Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/070431

Study the Behavior of Vortex-Antivortex Bundles in *He II*

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Received: 16 Dec. 2012, Revised: 21 Feb. 2013, Accepted: 9 Mar. 2013
Published online: 1 Jul. 2013

Abstract: We present a numerical simulation to compute the evolution of vortex filaments bundle in superfluid helium. We show that the vortex-antivortex bundles with sinusoidally have stable structures and each bundle rotates about its common center. Because of they have circulation in opposing directions, the two vortex bundles move down together parallel to each other. A three dimensional periodic cube is used. Thus, the vortex filament points move through one side of the periodic volume and re-entering on the other side. The two bundles move for long time without reconnection. We found that our results are in agreement with the finding of Koplik and Levine [Phys. Rev. Lett. 71, 1375 (1993)], who used the nonlinear Schrödinger equation (NLSE) model to study the cases of single vortices. This movement leads to stretching and flexure of vortex lines which cause changes in velocities, radius, number of points and total length.

Keywords: Numerical simulation, superfluid turbulence, vortex filament, three dimensional periodic cube

1 Introduction

The turbulent motion of fluids has captured the attention of observers of nature for most of recorded history. Quantum turbulence is the turbulent flow of quantized vortices which occurs in low temperature fluids such as Helium 3. Quantized vortices appearing in quantum fluids influence many properties of the systems [1]. Vortices are appeared in many fields of nature. These are not well-defined for a typical classical fluid and the relationship between vortices and turbulence remains indistinct. The liquid state of ^{4}He exists in two phases: a high temperature phase which called Helium I and a low temperature phase called Helium II [2]. Both of the two phases are separated by a transition called the lambda transition which occurs at the critical temperature $T = T_\lambda = 2.1768\ K$ at saturated vapor pressure [3]. The relative proportion of normal fluid and superfluid is determined by the absolute temperature T. At $T = 0$, Helium II is entirely superfluid where $\rho_s/\rho = 1$ and $\rho_n/\rho = 0$ with ρ_s, ρ_n and ρ denote the superfluid density, normal fluid density and total density, respectively. If the temperature is increased, the superfluid fraction decreases and the normal fluid fraction increases until $T = T_\lambda$. Then, Helium II becomes entirely normal and $\rho_s/\rho = 0$ and $\rho_n/\rho = 1$. Most configurations of quantized vortices which have been investigated in superfluid (Helium II) can be grouped into two types [4]: ordered vortex arrays and disordered vortex tangles. The key property of a superfluid vortex line was discovered by Onsager and developed by Feynman (see Ref. [5] and references therein). The circulation around each superfluid vortex filament is fixed by the condition that $\oint_c \mathbf{v}_s \cdot \mathbf{dl} = \Gamma$, where c is a circular path around the axis of the vortex, $\Gamma = \frac{h}{m} = 9.97 \times 10^{-4}\ \ \mathrm{cm}^2/\mathrm{s}$ is the quantum of circulation, h is Plank's constant and m is the mass of the helium atom.

In the early 1940s, experiments on the rotational motion of He II revealed more surprises. The quantization of the circulation, predicted by Onsager and Feynman (see [5]), was first observed by Vinen (see Ref. [6]). Vinen also performed the first experimental investigations of quantum turbulence [6].

The dynamics of quantized vortices can be described by using the model of nonlinear Schrödinger equation (NLSE), also called the Gross-Pitaevsk (GP) model. This model is applicable at very low temperatures where normal fluid is absent. In the present, work we used the model of Schwarz [7,8] where the vortex points move according to the Biot-Savart law. So the vortex lines are numerically discretized by a large variable number of points depends on the local radius of curvature as we will see in the next section.

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The paper is planned as follows. In Sec.2 a description of the numerical simulation is presented. The results are given and discussed in Sec. 3. The paper is concluded in Sec. 4.

