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Global exponential stability for a class of impulsive BAM neural networks with distributed delays

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Abstract: In this paper, the exponential stability is investigated for a class of BAM neural networks with distributed delays and nonlinear impulsive operators. By using Lyapunov functions and applying the Razumikhin technique, delay–independent sufficient conditions ensuring the global exponential stability of equilibrium points are derived. These results can easily be utilized to design and verify globally stable networks. An illustrative example is given to demonstrate the effectiveness of the obtained results.

Keywords: Impulsive BAM neural networks, Global exponential stability, Distributed delays, Lyapunov method, Razumikhin technique.

1 Introduction

Due to their wide range of applications in pattern recognition, associative memory and combinatorial optimization, bidirectional associative memory (abbreviated by BAM) neural networks and their various generalizations have attracted the attention of many mathematicians, physicists and computer scientists in the last two decades. A series of neural networks concerning BAM models have been first proposed by Kosko in [1,2,3]. These models are very general classes of neural network models. Indeed, some famous ecological systems and neural networks such as the Lotka–Volterra ecological system and the Hopfield neural networks have been under consideration.

In the design and applications of networks, it is of prime importance to ensure that the designed neural networks are stable. It should be noted that in both biological and man–made neural networks the delays occur due to the finite switching speed of the amplifiers and communication time [4]. However, time delays may lead to non–oscillation, divergence or instability which may be harmful to the system [4,5,6]. Therefore, the study of neural dynamics with the consideration of time delays has become extremely important to manufacture high quality neural networks. In the papers [7,8,9,10,11] some various stabilities have been studied for BAM neural networks with delays. The circuits diagram and the connection pattern implementation for the delayed BAM neural networks can be found in [10,11]. In reality, nevertheless, it is desirable that the neural network not only converges to an equilibrium point but also has a convergence rate which is as fast as possible. It is to be noted that the exponential stability gives a fast convergence rate to the equilibrium point. Therefore, it is crucial to determine the exponential stability and to estimate the exponential convergence rate.

On the other hand, impulsive effect likewise exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time. Such processes often appear in fields as medicine and biology, economics, mechanics, electronics and telecommunications, etc. As artificial electronic systems, neural networks such as Hopfield neural networks, bidirectional neural networks and recurrent neural networks are best described under impulsive perturbations which can affect dynamical behaviors of the systems just as time delays. Therefore, it would be more appropriate to consider both impulsive and delay effects on the stability of neural networks. Yet, few results have been developed in this direction for neural networks [12,13,14,18,19,20,21,22,24]. Although the use of constant fixed delays in

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models of delayed feedback provides a good approximation in simple circuits consisting of only a small number of cells, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Thus, it is common to have a distribution of propagation delays. Recently, some authors have investigated the stability of BAM neural networks with distributed delays but without impulses, see for instance the papers [11,15, 25,26] and the references quoted therein.

In this paper, inspired by Song and Cao in [25], we formulate a BAM neural network model with distributed delays and nonlinear impulsive operators. By means of piecewise continuous Lyapunov functions [17] and the Razumikhin technique [13,16,22,23] we establish criteria for global exponential stability of the equilibrium point. The conditions are independent of the form of specific delays and have important significance in both theory and applications. Thus, the results improve the ones established in the earlier literature. An example is given to demonstrate the effectiveness of the results.

2 The system, notations and definitions

Let $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}^n$ denote the n–dimensional Euclidean space and $\|y\| = \Big(\sum_{j=1}^{n} y_j^2\Big)^{1/2}$ define the norm of $y \in \mathbb{R}^n$.

Consider the following BAM impulsive system with distributed delays

$$\begin{cases} \dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ji} f_j\big(y_j(t)\big) \\ \qquad + \sum_{j=1}^{n} w_{ji} \int_{-\infty}^{t} K_{ji}(t-s) f_j\big(y_j(s)\big)\, ds + I_i,\ t \neq t_k, \\ \Delta x_i(t_k) = T_{ik}(x_i(t_k)),\ k = 1,2,\ldots, \\ \dot{y}_j(t) = -d_j y_j(t) + \sum_{i=1}^{m} b_{ij} g_i\big(x_i(t)\big) \\ \qquad + \sum_{i=1}^{m} h_{ij} \int_{-\infty}^{t} N_{ij}(t-s) g_i\big(x_i(s)\big)\, ds + J_j,\ t \neq t_k, \\ \Delta y_j(t_k) = U_{jk}(y_j(t_k)),\ k = 1,2,\ldots, \end{cases} \tag{2.1}$$

