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A Quick Convergent Inflow Algorithm for Solving Linear Programming Problems

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Abstract: This work provides a new method of obtaining optimizers of response functions. The method uses line search techniques and it is particularly useful for linear programming problems. The focus is to reach the optimizer in the fewest number of iterations. The method has been compared with simplex method, active set method, Linear Exchange algorithm (LEA), Quadratic exchange algorithm (QEA), and Minimum norm exchange algorithm (MNEA) and is found comparatively efficient. Numerical demonstrations prove effectiveness of the new method.

Keywords: Linear Programming; line search; optimizers; response functions.

1 Introduction

A standard LP problem is defined as an optimization problem of a linear objective function in n non-negative variables subject to m linearly independent constraints (where m < n, m = n or m > n). The objective function could be called the response function. When the interest of the experimenter is to locate the optimum of the response function, Response Surface Methodology (RSM) comes into play. According to Montgomery [7], Response Surface Methodology is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables x_1, x_2, …, x_n influence a dependent variable (response), say Z. The minimizer (or maximizer) of the objective function is a point $\underline{x}^* \in \tilde{X}$, where $\tilde{X}$ is a member of the experiment space $\{ \tilde{X}, F_x, \sum_x \}$, whose components are as defined below;

$\tilde{X} = \{ \underline{x} \}$ is the space of all possible trials of the experiment.

$F_x = \{ f(\underline{x}) \}$ is a space of finite dimensional continuous functions that can be defined on $\tilde{X}$.

$\sum_x = \{ \sigma_x^2 \}$ is the space of positive, continuous random observation error which can be defined on $\tilde{X}$.

In this work we assumed that the constraints are linear inequalities defined on convex feasible region. We seek therefore a method of solving linear programming problems using experimental design techniques. The method presented in this paper is such that would arrive at the desired optimum (minimum or maximum) in the fewest number of iterations.

2 Literature Review and Methodology

The procedure embodied in the search algorithm of Box [1] locates the optimum $x^* \in \tilde{X}$ of a response function using line search equation,

$$x^* = \underline{\bar{x}} \pm \rho \, \underline{d} \qquad (2.1)$$

where $\underline{\bar{x}}$ is the starting point of search, $\underline{d}$ is the direction of search and ρ is the step-length. Many methods have been provided for obtaining the components of the above line equation. These methods include the steepest descent (ascent) method, the Newton's method (see e.g Storey [11]). Umoren [12] considered the construction of exact D-optimum designs for constrained optimization problems. Umoren [13] presented a maximum norm exchange algorithm for solving linear programming problems. Umoren [14] applied optimal design theory to the solutions of a constrained optimization problems. Umoren [15] presented some optimality conditions for the existence of optimizers of a certain class of linear programming problems. Umoren [16] developed a quadratic exchange algorithm for solving linear programming problems. Other related works include Umoren [17] and Etukudo and Umoren [2].

One of the recent line search techniques that is a powerful tool for solving different optimization problems that are often encountered in mathematical programming is due to Onukogu and Chigbu [9]. The line search technique is built around the concept of super convergence. The algorithm locates the local optimizer of a response function in the fewest number of moves and hence the technique is called Super Convergent Line Series (SCLS). Etukudo and Umoren [3] have modified the Super Convergent Line Series algorithm for solving linear programming. This reduction has reduced the computational requirements of the initial algorithm. Many comparisons have been made using the modified algorithm and may be seen in Etukudo and Umoren [4], Umoren and Etukudo [19], [20], Etukudo, Umoren and Enang [6].

In this work, we develop a new approach of obtaining an optimizer of a response function using the line equation in 2.1. Specifically, we provide a new method of obtaining the direction of search. The direction is such that has minimum variance property and is based on designing the experiment to minimize the variance of the response function. Onukogu [8] has shown that the optimizer of a response function has a minimum variance. We seek then a line equation that searches in the direction of minimum variance. Thus, given an n-variate response function $f(\underline{x}) = \underline{g}'\underline{x}$ with k-component gradient vector $\underline{g}$, the gradient vector $\underline{g}$ can be transformed by an n x k full rank matrix of transformation, say T, so that $Z = T\underline{g}$, has minimum variance. The direction vector is then $\underline{d} = A^{-1} T\underline{g}$, where A is an n x n nonsingular symmetric matrix. In a minimization problem, the line equation becomes $\underline{x}^* = \underline{\bar{x}} - \rho\,\underline{d}$ and in a maximization of problem the line equation becomes $\underline{x}^* = \underline{\bar{x}} + \rho\,\underline{d}$. The components of the line equation, $\underline{\bar{x}}$, ρ, and $\underline{d}$ are optimally chosen. We present in section 2.1 the sequence of steps involved in the algorithm.

