_{n(t)}}S(x)) \xrightarrow[n]{[c,d]} N(W^*(x)-t(1-\alpha)\ell). \quad (9)$$

Therefore, applying Lemma 2.4 in [1] and by (7), (9) and (8) we deduce that there exist two real functions $\alpha(t)>0$ and $\beta(t)>0$ such that $N(W^*(x)-t(1-\alpha)\ell)=N(\tilde{W}(\alpha(t)|x|^{\beta})S(x))$, i.e.

$$W^*(x)=\tilde{W}(\alpha(t)|x|^{\beta(t)}S(x))+t(1-\alpha)\ell, \forall x\in[c,d]. \quad (10)$$

Now, if $\tilde{W}(-\infty)>-\infty$, then (10) yields $W^*(x)\geq\tilde{W}(-\infty)+t(1-\alpha)\ell$, for all arbitrary positive large value of t. Hence, by letting $t\to\infty$, we deduce that $W^*(x)=\infty, \forall x\in[c,d]$, which contradicts our assumptions. On the other hand , if $\tilde{W}(\infty)<\infty$, then we have (in view of (10)) $W^*(x)\leq\tilde{W}(\infty)+t(1-\alpha)\ell$, for all arbitrary negative values of t. Consequently, we get $W^*(x)=-\infty$, which again contradicts our assumptions. This completes the proof of Lemma 2.1.

We now turn to the proof of Theorem 2.1 . Since the proof is some what lengthy, we split it into several steps, some of which are of independent interest.

The Proof of Theorem 2.1.

Step 1. If there exist two real numbers $t'<t''$, such that $-\infty<t'<0<t''<\infty$, $c\leq(\alpha(t'')|c|^{\beta(t'')}S(c))<d$ and $c<(\alpha(t')|d|^{\beta(t')}S(d))\leq d$, then $\Psi^*(c)=0$ and $\Psi^*(d)=1$. (This means that the convergence in (7) will continue weakly to a nondegenerate df and in this case the proof of the theorem will follow at once from the result of [3]).

Proof. If two such real numbers t' and t'' exist then, by virtue of (10), we deduce that
$W^*(c)\leq W^*(\alpha(t'')|c|^{\beta(t'')}S(c))=\tilde{W}(\alpha(t'')|c|^{\beta(t'')}S(c))=W^*(c)-\ell(1-\alpha)t'',\ t''>0,$
$W^*(d)\geq W^*(\alpha(t')|d|^{\beta(t')}S(d))=\tilde{W}(\alpha(t')|d|^{\beta(t')}S(d))=W^*(d)-\ell(1-\alpha)t',\ t'<0,$
which is impossible unless $W^*(c)=-\infty$ *or* ∞ and $W^*(d)=-\infty$ *or* ∞. But the cases $W^*(c)=\infty$ and $W^*(d)=-\infty$ are impossible because they lead to $N(W^*(c))=1=\Psi^*(c)$ and $N(W^*(d))=0=\Psi^*(d)$, respectively. Therefore, $W^*(c)=-\infty$ and $W^*(d)=\infty$, which completes the proof of Step 1.

Step 2. Under the assumptions of Lemma 2.1, it is impossible to find $0<t''<\infty$, such that $c<d\leq(\alpha(t'')|c|^{\beta(t'')}S(c))$ and to find $-\infty<t'<0$ such that $\alpha(t')|d|^{\beta(t')}S(d)\leq c<d$.

