Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/070423

A Characterization of the Class of Statistically Pre-Cauchy Double Sequences of Fuzzy Numbers

Hemen Dutta[*]

Department of Mathematics, Gauhati University, Guwahati-781014, Assam, India

Received: 14 Dec. 2012, Revised: 27 Jan. 2013, Accepted: 13 Feb. 2013
Published online: 1 Jul. 2013

Abstract: In this article, we introduce the notion of statistically pre-Cauchy double sequence of fuzzy numbers. Then we present a characterization of the class of bounded statistically pre-Cauchy double sequences of fuzzy numbers with the help of Orlicz function. Further, we present a characterization of the class of bounded statistically convergent double sequences of fuzzy numbers and linked with Cesàro summability.

Keywords: statistical convergence, double sequence of fuzzy number, statistically pre-Cauchy double sequence, Orlicz function, Cesàro summability

1. Introduction

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability, and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [28] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [20] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [11], Dutta [8,9] and many others.

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $u : R \to [0,1]$which satisfies the following four conditions:

(i) u is normal, i.e., there exist an $x_0 \in R$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex,
i.e. $u[\lambda x+(1-\lambda)y] \geq \min\{u(x),u(y)\}$ for all $x,y \in R$ and for all $\lambda \in [0,1]$.

(iii) u is upper semi-continuous.

(iv) The set $[u]_0 = \overline{\{x \in R : u(x) > 0\}}$ is compact, where $\overline{\{x \in R : u(x) > 0\}}$ denotes the closure of the set $\{x \in R : u(x) > 0\}$ in the usual topology of R.

We denote the set of all fuzzy numbers on R by E^1 and called it as the space of fuzzy numbers. λ-level set $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda = \begin{cases} \{t \in R : u(t) \geq \lambda\}, & (0 < \lambda \leq 1), \\ \overline{\{t \in R : u(t) > \lambda\}}, & (\lambda = 0). \end{cases}$$

The set $[u]_\lambda$ is a closed, bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda$=$[u^-(\lambda),u^+(\lambda)]$. R can be embedded in E^1, since each $r \in R$ can be regarded as a fuzzy number

$$\bar{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \neq r. \end{cases}$$

Let W be the set of all closed bounded intervals A of real numbers such that $A = [A_1,A_2]$. Define the relation d on W as follows:

$$d(A,B) = \max\{|A_1 - B_1|, |A_2 - B_2|\}.$$

It is well known that (W,d) is a complete metric space (See Diamond and Kloeden [10]).

[*] Corresponding author e-mail: hemen_dutta08@rediffmail.com

A fuzzy double sequence is a double infinite array of fuzzy real numbers. We denote a fuzzy double sequence by (a_{mn}), where a_{mn} are fuzzy real numbers for each $m,n \in N$. The initial works on double sequences of real or complex terms is found in Bromwich [2]. Hardy [14] introduced the notion of regular convergence for double sequences of real or complex terms. The works on double sequence was further investigated by Basarir and Solancan [3], Moricz [22], Tripathy [27] and many others.

Let $\left(a_{k_1k_2}\right) \in {}_2w^F$, set of all double sequences of fuzzy numbers. Then the expression $\sum_{k_1}\sum_{k_1} a_{k_1k_2}$ is called a series corresponding to the double sequence $\left(a_{k_1k_2}\right)$of fuzzy number. Denote $S_{mn} = \sum_{k_1=1}^{m}\sum_{k_2=1}^{n} a_{k_1k_2}$ for all$m,n \in N$. If the sequence (S_{mn})converges to a fuzzy number u, then we say that the series $\sum_{k_1}\sum_{k_1} a_{k_1k_2}$ converges to u and write $\sum_{k_1}\sum_{k_1} a_{k_1k_2} = u$.