2.1 The Algorithm

The line search algorithm follows the following sequence of steps,

S_1: Form the design measure, ξ_N^k, by selecting N support points $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_N$ from $\tilde{X}$. The support points that make up the design measure must satisfy the m linear inequality constraints and must result in a non-singular information matrix. To obtain a non-singular information matrix, the number of support points say N, must satisfy the bound

$$n \le N \le \tfrac{1}{2} n(n+1)+1 \qquad (2.2)$$

(see e.g Onukogu [8], Pazman [10])

S_2: Obtain the optimal starting point $\underline{\bar{x}}^*$. For n-variates, say, $x_1, x_2, \ldots, x_n$ the average of N support points can be used as the optimal starting point. That is,

$$\underline{\bar{x}}^* = \left(\underline{\bar{x}}_1^*, \underline{\bar{x}}_2^* \underline{\bar{x}}_3^*, \ldots, \underline{\bar{x}}_n^*\right) \qquad (2.3)$$

where

$$\underline{\bar{x}}_1^* = \left(x_{11}, x_{12}, x_{13}, \ldots, x_{1n}\right) / N$$

$\underline{\bar{x}}_2^* = \left(x_{21}, x_{22}, x_{23}, \ldots, x_{2n}\right) / N$

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$\underline{\bar{x}}_n^* = \left(x_{n1}, x_{n2}, x_{n3}, \ldots, x_{nn}\right) / N$

S_3: Determine the information matrix, M_k corresponding to the design measure ξ_{N+k}^K ; k = 0., M_k = T,

S_4: Obtain the determinant of the information matrix, say, det (M_k).

S_5: Obtain the variance-covariance matrix, M_k^{-1}, of the information matrix, M_k, where $M^{-1} = A^{-1}$

S_6: Relate the coefficients of the objective function with the information matrix by

$$Z_k = \begin{bmatrix} Z_{k1} \\ Z_{k2} \end{bmatrix} = M_k \underline{g} \tag{2.4}$$

where $\underline{g}$ is the vector of the coefficients of the objective function.

S_7: Determine the direction of search $\underline{d}_k$, where $\underline{d}_k$ is an n-component vector defined by

$$\underline{d}_k = M_k^{-1} Z_k = \begin{bmatrix} d_1 \\ d_2 \\ . \\ . \\ . \\ d_n \end{bmatrix} \tag{2.5}$$

S_8: Obtain the normalized direction of search $\underline{d}_k^*$ such that $\underline{d}_k^{*1} \underline{d}_k^* = 1$

The normalized direction vector is defined as

$$\underline{d}_k^* = \frac{1}{\sqrt{d_1^2 + d_2^2 + \ldots + d_n^2}} \begin{bmatrix} d_1 \\ d_2 \\ . \\ . \\ . \\ d_n \end{bmatrix} = \begin{bmatrix} \underline{d}_1^* \\ \underline{d}_2^* \\ . \\ . \\ . \\ \underline{d}_n^* \end{bmatrix} \tag{2.6}$$

S_9: Determine the optimal steps-length, ρ^*, by

$$\min \rho_{kj}^* = \min \left[\frac{\left(C_{i1}, C_{i2}, \ldots, C_{in}\right) \left(\underline{\bar{x}}_0\right) - b_i}{\left(C_{i1}, C_{i2}, \ldots, C_{in}\right) \left(\underline{d}^*\right)} \right] \tag{2.7}$$

i = 1,2,3, …, n

j = 1,2,3, …, m

S_{10}: With $\underline{\bar{x}}^*$; ρ^* ; $\underline{d}^*$; make a move to $\underline{x}_{k+1}^* = \underline{\bar{x}}^*{}_k \pm \rho_k^* \underline{d}_k^*$,

where k = 0,1, …, q.

S_{11}: Evaluate $f\left(\underline{x}_{k+1}\right) = f_{k+1}$;

S_{12}: Setting k = k+1 and N = N+1 add

the points $\underline{x}^{*}_{k+1}$ in step S_{10} above to the design measure in step S_1 and if $\underline{x}^{*}_{k+1}$ satisfy the constraints, continue from step S_2 to step S_{11}. Thus obtaining $\underline{x}^{*}_{k+2}$.