for $t \geq 0,\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n$ where $x_i(t)$ and $y_j(t)$ correspond to the states of the $i-$th unit and $j-$th unit, respectively, at time t; c_i and d_j are positive constants; K_{ji} and N_{ij} are the delay kernels; w_{ji} and h_{ij} are the connection weights; f_j and g_i are the activation functions; I_i and J_j, denote the external inputs; T_{ik} and U_{jk} are the abrupt changes of the states at the impulsive moments t_k; by $\Delta x_i(t_k)$ and $\Delta y_j(t_k)$ we mean the differences $x_i(t_k+0) - x_i(t_k)$ and $y_j(t_k+0) - y_j(t_k)$, respectively, and the sequence $0 < t_1 < t_2 < \ldots$ is strictly increasing such that $\lim_{k\to\infty} t_k = \infty$. The numbers $x_i(t_k) = x_i(t_k - 0)$ and $x_i(t_k+0)$ are, respectively, the states of the $i-$th unit before and after the impulse perturbation at the moment t_k; the numbers $y_j(t_k) = y_j(t_k - 0)$ and $y_j(t_k+0)$ are, respectively, the states of the $j-$th unit before and after the impulse perturbation at the moment t_k.

Let $\varphi \in PCB[(-\infty,0],\mathbb{R}^m]$, $\varphi = (\varphi_1,\varphi_2,\ldots,\varphi_m)^T$ and $\phi \in PCB[(-\infty,0],\mathbb{R}^n]$, $\phi = (\phi_1,\phi_2,\ldots,\phi_n)^T$ where $PCB[(-\infty,0],\mathbb{R}^m]$ is the class of all piecewise continuous and bounded on $(-\infty,0]$ functions with points of discontinuity of the first kind at $t = t_k$, $k = 1,2,\ldots$, which they are continuous from the left. Denote by

$$col(x(t),y(t)) = col(x(t;0,\varphi),y(t;0,\phi)) \in \mathbb{R}^{m+n},$$

where

$$col\big(x(t;0,\varphi),y(t;0,\phi)\big) = \Big(x_1(t;0,\varphi),\ldots,x_m(t;0,\varphi),y_1(t;0,\phi),\ldots,y_n(t;0,\phi)\Big)^T$$

the solution of system (2.1), satisfying the initial conditions

$$\begin{cases} x_i(s;0,\varphi) = \varphi_i(s),\ -\infty < s \leq 0,\ i = 1,2,\ldots,m, \\ y_j(s;0,\phi) = \phi_j(s),\ -\infty < s \leq 0,\ j = 1,2,\ldots,n, \\ x_i(0^+,0,\varphi) = \varphi_i(0),\ y_j(0^+,0,\phi) = \phi_i(0). \end{cases} \tag{2.2}$$

The solution $col(x(t),y(t)) = col(x(t;0,\varphi),y(t;0,\phi)) \in \mathbb{R}^{m+n}$ of problem (2.1), (2.2) is a piecewise continuous function [22] with points of discontinuity of the first kind at $t = t_k$, $k = 1,2,\ldots$, which it is continuous from the left, i.e., the following relations are valid

$$\begin{cases} x_i(t_k+0) = x_i(t_k) + T_{ik}(x_i(t_k)),\ i = 1,2,\ldots,m, \\ y_j(t_k+0) = y_j(t_k) + U_{jk}(y_j(t_k)),\ j = 1,2,\ldots,n. \end{cases} \tag{2.3}$$

Throughout the paper, we make the following assumptions:

H2.1 The signal functions f_j and g_i $(i = 1,2,...,m;$ $j = 1,2,...,n)$ are Lipschitz continuous, that is, there exist constants $L_j > 0$ and $M_i > 0$ such that

$$\big|f_j(u) - f_j(v)\big| \leq L_j|u-v|,\quad \big|g_i(u) - g_i(v)\big| \leq M_i|u-v|$$

for all $u,v \in \mathbb{R}$, $i = 1,2,\ldots,m,\ j = 1,2,\ldots,n.$

H2.2 The delay kernels $K_{ji}, N_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ are real valued piecewise continuous nonnegative functions and there exist positive numbers r_{ji} and s_{ij} such that

$$\int_{-\infty}^{t} K_{ji}(t-s)\, ds \leq r_{ji} < \infty,\quad \int_{-\infty}^{t} N_{ij}(t-s)\, ds \leq s_{ij} < \infty$$

for all $t \geq 0$, $i = 1,2,\ldots,m,\ j = 1,2,\ldots,n.$

H2.3 The functions T_{ik} and U_{jk} are continuous on $\mathbb{R}$, $i = 1,2,\ldots,m,\ j = 1,2,\ldots,n,\ k = 1,2,\ldots.$

H2.4 $0 = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots$ and $t_k \to \infty$ as $k \to \infty$.