Proof. If two such real numbers exist then, in view of (10), we get
$\tilde{W}(\alpha(t'')|c|^{\beta(t'')}S(c))\geq\tilde{W}(c)=W^*(c)=\tilde{W}(\alpha(t'')|c|^{\beta(t'')}S(c))+\ell(1-\alpha)t'', t''>0,$
and $\tilde{W}(\alpha(t')|d|^{\beta(t')}S(d))\leq\tilde{W}(d)=W^*(d)=\tilde{W}(\alpha(t')|d|^{\beta(t')}S(d))+\ell(1-\alpha)t',\ t'<0,$
which are impossible unless $\tilde{W}(\alpha(t'')|c|^{\beta(t'')}S(c))=-\infty$ or ∞ and $\tilde{W}(\alpha(t')|d|^{\beta(t')}S(d))=-\infty$ or ∞. The value $-\infty$, in the preceding two cases, leads to $\Psi^*(d)=0$ (in the first case we have $0=\tilde{\Psi}(-\infty)=\tilde{\Psi}(\alpha(t'')|c|^{\beta(t'')}S(c))\geq\tilde{\Psi}(d)=\Psi^*(d)$, i.e. $\Psi^*(d)=0$ and in the second case we have $\tilde{W}(d)=W^*(d)=-\infty+\ell(1-\alpha)t'=-\infty$, i.e., $\Psi^*(d)=0$), which is impossible. The value ∞, in the above two cases , is also impossible because it leads to $\Psi^*(c)=1$ (in the first case we get $\infty\geq\tilde{W}(c)=W^*(c)=\infty+\ell(1-\alpha)t''=\infty$, i.e., $W^*(c)=\infty$, which implies that $\Psi^*(c)=1$, while in the second case we have $\tilde{\Psi}((\alpha(t')|d|^{\beta(t')}S(d))))=1\leq\tilde{\Psi}(c)=\Psi^*(c)$. This proves Step 2.

Step 3. By combining Steps 1 and 2, we deduce, immediately, the following assertion:

(i) If there exists a real number $0<t''<\infty$ such that $c \le \alpha(t'')|c|^{\beta(t'')} S(c)$, then $\Psi^*(c)=0$, which implies the continuation of the convergence in (7) to the left (i.e., to $-\infty$).

(ii) If there exists a real number $-\infty<t'<0$ such that $\alpha(t')|d|^{\beta(t')} S(d) \le d$, then $\Psi^*(d)=1$, which implies the continuation of the convergence in (7) to the right (i.e., to ∞).

(iii) If there exists two such numbers as defined above , the convergence in (7) will continue weakly, $\forall x$, to a nondegenerate df which coincides with $\Psi^*(x)$ on $[c,d]$.

Moreover, in this case, the theorem will follows immediately from the results of [3].

Step 4. Under our conditions there exist at least two growth points x_1 and x_2 in (c,d). Let us assume that $x_1<x_2$. Assume further that $d<\alpha(t)|d|^{\beta(t)} S(d) \forall t<0$. Then there exists $t'<0$ such that

$$\alpha(t')|c|^{\beta(t')} S(c) < x_1 < d < \alpha(t')|d|^{\beta(t')} S(d). \quad (11)$$

Proof. Since the limit $\tilde{\Psi}(x)$ in (8) is a nondegenerate df, it follows from the Chibisov results and the result of [3] that it must be of the form $N(\tilde{W}(x))$. Moreover, $\tilde{W}(x)$ must be one of the types (4). Hence, using Khintchine's convergence theorem it is easy to prove that

$$\tilde{W}(x)=\tilde{W}(\alpha(t)|x|^{\beta(t)} S(x))+\ell(1-\alpha)t, \forall t \text{ and } x.$$

Now, a quick check shows that $\beta(t)=e^{-\frac{\ell}{\beta}(1-\alpha)t}$ in Type 1 and Type 3, $\beta(t)=e^{\frac{\ell}{\beta}(1-\alpha)t}$ in Type 2 and Type 4 and $\beta(t)=1$ in Type 5 and Type 6. Also, $\alpha(t)=1$ in Types 1, 2, 3, and 4, $\alpha(t)=e^{-\ell(1-\alpha)t}$ in Type 5 and $\beta(t)=e^{\ell(1-\alpha)t}$ in Type 6. Let us define a new continuous function $g_c(t)=\alpha(t)|c|^{\beta(t)} S(c)$. Clearly $g_c(0)=c$. Hence there exists $\delta>0$ such that

$$g_c(t')-g_c(0)=\alpha(t')|c|^{\beta(t')} S(c)-c<x_1-c,$$

whenever $-\delta<t'<0$. This implies that for all $-\delta<t<0$ (there are infinity many t such that $-\delta<t<0$) we have $\alpha(t)|c|^{\beta(t)} S(c)<x_1$, which completes the proof of Step 4.