S_{13}: Is $\| \underline{x}^{*}_{k+2} - \underline{x}^{*}_{k+1} \| < \varepsilon > 0$?

if No, go to step S_{12} and continue the process.

If Yes, the optimizer of the objective function is $\underline{X}_{k+1}$

3 Numerical Demonstration

The working of the algorithm developed in section 2 shall be tested numerically using some linear programming problems.

3.I llustration 1

The problem here is to maximize the objective function

$$Z = 3x_1 + 2x_2 \quad (3.1)$$

Subject to

$4x_1 + 3x_2 \leq 12$

$4x_1 + x_2 \leq 8$

$4x_1 - x_2 \leq 8$

$x_1, x_2 \geq 0$

To enable us maximize the given objective function we select the support points $[(1,2), (1.5, 1), (0.5, 2)]$ satisfying the linear constraints and satisfying equation 2.2. With these support points, we form the initial design measure as

$$\xi_3^{(0)} = \begin{bmatrix} 1 & 2 \\ 1.5 & 1 \\ 0.5 & 2 \end{bmatrix}$$

Notice that each of the support points that make up the initial design measure satisfies the three constraints of the objective function. We highlight this briefly as follows;

Using the constraint $4x_1 + 3x_2 \leq 12$, the support point (1,2) yields 10, (1.5, 1) yields 9 and (0.5,2) yields 8. Using the constraints $4x_1 + x_2 \leq 8$ the support points (1,2) yields 6, (1.5,1) yields 7 and (0.5,2) yields 4.

Similarly, using the constraint $4x_1 - x_2 \leq 8$ the support point (1,2) yields 2, the support point (1.5,1) yields 5 and the support point (0.5,2) yields 0.

Hence, the selected points satisfy the constraints.

The design matrix associated with the initial design measure ($\xi_3^{(0)}$) is

$$X_0 = \begin{bmatrix} 1 & 2 \\ 1.5 & 1 \\ 0.5 & 2 \end{bmatrix}$$

The starting point of the search is obtained by evaluating the average of the three points selected. This yields

$$\underline{\bar{x}}_0 = \begin{bmatrix} \bar{x}_{01} \\ \bar{x}_{02} \end{bmatrix} = \begin{bmatrix} \frac{1 + 1.5 + 0.5}{3} \\ \frac{2 + 1 + 2}{3} \end{bmatrix} = \begin{bmatrix} 1.0000 \\ 1.6667 \end{bmatrix}$$

The corresponding information matrix, M_0, of the design matrix, X_0, is

$$M_0 = X_0^1 X_0 = \begin{bmatrix} 1 & 1.5 & 0.5 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1.5 & 1 \\ 0.5 & 2 \end{bmatrix} = \begin{bmatrix} 3.5 & 4.5 \\ 4.5 & 9 \end{bmatrix}$$

whose determinant value is det $M_0 = 11.25$

The variance covariance matrix, M_0^{-1} is

$$\begin{Bmatrix} 0.8 & -0.4 \\ -0.4 & 0.3111 \end{Bmatrix}$$

The vector of coefficient of the objective function, g, is

$$g = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

With $g = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $M_0 = \begin{pmatrix} 3.5 & 4.5 \\ 4.5 & 9 \end{pmatrix}$

we compute Z_0 *as*

$$Z_0 = M_0 \underline{g} = \begin{pmatrix} 3.5 & 4.5 \\ 4.5 & 9 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{Bmatrix} 19.5 \\ 31.5 \end{Bmatrix}$$

The direction of search is

$$d_0 = M_0^{-1} Z_0 = \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 0.3111 \end{bmatrix} \begin{bmatrix} 19.5 \\ 31.5 \end{bmatrix} = \begin{bmatrix} 3.0000 \\ 1.9997 \end{bmatrix}$$

The normalized direction of search

$$\underline{d}_0^* = \frac{1}{\sqrt{3.0000^2 + 1.9997^2}} \begin{bmatrix} 3.0000 \\ 1.9997 \end{bmatrix} = \begin{bmatrix} 0.8321 \\ 0.5547 \end{bmatrix}$$

The step-lengths are computed as below;
For the first constraint,

$$\rho_{01} = \frac{(4 \;\; 3) \begin{pmatrix} 1.0000 \\ 1.6667 \end{pmatrix} - 12}{(4 \;\; 3) \begin{pmatrix} 0.8321 \\ 0.5547 \end{pmatrix}} = -0.6009$$

