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Approximates Method for Solving an Elasticity Problem of Settled of the Elastic Ground with Variable Coefficients

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Abstract: In this paper, we have given numerical solutions of the elasticity problem of settled on the elastic ground with variable coefficient. Firstly, we calculate the generalized successive approximation of the given boundary value problem and we transform it into Padé series form, which give an arbitrary order for solving differential equation numerically. Secondly, we apply Homotopy Perturbation Method(HPM) to given boundary value problem. Then we compare HPM and the generalized successive approximation -Padé Approximates method by means of numerical solution of given boundary problem. Results reveal that HPM presents more effective and accurate solution for given boundary value problem.

Keywords: The generalized successive approximation method, Integral Equations, BVPs, Padé series, Homotopy Perturbation Method(HPM)

1 Introduction

A common method used for the solution of boundary value problem is the integral method[1,2]. With this method, we obtain an integral equation that is equivalent to the boundary value problem and the solution of the integral equation is defined as the solution of the boundary value problem. The equivalent integral equation is usually a Fredholm equation in the classical theory. In this study, we obtain a Fredholm-Volterra integral equation different from the classical theory and we compare Homotopy perturbation Method and The generalized successive approximation method. We applied these methods to an example which is the elasticity problem of unit length homogeny beam, which is a special form of boundary value problem.

The elasticity problem of settled of the elastic ground with variable coefficient has the form

$$\frac{d^4x}{dt^4}+a(t)x=f(t), \quad 0\le t\le T \tag{1}$$

$$\frac{d^2x(0)}{dt^2}=A_1, \quad \frac{d^3x(0)}{dt^3}=B_1 \tag{2}$$

$$x(T)=A_2, \quad \frac{dx(T)}{dt}=B_2 \tag{3}$$

where $a(t)$ and $f(t)$ are beforehand continuous functions on the interval $0\le t\le T$. We applied the successive approximations method to the problem and then convert it to Padé series [3,4].

2 An Equivalent Integral Equation

The linear equations

$$x(t)=f(t)+\int_0^T K(t,s)x(s)ds \tag{4}$$

$$x(t)=f(t)+\int_0^t K(t,s)x(s)ds \tag{5}$$

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$$x(t) = f(t) + \int_0^t K_1(t,s)x(s)ds + \int_0^T K_2(t,s)x(s)ds \quad (6)$$

are said to be Fredholm, Volterra and Volterra-Fredholm integral equations, respectively. In these equations, the function $f(t)$ is called free term of the equations, $K(t,s)$ and $K_i(t,s)(i=1,2)$ are kernels of the integral equations, and $x(t)$ is transmission or unknown function on the interval $0 \le t \le T$.
$C[0,T]$ is defined to be spaces of all sets of continuous functions on the closed interval $[0,T]$. Let $x(t) \in C[0,T]$, the norm of the $x(t)$ is defined to be a function $\| \, . \, \|$ with real value such that

$\| x \| = \max_{0 \le t \le T} | x(t) |$.

F_x and V_x are are defined as follows

$$F_x \equiv \int_0^T K(t,s)x(s)ds$$

and

$$V_x \equiv \int_0^t K(t,s)x(s)ds$$

on the $C[0,T]$, and these are known Fredholm and Volterra operator, respectively. If $F_x \in C[0,T]$ for $x(t) \in C[0,T]$, then it is said that operator F_x acts on $C[0,T]$.

If operator F_x acts from $C[0,T]$ to $\mathbf{R}$ then operator F_x is said to be a linear functional. Furthermore, if the function $K(t,s)$ can be written as

$$K(t,s) = \sum_{i=1}^{n} a_i(t)b_i(s) \quad (7)$$

Then $K(t,s)$ is called degenerated kernel. If kernel function of integral operator in the integral equation is degenerated, then this kind of integral equation is called integral equation with a degenerated kernel [5].