Step 5. Assume that $\forall t<0$, we have $d<\alpha(t)|d|^{\beta(t)} S(d)$. Then the convergence in (7) will continue weakly for all values of x to the right (i.e., $\forall x>d$).

Proof. Let Ω be the set of all $t<0$, which satisfy the condition (11) . Henceforth, we consider only those values $t \in \Omega$. Furthermore, let us consider the following three cases:

(I) there exist $t \in \Omega$ such that $\beta(t)<1$;

(II) there exist $t \in \Omega$ such that $\beta(t)=1$;

(III) $\forall\ t \in \Omega$ we have $\beta(t)>1$.

Case I. Clearly we have $d<\alpha(t)|d|^{\beta(t)} S(d)$. Now, if we show that the convergence of the sequence $\{\Psi_{r_n:n}(\alpha_n|x|^{\beta_n})\}_n$ continues to the point $D=\alpha(t)^{\frac{1}{1-\beta(t)}}$, then, in view of Step 3 (ii), the convergence will continue for all x to the right. Indeed, by (9) and (10), we have

$$\Psi_{r_n:n}(\alpha_{n(t)}|x|^{\beta_{n(t)}} S(x)) \xrightarrow[n]{[c,d]} \tilde{\Psi}(\alpha(t)|x|^{\beta(t)} S(x)). \quad (12)$$

Put $y=\alpha(t)|x|^{\beta(t)} S(x)$, we get

$$\Psi_{r_n:n}(\alpha_n^*(t)|y|^{\beta_n^*(t)}S(y)) \overset{[c_1,d_1]}{\underset{n}{\to}} \tilde{\Psi}(y), \tag{13}$$

where $\alpha_n^*(t)=\alpha_{n(t)}\{\alpha(t)\}^{-\frac{\beta_{n(t)}}{\beta(t)}}$, $\beta_{n(t)}^*=\dfrac{\beta_{n(t)}}{\beta(t)}$, $c_1=\alpha(t)|c|^{\beta(t)}S(c)$ and $d_1=\alpha(t)|d|^{\beta(t)}\ S(d)$. In view of the facts that $\tilde{\Psi}(x)=\Psi^*(x), \forall x\in[c,d]\bigcap[c_1,d_1]$, and $\Psi^*(x)$ has more than two different values in the interval $[c,d]\bigcap[c_1,d_1]$, and by applying Lemma 2.1.3 (see [4]) on (7) and (13), we get

$$\left(\frac{\alpha_n^*(t)}{\alpha_n}\right)^{\frac{1}{\beta_n}} \xrightarrow[n]{} 1 \text{ and } \frac{\beta_n^*(t)}{\beta_n} \xrightarrow[n]{} 1.$$

By another application of Lemma 2.1.3 (see [4]), $\{\alpha_n^*(t)\}_n$ and $\{\beta_n^*(t)\}_n$ in (13) may be changed to α_n and β_n, respectively. Hence, we get

$$\Psi_{r_n:n}(\alpha_n|y|^{\beta_n}S(x)) \xrightarrow[n]{[c_1,d_1]} \tilde{\Psi}(y),$$

which, in view of (7), leads obviously to $\Psi_{r_n:n}(\alpha_n|y|^{\beta_n}S(x)) \xrightarrow[n]{[c,d_1]} \tilde{\Psi}(y)$. Repeating this argument N times, we get the relation

$$\Psi_{r_n:n}(\alpha_n|y|^{\beta_n}S(x)) \xrightarrow[n]{[c,d_N]} \tilde{\Psi}(y),$$

where

$$d_N=\{\alpha(t)\}^{1+\beta(t)+\ldots+\beta^{N-1}(t)}|d|^{\beta^N(t)}S(d))=\{\alpha(t)\}^{\frac{1-\beta^N(t)}{1-\beta(t)}}|d|^{\beta^N(t)}S(d))$$

and $d_N \xrightarrow[N]{} D=\alpha(t)^{\frac{1}{1-\beta(t)}}$. Therefore, due to the continuity of the function $\tilde{W}(y), \forall y$, the proof of Step 5 follows in this case.