For the second constraint,

$$\rho_{02} = \frac{(4 \;\; 1) \begin{pmatrix} 1.0000 \\ 1.6667 \end{pmatrix} - 8}{(4 \;\; 1) \begin{pmatrix} 0.8321 \\ 0.5547 \end{pmatrix}} = -0.6009$$

For the third constraint,

$$\rho_{03} = \frac{(4 \;\; -1) \begin{pmatrix} 1.0000 \\ 1.6667 \end{pmatrix} - 8}{(4 \;\; -1) \begin{pmatrix} 0.8321 \\ 0.5547 \end{pmatrix}} = -2.0430$$

In order to avoid making a move that takes us away from the feasible region, we consider using the shortest step length as measured by its absolute value. Hence, the optimal step length is $\rho_0^* = |-0.6009| = 0.6009$.

With $\underline{x}_0^*, \rho_0^*, \underline{d}_0^*$, a move is made to

$$\underline{x}_1^* = \bar{\underline{x}}^*_0 + \rho_0^* \underline{d}_0^*$$

$$= \begin{bmatrix}1.0000\\1.6667\end{bmatrix} + [0.6009] \begin{bmatrix}-0.8321\\+0.5547\end{bmatrix} = \begin{bmatrix}1.0000\\1.6667\end{bmatrix} + \begin{bmatrix}0.5000\\0.3333\end{bmatrix} = \begin{bmatrix}1.5000\\2.0000\end{bmatrix}$$

$$\underline{x}_1^* = \begin{bmatrix}1.5000\\2.0000\end{bmatrix}$$

With $\underline{x}_1^* = \begin{bmatrix}1.5\\2.0\end{bmatrix}$, the value of the objective function is $f(\underline{x}_1^*) = 8.5$

Also, we notice that $\underline{x}_1^* = \begin{bmatrix}1.5\\2.0\end{bmatrix}$ satisfies the three linear constraints. Before checking for optimality we need to make a second move. In other to do that the point $\underline{x}_1^*$ is added to the initial design measure and hence yields a new design measure

$$\xi_4^{(1)} = \begin{bmatrix}1 & 2\\1.5 & 1\\0.5 & 2\\1.5 & 2\end{bmatrix}$$

The corresponding design matrix is

$$X_1 = \begin{bmatrix}1 & 2\\1.5 & 1\\0.5 & 2\\1.5 & 2\end{bmatrix}$$

The coordinate of the average of the four points are

$$\bar{\underline{x}}_1 = \{1.1250 \quad 1.7500\}$$

The corresponding information matrix is

$$X_1^1 X_1 = \mathrm{M}_1 = \begin{Bmatrix}5.75 & 7.5\\7.5 & 13\end{Bmatrix}$$

and the associated determinant is

$$\det \mathrm{M}_1 = 18.5$$

The variance covariance matrix is

$$M_1^{-1} = \begin{Bmatrix}0.7027 & -0.4054\\-0.4054 & 0.3108\end{Bmatrix}$$

With $g = \begin{pmatrix}3\\2\end{pmatrix}$ and $\mathrm{M}_1 = \begin{pmatrix}5.75 & 7.5\\7.5 & 13\end{pmatrix}$

we compute Z as

$$Z_1 = M_1 g = \begin{pmatrix}5.75 & 7.5\\7.5 & 13\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix} = \begin{Bmatrix}32.25\\48.50\end{Bmatrix}$$

The direction of search is

$$\underline{d}_1 = M_1^{-1} Z_1 = \begin{Bmatrix}3.0002\\1.9997\end{Bmatrix}$$

The normalized direction of search is

$$\underline{d}_1^* = \begin{Bmatrix}0.8321\\0.5546\end{Bmatrix}$$

We compute the step-length as follows:

For the first constraint

$$\rho_{11} = \frac{(4 \quad 3)\begin{pmatrix}1.1250\\1.7500\end{pmatrix} - 12}{(4 \quad 3)\begin{pmatrix}0.8321\\0.5546\end{pmatrix}} = -0.4507$$

For the second constraint

$$\rho_{12} = \frac{(4 \quad 1)\begin{pmatrix}1.1250\\1.7500\end{pmatrix} - 8}{(4 \quad 1)\begin{pmatrix}0.8321\\0.5546\end{pmatrix}} = -0.4507$$