Statistically convergence is a generalization of the usual notation of convergence that parallels the usual theory of convergence. The concept of statistical convergence was first introduced by Fast [12] and also independently by Buck [4] and Schoenberg [26] for real and complex sequences. Further this concept was studied by Salat [25], Fridy [13], Cannor [5,6] and many others. Connor, Fridy and Kline [7] studied the notion of statistically pre-Cauchy sequence of scalars in comparison with statistically convergent sequence.

A double sequence of fuzzy number (x_{kl}) is called statistically convergent to L if

$$\lim_{m,n\to\infty}\frac{1}{mn}\left|\{(j,k): d(x_{jk},L)\geq\varepsilon, j\leq m, k\leq n\}\right| = 0.$$

where the vertical bars indicate the number of elements in the set.

A double sequence of fuzzy number(x_{jk}) is called statistically pre-Cauchy if for every $\varepsilon > 0$ there exist $p = p(\varepsilon)$ and $q(\varepsilon)$ such that

$$\lim_{m,n\to\infty}\frac{1}{m^2n^2}\left|\{(j,k): d(x_{jk},x_{pq})\geq\varepsilon, j\leq m, k\leq n\}\right| = 0.$$

A double sequence of fuzzy number(x_{jk}) is said to be bounded if $\sup_{jk} d\left(x_{jk},\bar{0}\right) < \infty$ and we denote the class of such sequences by ${}_2\ell^F$.

An Orlicz Function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$and $M(x) \to \infty$ as $x \to \infty$.

Lindesstrauss and Tzafriri [18] used the idea of Orlicz sequence space and introduced the sequence space ℓ_M as follows:

$\ell_M = \{x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,$

for some $\rho > 0\}$.

For double sequence of fuzzy numbers, we can define Orlicz space as

$${}_2\ell_M^F = \{x = (x_{jk}) \in_2 w^F : \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} M\left(\frac{d\left(x_{kj},\bar{0}\right)}{\rho}\right) < \infty,$$

for some $\rho > 0\}$.

It is obvious that, if $x = \left(x_{jk}\right)$ is bounded, then for an Orlicz function M, $M\left(d\left(x_{jk},\bar{0}\right)\right)$ is bounded for each j and k.

The Cesàro means (also called Cesàro averages) of a sequence (a_n) are the terms of the sequence (c_n), where $c_n = \frac{1}{n}\sum_{i=1}^{n} a_i$is the arithmetic mean of the first n elements of (a_n). This concept is named after Ernesto Cesàro. It is known that, if (a_n) converges to l than (c_n) also converges to the same limit l(Hardy [15]). This means that the operation of taking Cesàro means preserves convergent sequences and their limits. This is the basis for taking Cesàro means as a summability method in the theory of divergent series. If the sequence of the Cesàro means is convergent, the series is said to be Cesàro summable. There are certainly many examples for which the sequence of Cesàro means converges, but the original sequence does not. For example, the sequence $(a_n) = (-1)^n$ which is Cesàro sum- mable to 0. Cesàro means are often used in applied mathematics and in particular to Fourier series (Katznelson [16]). For details about Cesàro summable spaces of scalar sequences, one may refer to Maddox [19]. For some recent works on statistical convergence as summability method and its applications, one may refer to Alotaibi and Mursaleen [1] and Mohiuddin and Aiyub [23]. The investigation of this paper will further encourage to study statistical convergence via summability techniques and applications in fuzzy setting.

Throughout the paper, ${}_2w^F$ denote the set of all double sequences of fuzzy numbers and $x = (x_{jk}) \in_2 w^F$be an element.

By ${}_2w^{FSC}$, we denote the space of strongly Cesàro sum- mable double sequences of fuzzy numbers, where

$${}_2w^{FSC} = \{x = \left(x_{jk}\right) \in_2 w^F : \lim_{m,n}\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} d(x_{jk},L) = 0,$$

$$\text{for some } L\}.$$

2. Main Results

In this section, we the help of Orlicz function we present a characterization of the class of bounded statistically pre-Cauchy double sequences of fuzzy numbers. Then we present a characterization of the class of bounded statistically convergent double sequences of fuzzy numbers. Further, we link Cesro summability with statistical convergence.