H2.5 There exists a unique equilibrium

$$col(x^*,y^*) = col(x_1^*,x_2^*,...,x_m^*,y_1^*,y_2^*,\dots,y_n^*)$$

of the system (2.1) such that

$$c_i x_i^* = \sum_{j=1}^{n} a_{ji} f_j(y_j^*) + \sum_{j=1}^{n} w_{ji} \int_{-\infty}^{t} K_{ji}(t-s) f_j\left(y_j^*\right) ds + I_i,$$

$$d_j y_j^* = \sum_{i=1}^{m} b_{ij} g_i(x_i^*) + \sum_{i=1}^{m} h_{ij} \int_{-\infty}^{t} N_{ij}(t-s) g_i\left(x_i^*\right) ds + J_j,$$

and

$$T_{ik}(x_i^*) = 0, \;\; U_{jk}(y_j^*) = 0,$$

where $i = 1,2,\dots,m,\;\; j = 1,2,\dots,n,\;\; k = 1,2,\dots$.

The problem of existence and uniqueness of equilibrium states of BAM neural networks with distributed delays without impulses have been investigated in [25]. Efficient sufficient conditions for the existence and uniqueness of an equilibrium of impulsive BAM neural networks with constant delays are given in [18, 24].

Definition 2.1. The equilibrium $col(x^*,y^*) = col(x_1^*,x_2^*,\dots,x_m^*,y_1^*,y_2^*,\dots,y_n^*)$ of system (2.1) is said to be *globally exponentially stable*, if there exist constants $\eta > 0$ and $\Lambda \geq 1$ such that

$$\|x(t)-x^*\| + \|y(t)-y^*\| \leq \Lambda e^{-\eta t}\left(\|\varphi - x^*\|_\infty + \|\phi - y^*\|_\infty\right)$$

for $t \geq 0$, where

$$\|\varphi - x^*\|_\infty = \sup_{s\in(-\infty,0]} \|\varphi(s) - x^*\|, \;\; \varphi \in PCB[(-\infty,0],\mathbb{R}^m],$$

and

$$\|\phi - y^*\|_\infty = \sup_{s\in(-\infty,0]} \|\phi(s) - y^*\|, \phi \in PCB[(-\infty,0],\mathbb{R}^n].$$

Let $G_k = (t_{k-1},t_k) \times \mathbb{R}^m \times \mathbb{R}^n$, $k = 1,2,\dots$; $G = \cup_{k=1}^{\infty} G_k$. In the further considerations, we shall use piecewise continuous auxiliary functions [17], which belong to the class $V_0 = \{V : [0,\infty) \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}_+ : V \in C[G,\mathbb{R}_+],\; t \in [0,\infty), V$ is locally Lipschitzian in $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ on each of the sets G_k, $V(t_k - 0,x,y) = V(t_k,x,y)$ and $V(t_k+0,x,y) = \lim_{\substack{t\to t_k\\ t>t_k}} V(t,x,y)$ exists $\}$.

For $V \in V_0$ and for any $(t,x,y) \in [t_{k-1},t_k) \times \mathbb{R}^m \times \mathbb{R}^n$, $k = 1,2,\dots,$ the upper right–hand derivative $D^+_{(2.1)}V(t,x(t),y(t))$ of the function V with respect to system (2.1) is defined by

$$D^+_{(2.1)}V(t,x(t),y(t)) =$$

$$\limsup_{h\to 0^+} \frac{1}{h}\Big[V(t+h,x(t+h),y(t+h)) - V(t,x(t),y(t))\Big].$$

For the sake of convenience, we shall also use the following notations in the sequel

$$x(t) = (x_1(t),x_2(t),\dots,x_m(t))^T,$$

$$y(t) = (y_1(t),y_2(t),\dots,y_n(t))^T,$$

$$f(y(s)) = (f_1(y_1(s)), f_2(y_2(s)),\dots, f_n(y_n(s)))^T,$$

$$g(x(s)) = (g_1(x_1(s)),g_2(x_2(s)),\dots,g_m(x_m(s)))^T,$$

$$C = diag(c_1,c_2,\dots,c_m),\;\; D = diag(d_1,d_2,\dots,d_n),$$

$$A = (a_{ji})_{n\times m},\;\; B = (b_{ij})_{m\times n},\;\; R = (r_{ji})_{n\times m},\;\; S = (s_{ij})_{m\times n},$$

$$M = diag(M_1,M_2,\dots,M_m),\;\; L = diag(L_1,L_2,\dots,L_n),$$

$$W = (w_{ji})_{n\times m},\;\; H = (h_{ij})_{m\times n},$$

$$I = (I_1,I_2,\dots,I_m)^T,\;\; J = (J_1,J_2,\dots,J_n)^T,$$

$\lambda_{min}(P)$ is the smallest eigenvalue of matrix P,

$\lambda_{max}(P)$ is the greatest eigenvalue of matrix P,

and

$\|P\| = \left[\lambda_{max}(P^T P)\right]^{\frac{1}{2}}$ is the norm of matrix P.