For the third constraint

$$\rho_{13} = \frac{(4 \quad -1)\begin{pmatrix}1.1250\\1.7500\end{pmatrix} - 8}{(4 \quad -1)\begin{pmatrix}0.8321\\0.5546\end{pmatrix}} = -1.8507$$

The optimal step-length is

$$|\rho_{11}| = |\rho_{12}| = 0.4507 = \rho_1^*$$

With $\bar{x}_1^*, \rho_1^*, \underline{d}_1^*$, a second move is made

$$\underline{x}_2^* = \bar{x}_1^* + \rho_1^* \underline{d}_1^*$$

$$= \begin{bmatrix}1.1250\\1.7500\end{bmatrix} + [0.4507] \begin{bmatrix}+0.8321\\+0.5547\end{bmatrix}$$

$$\underline{x}_2^* = \begin{bmatrix}1.5000\\2.0000\end{bmatrix}$$

With $\underline{x}_2^* = \begin{bmatrix}1.5\\2.0\end{bmatrix}$, the value of the objective function is $f(\underline{x}_2^*) = 8.5$

Checking for optimality (by considering the norm of the vector $\underline{x}_2^* - \underline{x}_1^*$). we have

$$\|\underline{x}_2^* - \underline{x}_1^*\| = \left\| \begin{pmatrix}1.5000\\2.0000\end{pmatrix} - \begin{pmatrix}1.5000\\2.0000\end{pmatrix} \right\| = \left\| \begin{matrix}0.0000\\0.0000\end{matrix} \right\|$$

$$= \sqrt{(0.0000)^2 + (0.0000)^2}$$

$$= 0.0000$$

This value satisfies the stopping rule. We notice that the value of the objective function at the first iteration is the same as the value of the objective function at the second iteration.
Hence, the global maximum of the objective function, f($\underline{x}$), is

$$\bar{x}_1^* = \begin{pmatrix}1.5000\\2.0000\end{pmatrix}$$

3.2 Illustration 2

The problem considered here is given by

minimize f($\underline{x}$) = $3x_1 + 2x_2$ (3.2)

subject to

$2x_1 + x_2 \geq 6$

$x_1 + x_2 \geq 4$

$x_1 + 2x_2 \geq 6$
$x_1, x_2 \geq 0.$
We shall solve this problem using our Quick Convergent Inflow Algorithm (QCIA). We begin by selecting points to go into the design measure. With the points (1, 4) and (4, 1) we form the initial design measure as

$$\xi_2^{(0)} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

Notice that each of the support points that make up the initial design measure satisfies the three constraints of the objective function. The design matrix associated with the initial design measure ($\xi_2^{(0)}$) is $X_0 = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$

The starting point of the search is obtained by evaluating the average of the two points selected. This yields $\bar{x}_0 = [2.5 \quad 2.5]$.
The corresponding information matrix is

$$M_0 = X_0^1 X_0 = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

whose determinant value is $\det M_0 = 225$
The variance covariance matrix, M_0^{-1} is

$$M_0^{-1} = \begin{Bmatrix} 0.075556 & -0.035556 \\ -0.035556 & 0.075556 \end{Bmatrix}$$

The vector of coefficient of the objective function, g, is

$$g = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

With $g = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $M_0 = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}$
we compute Z_0 *as*

$$Z_0 = M_0 g = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{Bmatrix} 67 \\ 58 \end{Bmatrix}$$

The direction of search is

$$d_0 = M_0^1 Z_0 = \begin{bmatrix} 0.075556 & -0.035556 \\ -0.035556 & 0.075556 \end{bmatrix}\begin{bmatrix} 67 \\ 58 \end{bmatrix} = \begin{bmatrix} 3.000004 \\ 1.999996 \end{bmatrix}$$

The normalized direction of search

$$\underline{d}_0^* = \frac{1}{\sqrt{3.000004^2 + 1.999996^2}} \begin{bmatrix} 3.000004 \\ 1.999996 \end{bmatrix} = \begin{bmatrix} 0.832051 \\ 0.554699 \end{bmatrix}$$

The step-lengths are computed as below;
For the first constraint,

$$\rho_{01} = \frac{(2 \quad 1)\begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix} - 6}{(2 \quad 1)\begin{bmatrix} 0.832051 \\ 0.554699 \end{bmatrix}} = \frac{1.5}{2.218801} = 0.067041$$