2 Numerical Simulation

The vortex filament model was pioneered by Schwarz [7] . In this model, a quantized vortex line is represented as a curve $\mathbf{X} = \mathbf{X}(\xi,t)$ in three dimensional space, where ξ is the arc-length and t is the time [3]. The vectors $\mathbf{X}'$, $\mathbf{X}''$ and $\mathbf{X}' \otimes \mathbf{X}''$ are perpendicular to each other and point along the tangent, principal normal and binormal directions, respectively. The prime denotes to the derivative with respect to ξ. The equation of motion of a point $\mathbf{X}$ on a vortex line which moves according to the Biot-Savart law is given by

$$\mathbf{U}_L = \frac{d\mathbf{X}}{dt} = \gamma \mathbf{X}' \times \mathbf{X}'' + \frac{\Gamma}{4\pi}\int_{\ell'} \frac{(\mathbf{X}-\mathbf{Z}) \times d\mathbf{X}}{|\mathbf{X}-\mathbf{Z}|^3}, \qquad (1)$$

where

$$\gamma = \frac{\Gamma}{4\pi} \ln\left(\frac{c\sqrt{\ell_i \ell_{i-1}}}{a}\right). \qquad (2)$$

with a is the vortex core radius (typically $a \approx 10^{-8}$ cm) [9]. The symbol ℓ' means that some local region about the point $\mathbf{Z} = \mathbf{X}_i$ (through calculating the integration) has been removed from the integral. We define ℓ_i and ℓ_{i-1}, as the arc-lengths of the curve between $\mathbf{X}_{i+1}$ and $\mathbf{X}_i$ and between $\mathbf{X}_i$ and $\mathbf{X}_{i-1}$, respectively. These lengths ℓ_i and ℓ_{i-1} are always much larger than a. The size of the local region about one order of magnitude smaller than the local radius of curvature. This gives accurate results with a reasonable number of calculations, see also [5,10,11].

Equation (1) is a nonlinear parabolic partial differential equation that can be solved numerically by constructing suitable finite difference equations for the discrete solutions. In order to solve it numerically, the parametric representation of the vortices $\mathbf{X}(\xi)$ at some time t_n is approximated by discrete points

$$\mathbf{X}(\xi_i,t_n) = \mathbf{X}_i^n = \begin{bmatrix} x(\xi_i,t_n) \\ y(\xi_i,t_n) \\ z(\xi_i,t_n) \end{bmatrix}. \qquad (3)$$

The vectors $\mathbf{X}'$ and $\mathbf{X}''$ are the first and second spatial derivatives which can be approximated by the central difference replacement. There are several algorithms to numerically integrate in time the solution of Eq. (1). A finite difference method is compatible with the differential equation as long as the truncation error disappear as $\Delta t \to 0$, where Δt is the time step size. Our finite difference scheme is the classical fourth order Runge-Kutta method which is a well known general purpose method for ordinary differential equations. This method is described in more detail in Refs. [5,10]. In our simulation, we use a Lagrangian grid where the vortex filament is represented by a series of mesh points circulated along the centerline of the vortex filament. The motion of the vortex lines is calculated by moving each of these mesh points. As mesh points are too close to one another, our code will remove some points need to be removed from some sections on the filament. Vice versa, some points can be added by the code when the vortex filaments are locally elongated. So the spacing between the mesh points will be changeable with time. For more details, we refer to Refs. [12,13,14].

3 Results and Discussion

According to Schwarz [7], when two quantized vortices approach, the long- range interactions tend to drive the cores together so to be antiparallel at the point of closest approach with some oscillation along the core [15]. We refer to this case as 180^0. It is known, that when the vortices have equal circulation as shown in Fig. (1), the vortices will move in opposite directions and circle around the mid-point between them [16]. While, if the vortices have circulation in opposing directions, as shown in Fig. (2), the vortices will move parallel to each other in a straight line (see Refs. [17,18,19] for details).

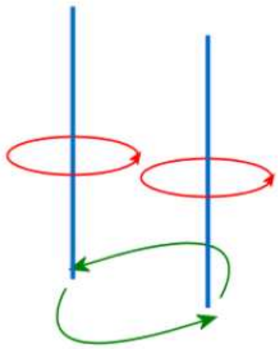

Fig. 1: Velocities of the flow around two vortices with equal circulation in one plane. Green arrow represents motion.

In this work, we study the behavior of two bundles (with sinusoidally) each one of them contains 5 of (initially) parallel vortex strands, set at 180^0 and one of them against the other (vortex with antivortex bundles). The distance between the closest point is approximately 4δ, where $\delta = 0.0155804$ cm is the radius of each bundle. In this case, we place four vortices at the corner of a square lie on a circle and one vortex in the middle. Where a cubic periodic box $-\mu \leq x,y,z \leq \mu$, with $\mu = 0.707107$ cm is considered, vortex bundles with sinusoidally are found to be stable structures. In addition, each bundle rotates about its common center, because

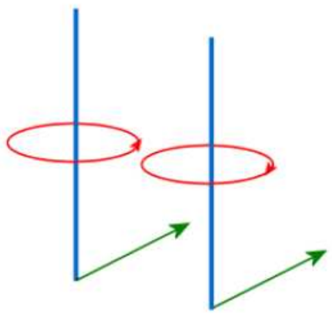

Fig. 2: Velocities of the flow around two vortices with opposite circulation (anti-parallel vortices) in one plane. Green arrow represents motion.