Proposition 2.1. Let $x \in_2 w^F \cap_2 \ell^F$ and U be an Orlicz function. Then the following statement holds:

"x is statistically pre-Cauchy if and only if $\lim_{m,n}\frac{1}{m^2n^2}\sum_{j,p\leq m}\sum_{k,q\leq n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right) = 0$ for some $\rho > 0$".

Proof. Suppose $\lim_{m,n}\frac{1}{m^2n^2}\sum_{j,p\leq m}\sum_{k,q\leq n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right) = 0$ for some $\rho > 0$.

For each $\varepsilon > 0$, $\rho > 0$ and $m,n \in N$, we have

$$\frac{1}{m^2n^2}\sum_{j,p\le m}\sum_{k,q\le n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right)$$

$$= \frac{1}{m^2n^2}\sum_{j,p\le m, d(x_{jk},x_{pq})<\varepsilon}\sum_{k,q\le n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right)$$

$$+\frac{1}{m^2n^2}\sum_{j,p\le m, d(x_{jk},x_{pq})\ge\varepsilon}\sum_{k,q\le n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right)$$

$$\ge \frac{1}{m^2n^2}\sum_{j,p\le m, d(x_{jk},x_{pq})\ge\varepsilon}\sum_{k,q\le n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right)$$

$$\ge U(\varepsilon)\left(\frac{1}{m^2n^2}|\{(j,k): d(x_{jk},x_{pq})\ge\varepsilon, j\le m, k\le n\}|\right)$$

$$\ge 0$$

Conversely suppose that x is statistically pre-Cauchy and that $\varepsilon > 0$ be such that $U(\delta) < \frac{\varepsilon}{2}$. Since x is bounded, there exist an integer B such that $U(x) < \frac{B}{2}$ for all $x \ge 0$. Note that, for each m, $n \in N$

$$\frac{1}{m^2n^2}\sum_{j,p\le m}\sum_{k,q\le n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right)$$

$$= \frac{1}{m^2n^2}\sum_{j,p\le m, d(x_{jk},x_{pq})<\delta}\sum_{k,q\le n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right)$$

$$+\frac{1}{m^2n^2}\sum_{j,p\le m, d(x_{jk},x_{pq})\ge\delta}\sum_{k,q\le n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right)$$

$$\le U(\delta)+\frac{1}{m^2n^2}\sum_{j,p\le m, d(x_{jk},x_{pq})\ge\delta}\sum_{k,q\le n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right)$$

$$\le \frac{\varepsilon}{2}+\frac{B}{2}\left(\frac{1}{m^2n^2}|\{(j,k): d(x_{jk},x_{pq})\ge\delta, j\le m, k\le n\}|\right)$$

$$\le \varepsilon + B\left(\frac{1}{m^2n^2}|\{(j,k): d(x_{jk},x_{pq})\ge\delta, j\le m, k\le n\}|\right) \quad (1)$$

Since x is statistically pre-Cauchy, there exists N such that R.H.S (1) is less than ε for all $m,n \in N$. Hence

$$\lim_{m,n}\frac{1}{m^2n^2}\sum_{j,p\le m}\sum_{k,q\le n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right) = 0.$$

Proposition 2.2. Let $x \in_2 w^F \cap_2 \ell^F$ and U be an Orlicz function. Then x is statistically convergent to L iff

$$\lim_{m,n}\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} U\left(\frac{d(x_{jk},L)}{\rho}\right) = 0.$$

Proof. The proof is similar to that of Proposition 2.1.