For the second constraint.

$$\rho_{02} = \frac{\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}2.5\\2.5\end{pmatrix}-4}{\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0.832051\\0.554699\end{pmatrix}} = \frac{1.0}{1.3867} = 0.7211$$

For the third constraint,

$$\rho_{03} = \frac{\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}2.5\\2.5\end{pmatrix}-6}{\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}0.832051\\0.554699\end{pmatrix}} = \frac{1.5}{1.9413} = 0.7727$$

Hence, the optimal step length is $\rho_0^* = |0.676041| = 0.676041$

With $\underline{\bar{x}}_0^*, \rho_0^*, \underline{d}_0^*$, a move is made

$$\underline{x}_1^* = \underline{\bar{x}}^*{}_0 - \rho_0^* \underline{d}_0^*$$

$$= \begin{bmatrix}2.5\\2.5\end{bmatrix} - [0.676041]\begin{bmatrix}0.832051\\0.554699\end{bmatrix} = \begin{bmatrix}1.937499\\2.125101\end{bmatrix} \approx \begin{bmatrix}1.9375\\2.1250\end{bmatrix}$$

$$\underline{x}_1^* = \begin{bmatrix}1.9375\\2.1250\end{bmatrix}$$

With $\underline{x}_1^* = \begin{bmatrix}1.9375\\2.1250\end{bmatrix}$, the value of the objective function is $f(\underline{x}_1^*) = 10.0625$

Also, we notice that $\underline{x}_1^* = \begin{bmatrix}1.9375\\2.1250\end{bmatrix}$ satisfies the three linear constraints.

Before checking for optimality we need to make a second move. In other to do that the point $\underline{x}_1^*$ is added to the initial design measure and hence yields a new design measure.

$$\xi_3^{(1)} = \begin{bmatrix}1 & 4\\4 & 1\\1.9375 & 2.1250\end{bmatrix}$$

The corresponding design matrix is

$$X_1 = \begin{bmatrix}1 & 4\\4 & 1\\1.9375 & 2.1250\end{bmatrix}$$

The coordinates of the average of the three points are

$$\underline{\bar{x}}_1 = \{2.3125,\ 2.3750\}$$

The corresponding information matrix is

$$M_1 = X_1^1 X_1 = \begin{Bmatrix}20.753906 & 12.117188\\12.117188 & 21.515625\end{Bmatrix}$$

and the associated determinant is

$$\det M_1 = 299.707014$$

The variance covariance matrix is

$$M_1^{-1} = \begin{Bmatrix}0.071789 & -0.040430\\-0.040430 & 0.069247\end{Bmatrix}$$

With $g = \begin{pmatrix}3\\2\end{pmatrix}$ and $M_1 = \begin{Bmatrix}20.753906 & 12.117188\\12.117188 & 21.515625\end{Bmatrix}$

we compute Z as

$$Z_1 = M_1 g = \begin{Bmatrix} 20.753906 & 12.117188 \\ 12.117188 & 21.515625 \end{Bmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{Bmatrix} 86.496094 \\ 79.382814 \end{Bmatrix}$$

The direction of search is

$$\underline{d}_1 = M_1^{-1} Z_1 = \begin{Bmatrix} 3.000021 \\ 1.999985 \end{Bmatrix}$$

The normalized direction of search is

$$\underline{d}_1^* = \begin{Bmatrix} 0.832054 \\ 0.554695 \end{Bmatrix}$$

We compute the step-length as follows:
For the first constraint,

$$\rho_{11} = \frac{(2 \quad 1) \begin{pmatrix} 2.3125 \\ 2.3750 \end{pmatrix} - 6}{(2 \quad 1) \begin{pmatrix} 0.832054 \\ 0.554695 \end{pmatrix}} = \frac{1}{2.218803} = 0.450693$$

For the second constraint,

$$\rho_{12} = \frac{(1 \quad 1) \begin{pmatrix} 2.3125 \\ 2.3750 \end{pmatrix} - 4}{(1 \quad 1) \begin{pmatrix} 0.832054 \\ 0.554695 \end{pmatrix}} = \frac{0.6875}{1.386749} = 0.495764$$

For the third constraint,

$$\rho_{13} = \frac{(1 \quad 2) \begin{pmatrix} 2.3125 \\ 2.3750 \end{pmatrix} - 6}{(1 \quad 2) \begin{pmatrix} 0.832054 \\ 0.554695 \end{pmatrix}} = \frac{1.0625}{1.941444} = 0.547273$$