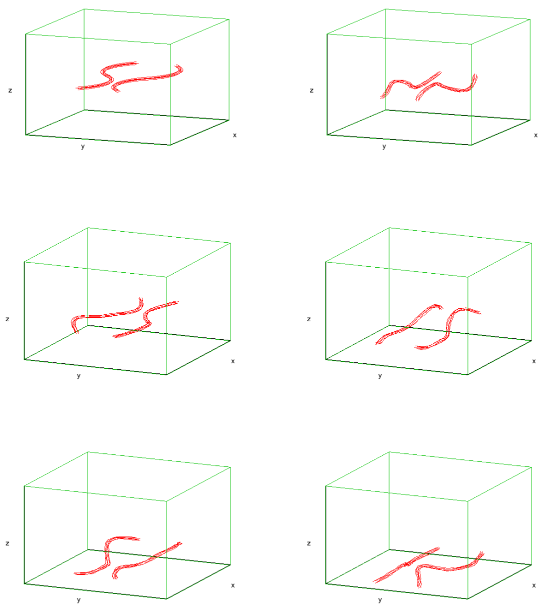

Fig. 3: Motion of vortex bundle and antivortex bundle (with sinusoidally) each one consist of 5 vortices: upper-left panel, $t = 0$ s; upper-right panel, $t = 50$ s; middle-left panel: $t = 97.74$ s; middle-right panel, $t = 147.74$ s; lower-left panel, $t = 197.97$ s; lower-right panel, $t = 247.84$ s.

each one contains 5 parallel vortices. Because of they have circulation in opposing directions, the two vortex bundles move down together parallel to each other in a semi-straight line as shown in Fig. (3).

In the present work, we have used a three dimensional periodic cube. This method as a matter of fact only ensures that vortex filament points moving through one side of the periodic volume and re-entering on the other side (see Fig (4)). It is clear from Fig. (5), that the two bundles move

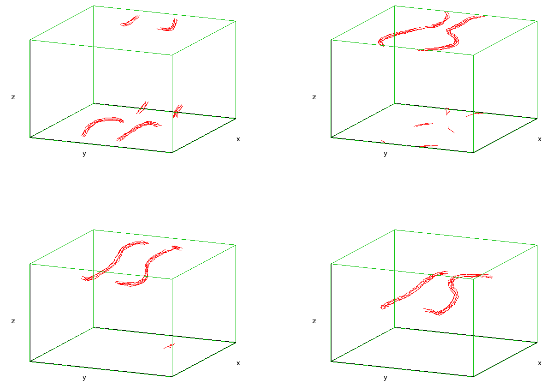

Fig. 4: vortex bundle and antivortex bundle move down together through one side of the periodic volume and re-entering on the other side: upper-left panel, $t = 272.86$ s; upper-right panel, $t = 322.91$ s; lower-left panel, $t = 372.96$ s; lower-right panel, $t = 568.18$ s.

for long time without reconnection until $T = 568.18$ s. But there is a potentiality to occurring of reconnection but need long time to bring the bundles close enough together for quantum mechanics to act and cause reconnection [15]. It has been shown in [20], that when the distance between the two vortex bundles is less than 2δ, the reconnection takes place in a short time. These results are in agreement with the finding of Koplik and Levine [15], who used the NLSE model to study the cases of single vortices.

As a result, each bundle rotates about its center (in different direction) and at the same time the bundles move parallel together. We note, that this movement leads to stretching and flexure of vortex lines. This makes the total length L increases in the upper-left panel of Fig. (5). The upper-right panel of Fig. (5) shows the average inverse radius of curvature, $< 1/R >$, obtained by computing $|\mathbf{X}''|$ at each discretization point $\mathbf{X}_j$ $(j = 1, 2, ..., N)$ and then averaging over all discretization points. As a result of the increase in L and the decrease of $< R >$, the number of discretization points (initially $N = 1999$) grows with time up to $N = 2450$ when we stop this particular calculation as shown in the lower-left panel of Fig. (5). The lower-right panel of Fig. (5) shows, that the decrease of $< R >$ causes the increase in the average velocity of vortex points.