Suppose that Eq.(4) Fredholm equation has kernel Eq.(7). Therefore equation Eq.(4) can be written as

$$x(t) = f(t) + \sum_{i=0}^{n} a_i(t) \int_0^T b_i(s)x(s)ds \quad (8)$$

Now, we investigate the solution of the integral equations (8), such that

$$x(t) = f(t) + \sum_{i=0}^{n} a_i(t)C_i.$$

To find C_j, we can write following system

$$C_i = \int_0^T b_i(s)f(s)ds + \sum_{j=0}^{n} \int_0^T a_i(s)b_i(s)f(s)dsC_j, \quad (i=1,...,n).$$

If the determinant of the above system is different than zero, that is $\Delta \neq 0$, then we find out

$$C_i = \frac{1}{\Delta} \sum_{j=1}^{n} \Delta_{ij} \int_0^T b_j(s)f(s)ds$$

where Δ_{ij} is algebraic complementary of determinant Δ. Δ_{ij} can be obtained by deleting ith row and jth of the determinant Δ. Therefore, if equation Eq.(4) has degenerated kernel, then the solution of the Eq.(4) is

$$x(t) = f(t) + \sum_{i,j=1}^{n} a_i(t) \frac{\Delta_{ij}}{\Delta} \int_0^T b_j(s)f(s)ds$$

or

$$x(t) = f(t) + \int_0^T \left[\sum_{i,j=1}^{n} a_i(t)b_j(s) \frac{\Delta_{ij}}{\Delta} f(s)ds \right].$$

3 Green Function and Solutions of Boundary Value Problems

Let us consider boundary values problem

$$\begin{aligned} x''(t) + b(t)x' + a(t)x &= f(t), \\ \alpha_0 x(0) + \beta_0 x'(0) &= \gamma_0, \\ \alpha_1 x(0) + \beta_1 x'(0) &= \gamma_1 \end{aligned} \quad (9)$$

where $a(t), b(t)$ and $f(t)$ $(0 \le t \le T)$ are beforehand functions, α_i, β_i and γ_i $(i = 0,1)$ are constants [6]. Appropriate homogeneous boundary value problem can be written for problem Eqs.(9) as follows

$$x''(t) + b(t)x' + a(t)x = 0, \quad (10)$$

$$\alpha_0 x(0) + \beta_0 x'(0) = 0, \quad (11)$$

$$\alpha_1 x(0) + \beta_1 x'(0) = 0. \quad (12)$$

3.1.Definition: A function $G(t,s)$ has following properties for its known value $s \in (0,T)$.

i. If $t \neq s$, then $G(t,s)$ is solution of the given problem with Eq.(10).

ii. If $t=s$, then $G(t,s)$ is continuous function with respect to t. Partial derivative of the $G(t,s)$ with respect to t has first kind of discontinuity and its jumping number 1.

That is,

$$\begin{aligned} G(s+0,s) &= G(s-0,s), \\ G'_t(s+0,s) - G'_t(s-0,s) &= 1 \end{aligned} \tag{13}$$

To establish Green function, let $x_1(t)$ and $x_2(t)$ be two linear independent solution of the Eq.(10). Furthermore, let solutions $x_1(t)$ and $x_2(t)$ satisfy boundary conditions (11) and (12), respectively. Now, let us consider following function

$$G(x,s) = \begin{cases} \varphi(s)x_1(t), & 0 \le t \le s, \\ \psi(s)x_2(t), & s < t \le T \end{cases} \tag{14}$$

Let us choose functions $\varphi(t)$ and $\psi(t)$ providing that condition (13). That is,

$$\psi(s)x_2(s) = \varphi(s)x_1(s), \quad \psi(s)x'_2(s) - \varphi(s)x'_1(s) = 1$$

By solving the above system we obtain functions $\varphi(s)$ and $\psi(s)$. If we substitute values of $\varphi(s)$ and $\psi(s)$ in Eq.(14), then function $G(x,s)$ is obtained which is Green function of Eqs.(10)-(12).

3.2.Theorem: If $G(x,s)$ are Green functions of problems Eqs.(10)-(12) and $f(t)$ is continious function, then function

$$x(t) = \int_0^T G(t,s) f(s) ds$$

is solution of non-homogeneous problem Eq.(9) [6].