The optimal step-length is

$$|\rho_{11}| = 0.450693 = \rho_1^*$$

With $\bar{\underline{x}}_1^*$, ρ_1^*, $\underline{d}_1^*$, a second move is made to

$$\underline{\mathbf{x}}_2^* = \bar{\underline{\mathbf{x}}}_1^* - \rho_1^* \underline{\mathbf{d}}_1^*$$

$$= \begin{bmatrix} 2.3125 \\ 2.3750 \end{bmatrix} - [0.450693] \begin{bmatrix} 0.832054 \\ 0.554695 \end{bmatrix} = \begin{bmatrix} 1.937499 \\ 2.125003 \end{bmatrix} \approx \begin{bmatrix} 1.9375 \\ 2.1250 \end{bmatrix}$$

$$\underline{x}_2^* = \begin{bmatrix} 1.9375 \\ 2.1250 \end{bmatrix}$$

With $\underline{x}_2^* = \begin{bmatrix} 1.9375 \\ 2.1250 \end{bmatrix}$, the value of the objective function is $f(\underline{x}_2^*) = 10.0625$

Checking for optimality (by considering the norm of the vector $\underline{x}_2^* - \underline{x}_1^*$).
we have

$$\left\| \underline{\mathbf{x}}_2^* - \underline{\mathbf{x}}_1^* \right\| = \left\| \begin{bmatrix} 1.9375 \\ 2.1250 \end{bmatrix} - \begin{bmatrix} 1.9375 \\ 2.1250 \end{bmatrix} \right\| = \left\| \begin{matrix} 0.0000 \\ 0.0000 \end{matrix} \right\|$$

$$= \sqrt{(0.0000)^2 + (0.0000)^2}$$

$$= 0.0000$$

This value satisfies the stopping rule. We notice that the value of the objective function at the first iteration is the same as the value of the objective function at the second iteration.
Hence, the global minimum of the objective function, f($\underline{x}$), is

$\underline{\bar{x}}_1^* = \begin{bmatrix} 1.9375 \\ 2.1250 \end{bmatrix}$.

4 Results

We present below the summary of search for the optimization problems considered in illustrations 1 and 2 of section 3. The summary provides information on the performance of the algorithm as measured by the number of iterations required to reach the optimum, the determinant value of the information matrix, the optimizer, the value of the objective function and the norm of the vectors of optimizers. From the summary statistics in Table 1, we see that the value of the determinant increases with addition of an optimal point to the initial design measure. The table also shows that the objective function is maximized at the first move.

Table 1: Summary Statistics for the maximization problems in illustration 1

Iteration	**Det information matrix**	**optimizer**	**Value of objective function**	**Norm**
1	11.2500	$\begin{pmatrix} 1.5 \\ 2.0 \end{pmatrix}$	8.5000	0
2	18.5000	$\begin{pmatrix} 1.5 \\ 2.0 \end{pmatrix}$	8.5000	0

Table 2: Summary Statistics for the minimization problem in illustration 2.

Iteration	**Det information matrix**	**optimizer**	**Value of objectivefunction**	**Norm**
1	225	$\begin{pmatrix} 1.9375 \\ 2.1250 \end{pmatrix}$	10.0625	0
2	299.707014	$\begin{pmatrix} 1.9375 \\ 2.1250 \end{pmatrix}$	10.0625	0

Also from the summary statistics in Table 2, we see that the value of the determinant increases with addition of an optimal point to the initial design measure. The table also shows that the objective function is minimized at the first move.We observe that the algorithm attempts to improve an initial design as measured by the determinant value of information matrix. Comparative study has been made with existing algorithms such as, the Simplex Method, Active Set, LEA, QEA and MNEA. The Quick Converging Inflow Algorithm (QCIA) has performed credibly well. We present in table 3 the summary of the comparative study using the minimization problem in illustration 2.

Table 3: Solutions using Existing Algorithms.

Technique	**Number of Iterations**	**Value of the Minimizer**	**Value of the Objective Function**
LEA	4	(1.87, 2.27)	10.15
QEA	4	(1.88, 2.24)	10.12
MNEA	4	(2.07, 1.97)	10.15
Active Set	2	(2.00, 2.00)	10.00
Simplex	2	(2.00, 2.00)	10.00

5 Conclusion

A Quick Convergent Inflow Algorithm (QCIA) has been presented for solving linear programming problems. The working of the algorithm has been presented for maximization problems as well as minimization problems. An important feature of the QCIA is that the starting

point, direction of search and the step length are optimally chosen. Specifically, the search moves in the direction of minimum variance and converges absolutely to the required optimum.