3 The main result

Theorem 3.1. *Assume that*

1. *Conditions* H2.1–H2.5 *hold.*
2. *There exist symmetric positively definite matrices* $P_{m\times m}$ *and* $Q_{n\times n}$ *such that*

$$\|A\|\,\|L\|\,\|P\| + \|B\|\,\|M\|\,\|Q\| + \|W\|\,\|R\|\,\|L\|\,\|P\|\left(\frac{\lambda_{max}(P)+\lambda_{min}(Q)}{\lambda_{min}(Q)}\right) + \|H\|\,\|S\|\,\|M\|\,\|Q\|\frac{\lambda_{max}(P)}{\lambda_{min}(P)} < \mu,$$

and

$$\|A\|\,\|L\|\,\|P\| + \|B\|\,\|M\|\,\|Q\| + \|H\|\,\|S\|\,\|M\|\,\|Q\|\left(\frac{\lambda_{min}(P)+\lambda_{max}(Q)}{\lambda_{min}(P)}\right) + \|W\|\,\|R\|\,\|L\|\,\|P\|\frac{\lambda_{max}(Q)}{\lambda_{min}(Q)} < \nu,$$

where $\mu,\nu = const > 0$.

3. *The functions* T_{ik} *and* U_{jk} *are such that*

$$T_{ik}(x_i(t_k)) = -\gamma_{ik}(x_i(t_k) - x_i^*),\quad 0 < \gamma_{ik} < 2,$$

and

$$U_{jk}(y_j(t_k)) = -\delta_{jk}(y_j(t_k) - y_j^*),\quad 0 < \delta_{jk} < 2,$$

for $i = 1,2,\dots,m,\; j = 1,2,\dots,n,\; k = 1,2,\dots$.

Then the equilibrium $col(x^*, y^*)$ *of* (2.1) *is globally exponentially stable.*

Proof. Set $u(t) = x(t) - x^*$ and $v(t) = y(t) - y^*$ and consider the following system

$$\begin{cases} \dot{u}_i(t) = -c_i u_i(t) + \sum_{j=1}^{n} a_{ji}[f_j(y_j^* + v_j(t)) - f_j(y_j^*)] \\ + \sum_{j=1}^{n} w_{ji} \int_{-\infty}^{t} K_{ji}(t-s)[f_j(y_j^* + v_j(s)) - f_j(y_j^*)]ds, \; t \neq t_k, \\ \dot{v}_j(t) = -d_j v_j(t) + \sum_{i=1}^{m} b_{ij}[g_i(x_i^* + u_i(t)) - g_i(x_i^*)] \\ + \sum_{i=1}^{m} h_{ij} \int_{-\infty}^{t} N_{ij}(t-s)[g_i(x_i^* + u_i(s)) - g_i(x_i^*)]ds, \; t \neq t_k, \\ \Delta u_i(t_k) = I_{ik}(u_i(t_k)), \; \Delta v_j(t_k) = J_{jk}(v_j(t_k)), \; k = 1, 2, \ldots, \end{cases} \tag{3.1}$$

where

$$I_{ik}(u_i(t_k)) = T_{ik}(u_i(t_k) + x_i^*)$$

and

$$J_{jk}(v_j(t_k)) = U_{jk}(v_j(t_k) + y_j^*),$$

for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots$.

We define a Lyapunov function

$$V(t, u(t), v(t)) = u^T(t)Pu(t) + v^T(t)Qv(t).$$

By virtue of condition 3 of Theorem 3.1, we obtain for $t = t_k$

$$\begin{aligned} & V(t_k+0, u(t_k+0), v(t_k+0)) \\ &= u^T(t_k+0)Pu(t_k+0) + v^T(t_k+0)Qv(t_k+0) \\ &= ((1-\gamma_{1k})u_1(t_k), \ldots, (1-\gamma_{mk})u_m(t_k))^T \\ &\times P((1-\gamma_{1k})u_1(t_k), \ldots, (1-\gamma_{mk})u_m(t_k)) \\ &+ ((1-\delta_{1k})v_1(t_k), \ldots, (1-\delta_{nk})v_n(t_k))^T \\ &\times Q((1-\delta_{1k})v_1(t_k), \ldots, (1-\delta_{nk})v_n(t_k)) \\ &< u^T(t_k)Pu(t_k) + v^T(t_k)Qv(t_k) \\ &= V(t_k, u(t_k), v(t_k)), \; k = 1, 2, \ldots. \end{aligned} \tag{3.2}$$

Let $t \geq 0$ and $t \neq t_k$, $k = 1, 2, \ldots$. Then from H2.1 and H2.2, for the upper right–hand derivative of the function V $D^+_{(3.1)}V(t, u(t), v(t))$ with respect to system (3.1) we get

$$\begin{aligned} & D^+_{(3.1)}V(t, u(t), v(t)) = \dot{u}^T(t)Pu(t) + u^T(t)P\dot{u}(t) \\ &+ \dot{v}^T(t)Qv(t) + v^T(t)Q\dot{v}(t) \\ &\leq \Big(-Cu(t) + ALv(t) + WRL \sup_{-\infty<s\leq t} v(s)\Big)^T Pu(t) \\ &+ u^T(t)P\Big(-Cu(t) + ALv(t) + WRL \sup_{-\infty<s\leq t} v(s)\Big) \\ &+ \Big(-Dv(t) + BMu(t) + HSM \sup_{-\infty<s\leq t} u(s)\Big)^T Qv(t) \\ &+ v^T(t)Q\Big(-Dv(t) + BMu(t) + HSM \sup_{-\infty<s\leq t} u(s)\Big). \end{aligned}$$