4 An Equivalent Fredholm-Volterra Integral Equations

Suppose that $F(t) = f(t) - a(t)x$. If we consider the boundary conditions (2) and the following equation

$$\frac{d^4x}{dt^4} = F(t)$$

is integrated four time between 0 and t, then the following equations can be obtained

$$x'''(t) = x'''(0) + \int_0^t F(s)ds,$$

$$x''(t) = x''(0) + x'''(0)t + \int_0^t (t-s)F(s)ds,$$

$$x'(t) = x'(0) + x''(0)t + \frac{x'''(0)t^2}{2} + \int_0^t \frac{(t-s)^2}{2} F(s)ds,$$

$$x(t) = x(0) + x'(0)t + \frac{x''(0)t^2}{2} + \frac{x'''(0)t^3}{6} + \int_0^t \frac{(t-s)^3}{6} F(s)ds$$

where

$$\begin{aligned} x(t) &= x(0) + x'(0)t + \frac{A_1 t^2}{2} \\ &+ \frac{B_1 t^3}{6} + \int_0^t \frac{(t-s)^3}{6} F(s)ds \end{aligned} \tag{15}$$

Nevertheless, boundary conditions (2)-(3) and $x(t), x'(t)$ are used,

$$A_2 = x(0) + x'(0)T + \frac{A_1 T^2}{2} + \frac{B_1 T^3}{6} + \int_0^T \frac{(T-s)^3}{6} F(s)ds$$

$$B_2 = x'(0) + A_1 T + \frac{B_1 T^2}{2} + \int_0^T \frac{(T-s)^3}{2} F(s)ds$$

are obtained. After solving above system, we have

$$\begin{aligned} x(0) &= A_2 - TB_2 + \frac{A_1 T^2}{2} + \frac{B_1 T^3}{3} \\ &+ \int_0^T \frac{(T-s)^3}{6} (2T+s) F(s)ds \\ x'(0) &= B_2 - A_1(T) - \frac{B_1 T^2}{2} - \int_0^T \frac{(T-s)^2}{6} F(s)ds \end{aligned} \tag{16}$$

If we used Eq.(16) in Eq.(15), then we obtain

$$x(t) = A_2 - TB_2 + \frac{A_1 T^2}{2} + \frac{B_1 T^3}{3}$$

$$+ \int_0^T \frac{(T-s)^2(2T+s)}{6} F(s)ds + \left(B_2 - A_1 T - \frac{B_1 T^2}{2} \right)$$

$$- \int_0^T \frac{(T-s)^2(t)}{2} F(s)ds + \frac{A_1 T^2}{2} + \frac{B_1 T^3}{6} + \int_0^t \frac{(t-s)^3}{6} F(s)ds$$

or

$$\begin{aligned} x(t) &= A_2 - TB_2 + \frac{A_1 T^2}{2} + \frac{B_1 T^3}{3} \\ &+ \left(B_2 - A_1 T - \frac{B_1 T^2}{2} \right) t + \frac{A_1 T^2}{2} + \frac{B_1 T^3}{6} \end{aligned}$$

$$+\int_0^T \left[\frac{(T-s)^2(2T+s)}{6} - \frac{t(T-s)^2}{2}\right] F(s)ds$$
$$+\int_0^t \frac{(t-s)^3}{6} F(s)ds.$$

Therefore, if we take into consideration

$$F(t) = f(t) - a(t)x$$

then

$$x(t) = A_2 - TB_2 + \frac{A_1T^2}{2} + \frac{B_1T^3}{3}$$
$$+\left(B_2 - A_1T - \frac{B_1T^2}{2}\right)t + \frac{A_1T^2}{2} + \frac{B_1T^3}{6}$$
$$+\int_0^T \left[\frac{(T-s)^2(2T+s)}{6} - \frac{t(T-s)^2}{2}\right] f(s)ds$$
$$+\int_0^t \frac{(t-s)^3}{6} f(s)ds$$
$$-\int_0^T \left[\frac{(T-s)^2(2T+s)}{6} - \frac{t(T-s)^2}{2}\right] a(s)x(s)ds$$
$$-\int_0^t \frac{(t-s)^3}{6} a(s)x(s)ds$$

where if we choose

$$h(t) = A_2 - TB_2 + \frac{A_1T^2}{2}$$
$$+\frac{B_1T^3}{3} + \left(B_2 - A_1T - \frac{B_1T^2}{2}\right)t + \frac{A_1T^2}{2} + \frac{B_1T^3}{6}$$
$$+\int_0^T \left[\frac{(T-s)^2(2T+s)}{6} - \frac{t(T-s)^2}{2}\right] f(s)ds$$
$$+\int_0^t \frac{(t-s)^3}{6} f(s)ds$$

then we obtain

$$x(t) = h(t) - \int_0^t \frac{(t-s)^3}{6} a(s)x(s)ds$$
$$-\int_0^T \left[\frac{(T-s)^2(2T+s)}{6} - \frac{t(T-s)^2}{2}\right] a(s)x(s)ds \tag{17}$$