References

[1] Box, M. J. (1966). A Comparison of several Current Optimization methods, and the use of Transformations in Constrained Problems. *The Computer Journal*, **9**(1), 67-77.

[2] Etukudo, I. A., and Umoren, M. U. (2003). A Quadratic Programming Model for Crop Combination in Intercropping. *Global Journal of Mathematical Sciences*, **2**(1) ,73 – 79.

[3] Etukudo, I. A. and Umoren. M. U. (2008). A Modified Super Convergent Line Series algorithm for Solving Linear Programming Problems. *Journal of Mathematical Sciences*, International Centre for Advance Studies, West Bengal; **19** (1), 73 – 88.

[4] Etukudo, I. A., and Umoren, M. U. (2009). Invariant Property of the Direction Vector of Linearly Constrained Optimization Problems with Linear Objective Function. *Journal of Mathematical Sciences*. International Centre for Advance Studies, West Bengal, **20**(1), 67 – 74.

[5] Etukudo, I. A., and Umoren, M. U. (2009). A Comparison of Modified Super Convergent Line Series Algorithm and Modified Simplex Method for Solving Quadratic Programming Problems. *ICASTOR Journal of Mathematical Sciences*. **3** (1), 41 – 61.

[6] Etukudo, I. A., Umoren, M. U. , and Enang, E. I. (2009). A Comparison of Modified Super Convergent Line Series Algorithm and Karmarkar's Interior Point Method for Solving Linear Programming Problems. *International Journal of Physical Sciences*, **1** (1), 88 – 103.

[7] Montgomery, D.C. (2009). *Design and Analysis of Experiments*. 7th edition, John Wiley.

[8] Onukogu, I. B. (1997). *Foundation of Optimal Exploration of Response Surfaces*. Ephrata Press, Nsukka.

[9] Onukogu, I. B. and Chigbu, P. E. (2002). *Super Convergent Line Series (in optimal Design of Experiment and Mathematical Programming)*, AP Express, Publishing, Nsukka.

[10] Pazman, A. (1986) *Foundation of optimum Experimental Design*, D. Riedel Publishing Company.

[11] Storey, C. (1985). Workshop on Mathematics in Industry, International Centre for Theoretical Physics.

[12] Umoren, M. U. (1998) Construction of Exact D-Optimum Designs for a Constrained Optimization Problem. Nigerian *Journal of Science and Science Education.* **5**, 76 – 86.

[13] Umoren, M. U. (1999). A Maximum Norm Exchange Algorithm for Solving Linear Programming Problems. *Journ. Nig. Stat. Assoc.* **13**, 39 – 56.

[14] Umoren, M. U. (2000). Application of Optimal Design Theory to the Solution of a Constrained Optimization Problem. *Journal of Natural and Applied Sciences* **1**, 78 – 89.

[15] Umoren, M. U. (2001). Some Optimality Conditions for the Existence of Optimizers of Certain Class of Linear Programming Problems. Global *Journal of Pure and Applied Sciences*, 7(2), 389 -396.

[16] Umoren, M. U. (2002). A Quadratic Exchange Algorithm for Solving Linear Programming Problems. *Journal of the Nigerian Statistical Association.* 15, 44 – 57.

[17] Umoren, M. U. (2003). A Necessary and Sufficient Condition for the Existence of Optimizers of Linear Programming Problems. *Global of Journal of Mathematical Sciences*, **2**(1), 9 – 16.

[18] Umoren, M. U. (2009). Application of Optimal Experimental Design Theory to the Solution of Certain Class of Quadratic Programming Problems. *International Journal of Numerical Mathematics*, **4**(1), 44-58.

[19] Umoren, M. U., and Etukudo, I. A. (2009). A Modified Super Convergent Line Series Algorithm for Solving Quadratic Programming Problems. *Journal of Mathematical Sciences*, International Centre for Advance Studies, West Bengal, **20**(1), 55 – 66.

[20] Umoren, M. U. and Etukudo, I. A. (2010). A Modified Super Convergent Line Series Algorithm for Solving Unconstrained Optimization Problems. *Journal of Modern Mathematics and Statistics* **4**(4), 115-122.