Since the matrices $CP + PC$ and $DQ + QD$ are positively definite, then there exist $\mu > 0$ and $\nu > 0$ such that

$$\begin{aligned} & D^+_{(3.1)}V(t, u(t), v(t)) \leq -\mu\|u(t)\|^2 - \nu\|v(t)\|^2 \\ &+ 2\|A\|\,\|L\|\,\|P\|\,\|v(t)\|\,\|u(t)\| \\ &+ 2\|B\|\,\|M\|\,\|Q\|\,\|v(t)\|\,\|u(t)\| \\ &+ 2\|P\|\,\|W\|\,\|R\|\,\|L\|\,\|\sup_{-\infty<s\leq t} v(s)\|\,\|u(t)\| \\ &+ 2\|H\|\,\|S\|\,\|M\|\,\|Q\|\,\|\sup_{-\infty<s\leq t} u(s)\|\,\|v(t)\|. \end{aligned}$$

Using the inequality $2|a||b| \leq a^2 + b^2$, we get for $t \neq t_k$, $k = 1, 2, \ldots$.

$$\begin{aligned} & D^+_{(3.1)}V(t, u(t), v(t)) \leq -\mu\|u(t)\|^2 - \nu\|v(t)\|^2 \\ &+ \Big(\|A\|\,\|L\|\,\|P\| + \|B\|\,\|M\|\,\|Q\|\Big)\Big(\|v(t)\|^2 + \|u(t)\|^2\Big) \\ &+ \|P\|\,\|W\|\,\|R\|\,\|L\|\Big(\|\sup_{-\infty<s\leq t} v(s)\|^2 + \|u(t)\|^2\Big) \\ &+ \|H\|\,\|S\|\,\|M\|\,\|Q\|\Big(\|\sup_{-\infty<s\leq t} u(s)\|^2 + \|v(t)\|^2\Big). \end{aligned} \tag{3.3}$$

Since for the function $V(t, u(t), v(t))$, we have

$$\begin{aligned} & \lambda_{min}(P)\|u(t)\|^2 + \lambda_{min}(Q)\|v(t)\|^2 \\ &\leq u^T(t)Pu(t) + v^T(t)Qv(t) \\ &\leq \lambda_{max}(P)\|u(t)\|^2 + \lambda_{max}(Q)\|v(t)\|^2, \; t \geq 0, \end{aligned} \tag{3.4}$$

then for $u(t)$ and $v(t)$ that satisfy the Razumikhin condition

$$V(s, u(s), v(s)) \leq V(t, u(t), v(t)), \;\; -\infty < s \leq t,$$

we obtain

$$\begin{aligned} \lambda_{min}(P)\|u(s)\|^2 + \lambda_{min}(Q)\|v(s)\|^2 & \\ &\leq u^T(s)Pu(s) + v^T(s)Qv(s) \\ &\leq u^T(t)Pu(t) + v^T(t)Qv(t) \\ &\leq \lambda_{max}(P)\|u(t)\|^2 + \lambda_{max}(Q)\|v(t)\|^2, \end{aligned}$$

and hence

$$\begin{cases} \|u(s)\|^2 \leq \dfrac{\lambda_{max}(P)\|u(t)\|^2 + \lambda_{max}(Q)\|v(t)\|^2}{\lambda_{min}(P)}, \\ \|v(s)\|^2 \leq \dfrac{\lambda_{max}(P)\|u(t)\|^2 + \lambda_{max}(Q)\|v(t)\|^2}{\lambda_{min}(Q)}, \end{cases} \tag{3.5}$$

for $-\infty < s \leq t$, $t \geq 0$.