Eq.(17) is called that linear Volterra-Fredholm integral equation in which Fredholm operator has degenerated kernel. Let

$$Vx \equiv -\int_0^t \frac{(t-s)^3}{6} a(s)x(s)ds$$

$$F_1x \equiv -\int_0^T \left[\frac{(T-s)^2(2T+s)}{6}\right] a(s)x(s)ds$$

$$F_2x \equiv \int_0^T \left[\frac{(T-s)^2}{2}\right] a(s)x(s)ds.$$

In this case the Eq.(17) can be written as follows:

$$x(t) = h(t) + Vx + F_1x + tF_2x \tag{18}$$

because of F_1x, F_2x Fredholm and V_x Volterra operators. Thus, problem (1)-(3) is equivalent to the integral equation Eq.(18) [6].

5 The Generalized Successive Approximation Method

An approximation for the Volterra-Fredholm integral equation (18) can be obtained by the following formula

$$x_n(t) = h(t) + Vx_{n-1} + F_1x_{n-1} + tF_2x_{n-1} \tag{19}$$

where $(n = 0, 1, 2, \ldots), h(t) = x_0(t)$ is an arbitary and continuous function. To find the approximation $x_n(t)$ from those equations, we must solve the linear Volterra-Fredholm integral equation

$$y(t) = \tilde{h}(t) + F_1y + tF_2y \tag{20}$$

which has a degenerated kernel. The last equation Eq.(20) has a solution

$$y(t) = \tilde{h}(t) + C_1 + tC_2, \tag{21}$$

where the unknown terms C_1 and C_2 can be calculated by solving the following Linear equation system :

$$\begin{aligned} (1-F_1 1)C_1 - (F_1 t)C_2 &= F_1\tilde{h} \\ -(F_2 1)C_1 + (1-F_2 t)C_2 &= F_2\tilde{h} \end{aligned} \tag{22}$$

Suppose the determinant of the coefficient matrix of this system is different than zero, that is,

$$\Delta = (1-F_1 1)(1-F_2 t) - (F_1 t)(F_2 1)$$
$$= \left(1 + \frac{1}{6}\int_0^T (T-s)^2(2T+s)a(s)ds\right)$$

$$\times\left(1-\frac{1}{2}\int_0^T s(T-s)^2a(s)ds\right)$$
$$+\frac{1}{12}\left(\int_0^T s(T-s)^2(2T+s)a(s)ds\right)$$
$$\times\left(\int_0^T (T-s)^2a(s)ds\right)\neq 0.$$

Therefore we can compute C_1 and C_2 as

$$C_1=\frac{1}{\Delta}\left[(F_1\tilde{h})(1-F_2t)+(F_1t)(F_2\tilde{h})\right]$$

$$C_2=\frac{1}{\Delta}\left[(1-F_11)(F_2\tilde{h})+(F_1\tilde{h})(F_21)\right].$$

If we substitute C_1 and C_2 into the Eq.(21) we get the solution of the Eq.(20). That is,

$$y(t)=\tilde{h}+\frac{1}{\Delta}[1-F_2t+tF_21](F_1\tilde{h})$$
$$+\frac{1}{\Delta}[(1-F_11)t+F_1t](F_2\tilde{h}). \tag{23}$$

If we use Eq.(19) and the equality

$$\tilde{h}(t)=h(t)+Vx_{n-1}$$

we obtain the approximation $x_n(t)$ by the following formula:

$$x_n(t)=h^*(t)+\frac{1-F_2t+tF_21}{\Delta}F_1Vx_{n-1}$$
$$+\frac{(1-F_11)t+F_1t}{\Delta}F_2Vx_{n-1}+Vx_{n-1}, \tag{24}$$

where

$$h^*(t)=h(t)+\frac{1-F_2t+tF_21}{\Delta}F_1h$$
$$+\frac{(1-F_11)t+F1t}{\Delta}F_2h \tag{25}$$