Lemma 2.1. Let $x \in_2 w^F \cap_2 \ell^F$ and U be an Orlicz function. Then we have

$$\lim_{m,n}\frac{1}{m^2n^2}\sum_{j=1}^{m}\sum_{k=1}^{n} d(x_{jk},x_{pq}) = 0$$

if and only if

$$\lim_{m,n}\frac{1}{m^2n^2}\sum_{j=1}^{m}\sum_{k=1}^{n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right) = 0.$$

Proof. Let $A = \sup_{j,k} d\left(x_{jk},\overline{0}\right)$ and define

$$U(x) = (1+2A)\frac{x}{1+x}$$

Then

$$U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right) \le (1+2A)d(x_{jk},x_{pq})$$

and

$$U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right) = (1+2A)\frac{U(x_{jk},x_{pq})}{1+d(x_{jk},x_{pq})}$$

$$\ge \frac{(1+2A)d(x_{jk},x_{pq})}{1+d(x_{jk},0)+d(x_{pq},0)}$$

$$\ge \frac{(1+2A)d(x_{jk},x_{pq})}{1+2A}$$

$$= d(x_{jk},x_{pq})$$

Hence $\lim_{m,n}\frac{1}{m^2n^2}\sum_{j=1}^{m}\sum_{k=1}^{n} d(x_{jk},x_{pq}) = 0$ if and only if $\lim_{m,n}\frac{1}{m^2n^2}\sum_{j=1}^{m}\sum_{k=1}^{n} U\left(\frac{d(x_{jk},x_{pq})}{\rho}\right) = 0.$

Now, we are ready to give the main result of this paper as follows.

Theorem 2.3. Let $x \in_2 w^F \cap_2 \ell^F$. Then the following statement holds:

"x is statistically pre-Cauchy if and only if $\lim_{m,n}\frac{1}{m^2n^2}\sum_{j=1}^{m}\sum_{k=1}^{n} d(x_{jk},x_{pq}) = 0$".

Proof. Applying Lemma 2.1 and Proposition 2.1, we have the desired result.

Theorem 2.4. Let $x \in_2 w^F \cap_2 \ell^F$. Then the following statement holds:

"x is statistically convergent to L if and only if $\lim_{m,n}\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} d(x_{jk},L) = 0$".

Proof. If we consider $U(x) = (1+A+L)\frac{x}{1+x}$, then the proof is similar to that of Theorem 2.3.

In view of the above theorem and the notion of strongly Cesàro summable double sequences, we can now link the statistical convergence and Cesàro summability as follows:

Corollary 2.5. Let$x \in_2 w^F \cap_2 \ell^F$. Then the following statement holds:

"x is statistically convergent to L if and only if x is strongly Cesàro summable to L".

or

"The class of bounded and statistically convergent double sequences of fuzzy numbers is equivalent to the class of strongly Cesàro summable double sequences of fuzzy numbers".

3. Conclusion

In this paper, we first prove that x is bounded and statistically pre-Cauchy then $\lim_{m,n} \frac{1}{m^2n^2} \sum_{j=1}^{m} \sum_{k=1}^{n} d(x_{jk}, x_{pq}) = 0$ holds. Again if $\lim_{m,n} \frac{1}{m^2n^2} \sum_{j=1}^{m} \sum_{k=1}^{n} d(x_{jk}, x_{pq}) = 0$, then x is statistically pre-Cauchy. We know that statistically convergence is a generalization of the usual notation of convergence and Cesàro summability does not guarantee usual convergence (in fact the notion of Cesàro summability was introduced to assign limit to non-convergent sequences). In this context, the last result of this paper assures about the existence of a class of convergence which is equivalent with a class of Cesàro summability for double sequences of fuzzy numbers.

Acknowledgement

The author is grateful to the anonymous three referees for careful reading of the paper and for helpful comments that improved the presentation of the paper.

References

[1] A. Alotaibi and M. Mursaleen, *A-statistical summability of Fourier series and Walsh-Fourier series*, Appl. Math. Inf. Sci. **6(3)**, 535-538 (2012).

[2] T.J.I. Bromwich, An Introduction to the Theory of Infinite Series, Macmillan & Co., New York, 1965.

[3] M. Basarir, O. Solancan, *On some double sequence spaces*, J. Indian Acad. Math. **21**, 193-200 (1999.