Case II. In this case, we can easily show that $\alpha(t)\neq 1, \forall t\in\Omega$ (in fact $\alpha(t)<1$ if $d<0$ and $\alpha(t)>1$ if $d>0$). Hence, if we put $y=\alpha(t)|x|S(x)$ in (12) (with $\beta(t)=1$), we get

$$\Psi_{r_n:n}(\alpha_n^{**}(t)|y|^{\beta_n^{**}(t)}S(y)) \xrightarrow[n]{[c_1',d_1']} \tilde{\Psi}(y), \tag{14}$$

where $c_1'=\alpha(t)|c|S(c), d_1'=\alpha(t)|d|S(d)$, $\alpha_n^{**}(t)=\dfrac{\alpha_{n(t)}}{(\alpha(t))^{\beta_{n(t)}}}$ and $\beta_n^{**}(t)=\beta_{n(t)}$. An application of Lemma 2.1.3 (see [4]) to (7) and (14), thus yields

$$\left(\frac{\alpha_n^{**}(t)}{\alpha_n}\right)^{\frac{1}{\beta_n}} \xrightarrow[n]{} 1 \text{ and } \frac{\beta_n^{**}(t)}{\beta_n} \xrightarrow[n]{} 1.$$

Note that $\tilde{\Psi}(x)=\Psi^*(x)\ \ \forall x\in[c,d]\bigcap[c_1',d_1']$ and $\Psi^*(x)$ has more than two different values in the interval $[c,d]\bigcap[c_1',d_1']$. By applying Lemma 2.1.3, the two sequences $\alpha_n^{**}(t)$ and $\beta_n^{**}(t)$ in (14) may be changed to α_n and β_n, respectively. Therefore, we get $\Psi_{r_n:n}(\alpha_n|y|^{\beta_n}S(y)) \overset{[c_1',d_1']}{\underset{n}{\to}} \tilde{\Psi}(y)$, which obviously leads to $\Psi_{r_n:n}(\alpha_n|y|^{\beta_n}S(y)) \xrightarrow[n]{[c,d_1']} \tilde{\Psi}(y)$. By using the last procedure N times, we deduce that

$$\Psi_{r_n:n}(\alpha_n |y|^{\beta_n} S(y)) \xrightarrow[n]{[c',d'_N]} \tilde{\Psi}(y),$$

where

$$d'_N = \alpha^N(t)|d|S(d) \xrightarrow[N]{} \begin{cases} 0, if & \alpha(t) < 1, \\ \infty, if & \alpha(t) \geq 1. \end{cases}$$

The proof in this case follows immediately in view of the continuity $\tilde{W}(y)$, for all y and the fact that $\alpha(t) \neq 1$.

Case III. In this case we have $d < \alpha(t)|d|^{\beta(t)} S(d)$. Using the same argument, which was applied in Case I, we get

$$\Psi_{r_n:n}(\alpha_n |y|^{\beta_n} S(y)) \xrightarrow[n]{[c,d_N]} \tilde{\Psi}(y),$$

where

$$d_N = \{\alpha(t)\}^{\frac{1}{1-\beta(t)}} \{\alpha(t)|d|^{\beta(t)-1} S(d)\}^{\frac{\beta^N(t)}{\beta(t)-1}} \xrightarrow[N]{} \begin{cases} 0, if & d < 0, \\ \infty, if & d \geq 0. \end{cases}$$

Then the convergence in (7) will continue to $\tilde{\Psi}(x), \forall x > d$ (to the right), which completes the proof.

Step 6. Let $\alpha(t)|c|^{\beta(t)} S(c) < c \ \forall t > 0$. Then there exist at least one $t'' > 0$ such that $\alpha(t'')|c|^{\beta(t'')} S(c) < c < x_2 < \alpha(t'')|d|^{\beta(t'')} S(d)$.

Proof. The proof of this step is exactly the same as the proof of Step 4 (with only the obvious modifications).