To show that the approximations $x_n(t)$ approach to the solution of the problem (1)-(3) it is enough to check that the linear operator

$$A(x)=\frac{1-F_2t+tF_21}{\Delta}F_1Vx$$
$$+\frac{(1-F_11)t+F_1t}{\Delta}F_2Vx+Vx$$

satisfies the inequalities

$$\|A(x)\|\leq\beta\|x\|,$$

$$\beta=\left(\frac{|1-F_2t|+T|F_21|}{6|\Delta|}\int_0^T(T-s)^2(2T+s)|a(s)|ds\right.$$
$$\left.+\frac{T|1-F_11|+|F_1T|}{2|\Delta|}\int_0^T(T-s)^2|a(s)|ds+1\right)$$
$$\times\left(\frac{1}{6}\int_0^T(T-s)^3|a(s)|ds\right)<1.$$

Thus, the convergence velocity of the approximations to the problem satisfies the inequalities

$$\|x_n-x\|\leq\beta^n\|x_0-x\|$$

or

$$\|x_n-x\|\leq\frac{\beta^n}{1-\beta}\|x_1-x_0\|.$$

6 Padé Series

The Power series can be transformed into Padé series easily. Padé series is defined in the following:

$$a_0+a_1x+a_2x^2+\ldots=\frac{p_0+p_1x+\ldots+p_Mx^M}{1+q_1x+\ldots+q_Lx^L} \tag{26}$$

Multiply both sides of Eq.(26) by the denominator of right-hand side in Eq.(26) and compare the coefficients of both sides in Eq.(26). We have

$$a_l+\sum_{k=1}^{M}a_{l-k}q_k=p_1,\quad (l=0,\ldots,M) \tag{27}$$

$$a_l+\sum_{k=1}^{L}a_{l-k}q_k=0,\quad (l=M+1,\ldots,M+L). \tag{28}$$

Solving the linear equation in Eq.(28), we have q_k $(k=1,\ldots,L)$. And substituting q_k into Eq.(26), we have p_k, $(l=0,\ldots,M)$[1,2].

7 Homotopy Perturbation Method

Homotopy Perturbation method was introduced by Chineese mathematician J.H.He. HPM is very effective method and it can be applied various kinds of problems in the literature. One wants to learn more details about the HPM can see to the [7,9].

Instead of ordinary perturbation methods, this method doesn't need a small parameter in an equation. According to this method, a homotopy with an embedding parameter is constructed and the embedding parameter is considered as a "small parameter". Thus, this method is called the homotopy perturbation method.

To illustrate the homotopy perturbation method, consider the following nonlinear differential equation

$$A(u) = f(r), r \in \Omega \tag{29}$$

with boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, r \in \Gamma \tag{30}$$

Here A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. Generally speaking, the operator A can be divided into two parts L and N, where L is a linear and N is a nonlinear operator. Therefore, Eq.(29) can be rewritten as follows:

$$L(u) + N(u) = f(r). \tag{31}$$

By using homotopy tecnique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \to \mathbf{R}$ which satisfies

$$\begin{aligned} H(v,p) = (1-p)[L(v) - L(u_0)] + \\ p[A(v) - f(r)] = 0, p \in [0,1], r \in \Omega \end{aligned} \tag{32}$$

where u_0 is an initial approximation of Eq.(29) which satisfies the boundary conditions Eq.(30). Obviously, from Eq.(32) we have

$$\begin{aligned} H(v,0) = [L(v) - L(u_0)] = 0, \\ H(v,1) = A(v) - f(r) = 0. \end{aligned} \tag{33}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. We consider v as following:

$$v = v_0 + pv_1 + p^2 v_2 + \dots \quad . \tag{34}$$

According to the HPM, the best approximate solution of Eq.(31) can be explained as a series of powers of p

$$u = \lim_{p\to 1} = v_0 + v_1 + v_2 + \dots \quad . \tag{35}$$

The above convergence is given in [10].