From (3.3) and (3.5), we obtain

$$D^+_{(3.1)}V(t,u(t),v(t)) \le -\mu\|u(t)\|^2 - \nu\|v(t)\|^2$$
$$+ (\|A\|\,\|L\|\,\|P\| + \|B\|\,\|M\|\,\|Q\|)(\|v(t)\|^2 + \|u(t)\|^2)$$
$$+ \|P\|\,\|W\|\,\|R\|\,\|L\|$$
$$\times \left(\frac{\lambda_{max}(P)\|u(t)\|^2 + \lambda_{max}(Q)\|v(t)\|^2}{\lambda_{min}(Q)} + \|u(t)\|^2\right)$$
$$+ \|H\|\,\|S\|\,\|M\|\,\|Q\|$$
$$\times \left(\frac{\lambda_{max}(P)\|u(t)\|^2 + \lambda_{max}(Q)\|v(t)\|^2}{\lambda_{min}(P)} + \|v(t)\|^2\right)$$
$$= \Bigg[-\mu + \|A\|\,\|L\|\,\|P\| + \|B\|\,\|M\|\,\|Q\|$$
$$+ \|P\|\,\|W\|\,\|R\|\,\|L\|\left(\frac{\lambda_{max}(P)+\lambda_{min}(Q)}{\lambda_{min}(Q)}\right)$$
$$+ \|H\|\,\|S\|\,\|M\|\,\|Q\|\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\Bigg]\|u(t)\|^2$$
$$+ \Bigg[-\nu + \|A\|\,\|L\|\,\|P\| + \|B\|\,\|M\|\,\|Q\|$$
$$+ \|H\|\,\|S\|\,\|M\|\,\|Q\|\left(\frac{\lambda_{min}(P)+\lambda_{max}(Q)}{\lambda_{min}(P)}\right)$$
$$+ \|P\|\,\|W\|\,\|R\|\,\|L\|\frac{\lambda_{max}(Q)}{\lambda_{min}(Q)}\Bigg]\|v(t)\|^2,\ t \ne t_k,\ k = 1,2,\ldots.$$

From condition 2 of Theorem 3.1, we derive for $t \ne t_k$, $k = 1,2,\ldots$

$$D^+_{(3.1)}V(t,u(t),v(t)) < -p\|u(t)\|^2 - q\|v(t)\|^2$$
$$\le -k_1\left(\|u(t)\|^2 + \|v(t)\|^2\right), \qquad (3.6)$$

where $p,q = const > 0$ and $k_1 = min\{p,q\} > 0$.

Using (3.4), we get

$$\alpha\left(\|u(t)\|^2 + \|v(t)\|^2\right) \le V(t,u(t),v(t))$$
$$\le \beta\left(\|u(t)\|^2 + \|v(t)\|^2\right),\ t \ge 0, \qquad (3.7)$$

where

$$\alpha = min\Big\{\lambda_{min}(P),\lambda_{min}(Q)\Big\},\ \beta = max\Big\{\lambda_{max}(P),\lambda_{max}(Q)\Big\}.$$

Then, from the inequalities (3.7), (3.6) and (3.2), we obtain

$$V(t,u(t),v(t)) \le e^{-\frac{k_1 t}{\beta}}V(0,u(0),v(0))$$

for all $t \ge 0$.

Since

$$\alpha(\|u(t)\|^2 + \|v(t)\|^2) \le V(t,u(t),v(t))$$
$$\le e^{-\frac{k_1 t}{\beta}}V(0,u(0),v(0))$$
$$\le e^{-\frac{k_1 t}{\beta}}\beta(\|u(0)\|^2 + \|v(0)\|^2),\ t \ge 0,$$

then

$$\|u(t)\|^2 + \|v(t)\|^2 \le e^{-\frac{k_1 t}{\beta}}\frac{\beta}{\alpha}\left(\|u(0)\|^2 + \|v(0)\|^2\right),\ t \ge 0.$$

Using the inequalities

$$(a^2+b^2)^{1/2} \le a+b \le \sqrt{2}(a^2+b^2)^{1/2},$$

we get

$$||u(t)|| + ||v(t)|| = \left(\sum_{i=1}^m u_i^2(t)\right)^{1/2} + \left(\sum_{j=1}^n v_j^2(t)\right)^{1/2}$$
$$\le \sqrt{2}\left(\sum_{i=1}^m u_i^2(t) + \sum_{j=1}^n v_j^2(t)\right)^{1/2}$$
$$= \sqrt{2}\left(\|u(t)\|^2 + \|v(t)\|^2\right)^{1/2}$$
$$\le \sqrt{2}\left(e^{-\frac{k_1 t}{\beta}}\frac{\beta}{\alpha}(\|u(0)\|^2 + \|v(0)\|^2)\right)^{1/2}$$
$$\le \sqrt{\frac{2\beta}{\alpha}}e^{-\frac{k_1 t}{2\beta}}\left(\|u(0)\| + \|v(0)\|\right),\ t \ge 0$$

or

$$\|x(t)-x^*\| + \|y(t)-y^*\| \le \Lambda e^{-\eta t}\left(\|\varphi - x^*\|_\infty + \|\phi - y^*\|_\infty\right),$$

for $t \ge 0$, where $\Lambda = \sqrt{\frac{2\beta}{\alpha}}$ and $\eta = \frac{k_1}{2\beta}$. This completes the proof of the theorem.