Step 7. Let $\alpha(t)|c|^{\beta(t)} S(c) < c, \forall t > 0$. Then the convergence in (7) will continue weakly for all values of x to the left (i.e., to $-\infty$).

Proof. The proof of this step is exactly the same as the proof of Step 5 (with only the obvious modifications).

Proof of Theorem 2.1 . The proof of Theorem 2.1 in this case follows immediately by combining Lemma 2.1 and Steps 1-7.

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References

[1] H. M. Barakat. On the continuation of the limit distribution of the extreme and central terms of a sample. *Test*, **6**(2), 351-368. (1997).

[2] H. M.Barakat. New versions of the extremal types theorem. *South African Statist. J.*, ***34**(1), 1-20. (2000).*

[3] H. M.Barakat. and A. R Omar . Limit theorems for order statistics under nonlinear normalization. *J. Statist. Plann. Inference. 141, 524-535. (2011).*

[4] H. M. Barakat. and E. M. Nigm. Some Asymptotic stability properties of extreme order statistics under power normalization. *Kuwait journal of Science and Engineering. **30**(1), 15-27. (2003).*

[5] H. M. Barakat. and B.Ramchandran. Continuability / Identifiability of local weak limits for certain normalized intermediate / central rank sequences of order statistics. J. of Indian Statist. Assoc., **39**, 1-31. (2001).

[6] H. M. Barakat., E. M. Nigm and E. M . El-Adll. Continuation theorems of the extremes under power normalization. *J. Appl. Math. and Computing.* ***10*** *(1-2), 1-15. (2002).*

[7] H. M. Barakat., E. M. Nigm and E. M . El-Adll. Comparison between the rates of convergence of extremes under linear and under power normalization. *Statist. Papers.* ***51****(1), 149-164. (2010).*

[8] D. M. Chibisov. On limit distributions of order statistics. *Theory Prob. Appl. 9, 142-148. (1994).*

[9] B. V. Gnedenko. On some stability theorems. Stability Problems for Stochastic Models. *Proc. 6th Seminar, Moscow, ed. V. V. Kalashnikov and V.M. Zolotarev, Lecture Notes in Math.,* ***982****, Springer-Verlag, Berlin, 24-31, (1982).*

[10] B. V. Gnedenko and L.Senocy Bereksy. On one characteristic of the limit distributions for the maximum and minimum of variational series. *Dokl. Akad. Nauk. SSSR ,* ***267****(5), 1039-1040. (1982).*

[11] B. V. Gnedenko and L.Senocy Bereksy. On the continuation property of the limit distributions of maxima of variational series. *Vestnik. Moskov. Univ. Ser. Mat. Mch. No. 3, , 11-20. (1983). Translation : Moscow Univ. Matm. Bull. Moscow University. Mathematics Bulletin ,(New York).*

[12] N. R. Mohan and S.Ravi. Max domains of attraction of univariate and multivariate p-max stable laws. *Theory Prob. Appl.* ***37****, 632-643. (1992).*

[13] E.Pantcheva. Limit theorems for extreme order statistics under nonlinear normalization. *Lecture Notes in Math., No. 1155, 284-309. (1984).*

[14] E.Pantcheva. Extreme value limit theory with nonlinear normalization. In: Galambos, J., et al. (eds.) Extreme Value Theory and Application. *Kluwere, Boston, 305-318. (1994).*

[15] M. A. Riedel. A new version of the central limit theorem. *Teor. Verojatnost. i Primenen.* ***22****(1), 187. (1977).*

[16] R H. J. Rossberg. Limit theorems involving restricted convergence. *Theory Prob. Appl.* ***39****(2), 298-314. (1995).*

[17] R H. J. Rossberg.and G. Siegel. *Continuation of convergence in the central limit theorem. Teor. Verojatnost. i Primenen, (1976).* ***20****(4), 885-887. (1976).*

[18] S. B. Weinstein. Theory and application of some classical and generalized asymptotic distributions of extreme values. *IEEE Trans. Information Theory IT-19. No. 2, 148-154. (1973).*

[19] C. Y. Wu. The types of limit distributions for terms of variational series. *Statistica. Sincia 15, 749-762. (1966).*