4 Conclusion

The dynamics of quantized vortices is investigated by using the Schwarz model [7] where the vortex points move according to the Biot-Savart law. It has shown that the vortex-antivortex bundles with sinusoidally are stable structures. Each bundle rotates about its common center

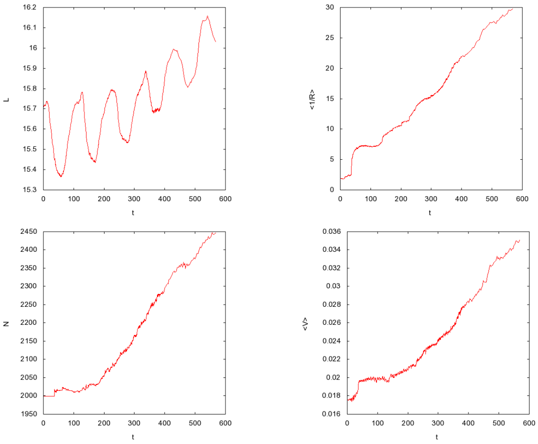

Fig. 5: Corresponding to the evolution shown in Figs. (3) and (4): upper-left panel, total vortex length L versus time t; upper-right panel, average inverse radius of curvature $< 1/R >$ versus time t; lower-left panel, number of discretization points N versus time t; lower-right panel, average velocity of vortex points $< v >$ versus time t.

because each one contains 5 parallel vortices. Since the two vortex bundles have circulation in opposing directions, they move down together parallel to each other. In this work, we used a three dimensional periodic cube. So the vortex bundles move through one side of the periodic volume and re-entering on the other side. They move for long time without reconnection. This movement leads to stretching and flexure of vortex lines which cause change in velocities, radius, number of points and total length. The results of this work are in agreement with the finding of Koplik and Levine [15], who used the NLSE model to study the case of single vortices.

Acknowledgement

I am grateful to Prof. Eed M. Darwish for useful discussions and a critical reading of the manuscript.

References

[1] Donnelly, R .J., *Quantized Vortices In Helium II*, Cambridge University Press, Cambridge (1991).
[2] Barenghi, C. F., Donnelly, R . J. and Vinen, W. F., *Quantized Vortex Dynamics and Superuid Turbulence*, Springer (2001).
[3] Alamri, S. Z., Youd, Anthony J. and Barenghi, C. F., Phys. Rev. Lett. **101**, 215302 (2008).
[4] Tsubota, M., Araki, T. and Nemirowskii, S. K., Phys. Rev. B **62**, 11751 (2000).
[5] Aarts, R., *A Numerical Study of Quantized Vortices in He II*, PhD Thesis, Eindhoven University (1993).
[6] Baggaley, A.W. and Barenghi, C.F., J. Phys.: Conf. Ser. **318**, 062001 (2011).
[7] Schwarz , K.W., Phys. Rev. B **31**, 5782 (1985).
[8] Schwarz, K.W., Phys. Rev. B **38**, 2398 (1988).
[9] Maggioni, F., Alamri, S. Z., Barenghi, C. and Ricca, R., Il Nuovo Cimento **32** C, 133 (2009).
[10] Berry, M. V. and Dennis, M. R., Eur. J. Phys. **33**, 723 (2012).
[11] Barenghi, C. F., Eur. J. Mech. B **23**, 415 (2004).
[12] Ismail, M.S., Alamri S.Z., Int. J. Comp. Math. **81**, 333 (2004).
[13] Maggioni, F., Alamri, S. Z., Barenghi, C. and Ricca, R., Phys. Rev. E **82**, 026309 (2010).
[14] Baggaley, A.W., Barenghi, C. F., and Sergeev, Y. A., Phys. Rev. B **85**, 060501(R) (2012).
[15] Koplik, J. and Levine, H., Phys. Rev. Lett. **71**, 1375 (1993).
[16] Alamri, S.Z., *A Numerical Study of Quantum Turbulence*, PhD Thesis, Newcastle University (2009).
[17] Baggaley, A.W. and Barenghi, C.F., Low Temp. Phys. **166**, 3 (2012).
[18] Baggaley, A.W. and Barenghi, C. F., Phys. Rev. E **84**, 067301 (2011).
[19] White, A.C., Barenghi, C.F. and Proukakis, N.P., Phys. Rev. A **86**, 013635 (2012).
[20] Alamri, S.Z. and Alenezi, A.A., submitted for publication (2013).

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