4 An example

Let $t \ge 0$. Consider the impulsive BAM neural network

$$\begin{cases} \dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^2 a_{ji} f_j(y_j(t)) \\ + \sum_{j=1}^2 w_{ji}\int_{-\infty}^t K_{ji}(t-s) f_j(y_j(s))\,ds + I_i,\ i = 1,2,\ t \ne t_k, \\ \dot{y}_j(t) = -d_j y_j(t) + \sum_{i=1}^2 b_{ij} g_i(x_i(t)) \\ + \sum_{i=1}^2 h_{ij}\int_{-\infty}^t N_{ij}(t-s) g_i(x_i(s))\,ds + J_j,\ j = 1,2,\ t \ne t_k, \end{cases} \qquad (4.1)$$

with impulsive perturbations of the form

$$\begin{cases} x_1(t_k+0) = \frac{0.125 + x_1(t_k)}{2},\ k = 1,2,\ldots, \\ x_2(t_k+0) = \frac{0.25 + x_2(t_k)}{3},\ k = 1,2,\ldots, \\ y_1(t_k+0) = \frac{0.25 + 2y_1(t_k)}{3},\ k = 1,2,\ldots, \\ y_2(t_k+0) = \frac{0.75 + 2y_2(t_k)}{5},\ k = 1,2,\ldots, \end{cases} \qquad (4.2)$$

where the impulsive moments are such that $0 < t_1 < t_2 < \ldots$, $\lim_{k\to\infty} t_k = \infty$, and

$$K_{ji} = N_{ij} = e^{-t}, \ \ i, j = 1,2,$$

$$f_j(u) = g_i(u) = \frac{1}{2}(|u+1| - |u-1|), \ \ i, j = 1,2, \, u \in \mathbb{R},$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, C = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix},$$

$$D = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}, B = \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix},$$

$$W = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}, H = \begin{pmatrix} -1/3 & 1/3 \\ 1/3 & 1/3 \end{pmatrix},$$

and

$$I = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, J = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}.$$

Upon substituting $I_1 = I_2 = 0.875$ and $J_1 = J_2 = 1.416667$, we find that system (4.1), (4.2) has an equilibrium $x_1^* = x_2^* = 0.125$, $y_1^* = y_2^* = 0.25$.

Let $P = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Since $L = M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $R = S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then $CP + PC = \begin{pmatrix} 36 & 0 \\ 0 & 36 \end{pmatrix}$, $DQ + QD = \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$ and for $\mu = 36$ and $\nu = 12$, we have

$$\begin{aligned} &-\mu + \|A\| \, \|L\| \, \|P\| + \|B\| \, \|M\| \, \|Q\| \\ &+ \|W\| \, \|R\| \, \|L\| \, \|P\| \left(\frac{\lambda_{max}(P) + \lambda_{min}(Q)}{\lambda_{min}(Q)} \right) \\ &+ \|H\| \, \|S\| \, \|M\| \, \|Q\| \frac{\lambda_{max}(P)}{\lambda_{min}(P)} = -36 + 8\sqrt{2} < 0 \end{aligned}$$

and

$$\begin{aligned} &-\nu + \|A\| \, \|L\| \, \|P\| + \|B\| \, \|M\| \, \|Q\| \\ &+ \|H\| \, \|S\| \, \|M\| \, \|Q\| \left(\frac{\lambda_{min}(P) + \lambda_{max}(Q)}{\lambda_{min}(P)} \right) \\ &+ \|W\| \, \|R\| \, \|L\| \, \|P\| \frac{\lambda_{max}(Q)}{\lambda_{min}(Q)} = -12 + \frac{13}{3}\sqrt{2} < 0. \end{aligned}$$

Moreover, one can easily deduce that $\gamma_{1k} = \frac{1}{2}$, $\gamma_{2k} = \frac{2}{3}$, $\delta_{1k} = \frac{1}{3}$ and $\delta_{2k} = \frac{3}{5}$. Thus, all conditions of Theorem 3.1 are satisfied. This implies that the equilibrium $x_1^* = x_2^* = 0.125$, $y_1^* = y_2^* = 0.25$ of (4.1) is globally exponentially stable.

On the other hand, if we consider again system (4.1) but with impulsive perturbations of the form

$$\begin{cases} x_1(t_k + 0) = \dfrac{0.125 + x_1(t_k)}{2}, \, k = 1,2,\ldots, \\ x_2(t_k + 0) = 4x_2(t_k) - 0.75, \, k = 1,2,\ldots, \\ y_1(t_k + 0) = \dfrac{0.25 + 2y_1(t_k)}{3}, \, k = 1,2,\ldots, \\ y_2(t_k + 0) = \dfrac{0.75 + 2y_2(t_k)}{5}, \, k = 1,2,\ldots, \end{cases} \tag{4.3}$$

then the point $x_1^* = x_2^* = 0.125$, $y_1^* = y_2^* = 0.25$ will be again an equilibrium of (4.1), (4.3) but there is nothing we can say about its exponential stability because $\gamma_{2k} = -3 < 0$.

This example shows that by means of appropriate impulsive perturbations, we can control the stability behavior of the neural networks.