[4] R.C. Buck, *Generalized asymptote density*, Amer. J. Math. **75**, 335-346 (1953).

[5] J.S. Connor, *The statistical and strong p-Cesáro convergence of sequences*, Analysis **8**, 47-63 (1988).

[6] J.S. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull. **32**, 194-198 (1989).

[7] J.S. Connor, J. Fridy and J. Kline, *Statistically pre-Cauchy sequences*, Analysis **14**, 311-317 (1994).

[8] H. Dutta, *On some complete metric spaces of strongly summable sequences of fuzzy numbers*, Rend. Semin. Mat. Univ. Politec. Torino **68**, 29-36 (2010).

[9] H. Dutta, *On some new type of summable and statistically convergence difference sequences of fuzzy numbers*, J. Fuzzy Math. **18**, 845-852 (2010).

[10] P. Diamond, P. Kloeden, *Metric spaces of fuzzy sets*, Fuzzy Sets and Systems **35**, 241-249 (1990).

[11] P. Diamond and P. Kloeden, Metric Spaces of Fuzzy Sets, Theory and Applications, World Scientific, Singapore, 1994.

[12] H. Fast, *Sur la convergence statistiue*, Colloq. Math. **2**, 241-244 (1951).

[13] J.A. Fridy, *On statistical convergence*, Analysis **5**, 301-313 (1985).

[14] G.H. Hardy, *On the convergence of certain multiple series*, Proc. Camb. Phil. Soc. **19**, 86-95 (1917).

[15] G. H. Hardy, Divergent Series, Providence: American Mathematical Society, 1992.

[16] Y. Katznelson, An Introduction to Harmonic Analysis, Dover Publications, New York, 1976.

[17] Vakeel A. Khan and Q.M. Danish Lohani, *Statistically pre-Cauchy sequences and Orlicz functions*, Southeast Asian Bull. Math. **31**, 1107-1112 (2007).

[18] J. Lindenstrauss and L. Tzafiri, *On Orlicz sequence spaces*, Israel J. Math. **10**, 379-390 (1971).

[19] I.J. Maddox, Elements of Functional Analysis, Cambridge University Press, 1970.

[20] M. Matloka, *Sequences of fuzzy numbers*, BUSEFAL **28**, 28-37 (1986).

[21] I.J. Maddox, *Sequence spaces defined by modulus*, Math. Proc. Camb. Soc. **100**, 161-166 (1986).

[22] F. Moricz, *Extension of spaces c and c_0 from single to double sequences*, Acta Math. Hung. **57**, 129-136 (1991).

[23] S.A. Mohiuddin and M. Aiyub, *Lacunary statistical convergence in random 2-normed spaces*, Appl. Math. Inf. Sci. **6(3)**, 581-585 (2012).

[24] Ö. Talo and F. Başar, *Determination of the duals of classical sets of sequences of fuzzy numbers and related matrix transformations*, Comput. Math. Appl. **58**, 717-733 (2009).

[25] T. Salat, *On statistically convergent sequences of real numbers*, Math. Soovaca **30**, 139-150 (1980).

[26] I.J. Schoenberg, *The integrability of certain functions and related sumability methods*, Amer. Math. Monthly **66**, 361-375 (1959).

[27] B.C. Tripathy, *Statistically convergent double sequences*, Tamkang J. Math. **34**, 231-237 (2003).

[28] L.A. Zadeh, *Fuzzy sets*, Information and Control **8**, 338-353 (1965).

Hemen Dutta is Assistant Professor of Mathematics at University of Gauhati of Assam. His research interests are: summability techniques, convergence and completeness related properties, duality theory, matrix transformation, fixed point theory and applications, n-norm and fuzzy sets. He has to his credit many research publications in reputed journals including SCI/SCI(Exp.) accredited journals. He is referee of several international journals in the frame of pure and applied mathematics and Editor of few journals covering the subject mathematics.