Conclusions

In this paper, we have obtained a matrix format sufficient conditions for the global exponential stability of the equilibrium point of a general class of BAM neural network model with distributed delays and nonlinear impulsive operators. Although, the matrix format sufficient conditions are easy to be resolved, a few authors have studied the stability of the delayed BAM neural networks with impulses using matrix theory. The main result is established by using a suitable piecewise continuous Lyapunov function and by applying the Razumikhin technique. We show that by means of appropriate impulsive perturbations we can control the stability behavior of the neural networks. The technique can be extended to study other types of impulsive delayed systems.

References

[1] B. Kosko, Adaptive bidirectional associative memories, *Appl. Opt.* **26** (1987), 4947–4960.

[2] B. Kosko, Bi–directional associative memories, *IEEE Trans. Syst. Man Cybern.* **18** (1988), 49–60.

[3] B. Kosko, Neural Networks and Fuzzy Systems–A Dynamical Systems Approach to Machine Intelligence, Prentice–Hall, Englewood Cliffs, NJ, 1992.

[4] C. M. Marcus, R. M. Westervelt, Stability of analog neural networks with delay, *Phys. Rev. A* **39** (1989), 347–359.

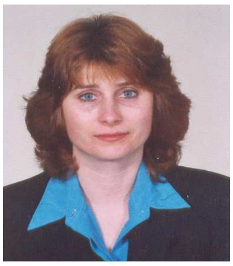
[5] P. Baldi, A. F. Atiya, How delays affect neural dynamics and learning, *IEEE Trans. Neural Networks* **5** (1994), 512–621.

[6] S. I. Niculescu, Delay Effects on Stability: A Robust Control Approach, Vol. LNCIS, 269, Heindelberg: Springer, 2001.

[7] S. Aric, V. Tavsanoglu, Global asymptotic stability analysis of bidirectional associative memory neural networks with constant time delays, *Neurocomputing* **68** (2005), 161–176.

[8] J. D. Cao, Global asymptotic stability of delayed bi–directional associative memory networks, *Appl. Math. Comput.* **142** (2003), 333–339.

[9] J. D. Cao, M. Dong, Exponential stability of delayed bi–directional associative memory neural networks, *Appl. Math. Comput.* **135** (2003), 105–112.
[10] J. D. Cao, L. Wang, Exponential stability and periodic oscillatory solution in BAM networks with delays, *IEEE Trans. Neural Networks* **13** (2002), 45–463.
[11] A. P. Chen, J. D. Cao, L. H. Huang, Exponential stability of BAM neural networks with transmission delays, *Neurocomputing* **57** (2004), 435–454.
[12] H. Akca, R. Alassar, V. Covachev, Z. Covacheva, E. Al–Zahrani, Continuous–time additive Hopfield-type neural networks with impulses, *J. Math. Anal. Appl.* **290** (2004), 436–451.
[13] S. Ahmad, I. M. Stamova, Global exponential stability for impulsive cellular neural networks with time–varying delays, *Nonlinear Anal.* **69** (2008), 786–795.
[14] M. A. Arbib, *Brains, Machines, and Mathematics*, New York: Springer–Verlag, 1987.
[15] K. Gopalsamy, X. He, Delay–independent stability in bidirectional associative memory networks, *IEEE Trans. Neural Networks* **5** (1994), 998–1002.
[16] J. K. Hale, *Theory of Functional Differential Equations*, Springer–Verlag, New York, Heidelberg, Berlin, 1977.
[17] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, New Jersey, London, 1989.
[18] Y. Li, Global exponential stability of BAM neural networks with delays and impulses, *Chaos Solutions Fractals* **24** (2005), 279–285.
[19] X. Y. Lou, B. T. Cui, Global asymptotic stability of delay BAM neural networks with impulses, *Chaos Solutions Fractals* **29** (2006), 1023–1031.
[20] G. Tr. Stamov, Existence of almost periodic solutions for impulsive cellular neural networks, *Rocky Mount. J. Math.* **4** (2008), 1271–1285.
[21] G. Tr. Stamov, I. M. Stamova, Almost periodic solutions for impulsive neural networks with delay, *Appl. Math. Model.* **31** (2007), 1263–1270.
[22] I. M. Stamova, *Stability Analysis of Impulsive Functional Differential Equations*, Walter de Gruyter, Berlin, New York, 2009
[23] J. Yan, J. Shen, Impulsive stabilization of impulsive functional differential equations by Lyapunov-Razumikhin functions, *Nonlinear Anal.* **37** (1999), 245–255.
[24] Q. Zhou, L. Wan, Impulsive effects on stability of Cohen–Grossberg–type bidirectional associative memory neural networks with delays, *Nonlinear Anal. RWA* **10** (2009), 2531–2540.
[25] Q. K. Song, J. D. Cao, Global exponential stability of bidirectional associative memory neural networks with distributed delays, *J. Comp. Appl. Math.* **202** (2007), 266–279.
[26] H. Zhao, Global stability of bidirectional associative memory neural networks with distributed delays, *Phys. Lett. A* **30**, (2002) 519–546.



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