8 An Example

Consider elasticity problem of homogeneous beam with unit length. Suppose that left end of beam is free and right end of it is fixed. Let loads of beam be smooth, i.e, $f(t) = t^2$ and elasticity coefficient$a(t) = 1$. Therefore, boundary value problem can be written as

$$\begin{aligned} \frac{d^4 x}{dt^4} + x = t^2, \\ x''(0) = 0, x'''(0) = 0, \\ x(1) = 0, x'(1) = 1 \end{aligned} \tag{36}$$

Now, we calculate approximate solution by using the The Generalized Successive Approximation- Padé Approximates Method Algoritm and Eq.(24). Therefore,

$$\begin{aligned} x_3(t) = 0.000002t^9 - 0.000022t^8 + 0.002777t^6 \\ -0.007242t^5 + 0.037593t^4 + 0.86914t - 0.902233 \end{aligned}$$

is an approximate solution of the problem Eq.(36) with $\Delta = 1.083680556 \neq 0$. We transform $x_3(t)$ into Padé series form as follows:

$$\begin{aligned} [7/6] = (-0.9022336418 + 0.8562379857t \\ +0.008166896781t^2 + 0.004239419663t^3 \\ +0.03726935617t^4 - 0.006513660254t^5 \\ +0.002849747602t^6)/(1 + 0.01430226793t \\ +0.004725831968t^2 - 0.0001463043916t^3 \\ +0.0002178515553t^4 - 0.000002415627288t^5) \end{aligned}$$

Let's apply Homotopy Perturbation method to the given boundary value problem. Construct the homotopy in like Eq.(32) and solve Eq.(36). From here, we obtained

$$v_0 = -1 + t$$

$$v_1 = \frac{1}{360}(38 - 51t + 15t^4 - 3t^5 + t^6);$$

$$\begin{aligned} v_2 = (1 - 15656 + 21535t - 7980t^4 + 2142t^5 \\ -45t^8 + 5t^9 - t^{10}) \times (1814400)^{-1}; \end{aligned}$$

$$\begin{aligned} v_3 = (30433332 - 41890863t + 15671656t^4 - 4311307t^5 \\ +114114t^8 - 17017t^9 + 91t^{12} - 7t^{13} \\ +t^{14}) \times (43589145600)^{-1}. \end{aligned}$$

Then approximation solution is obtained as follows :

$$\begin{aligned} x(t) = v(t) &= \lim_{p\to 1} v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots \\ &= v_0 + v_1 + v_2 + v_3 + \dots \quad . \end{aligned}$$

Table 1: $x(t_i)$ is numerical solution,$x(t_i)_{[7/6]}$ is the Padé series of $x(t_i)$, $x_{HPM}(t_i)$ is HPM solution for Eq.(36).

t_i	$x(t_i)$	$x(t_i)_{[7/6]}$	$x_{HPM}(t_i)$
0	-0.902427266	-0.9022336418	-0.902375006
0.1	-0.815492260	-0.8153157545	-0.81544719
0.2	-0.728506616	-0.7283472377	-0.728468698
0.3	-0.641344324	-0.6412021225	-0.641313447
0.4	-0.553802224	-0.5536772731	-0.553778129
0.5	-0.465603669	-0.4654961076	-0.465585932
0.6	-0.376400304	-0.3763103551	-0.376388315
0.7	-0.285771959	-0.2856998765	-0.285764836
0.8	-0.193224425	-0.1931705935	-0.19322109
0.9	-0.098185658	-0.0981505540	-0.098184782
1.0	0	0.00001580208	0

Table 2: Comparison of two methods(HPM) and Generalized Successive Approximation-Padé Approximates Method.

t_i	$\lvert x(t_i) - x(t_i)_{[7/6]} \rvert$	$\lvert x(t_i) - x_{HPM}(t_i) \rvert$
0	0.0001936154	0.0000522602
0.1	0.0001764969	0.0000450698
0.2	0.0001593673	0.0000379179
0.3	0.0001421993	0.0000308768
0.4	0.0001249510	0.0000240947
0.5	0.0001075655	0.0000177365
0.6	0.0000899721	0.0000119889
0.7	0.0000720919	0.0000007122
0.8	0.0000538339	0.0000003335
0.9	0.00003510447	0.0000000876
1.0	0.00001580208	0

Conclusion

The fundamental goal of this study has been to construct approximations to numerical solutions of the elasticity problem of settled of the elastic ground with variable coefficients. We show in Tables 1-2 the solutions of Eq.(36) by numerical methods. The numerical values on the Tables 1-2 are coincide with the exact solutions of Eq.(36) . As it is seem in Tables 1-2, Homotopy Perturbation Method is more accurate and effective than Generalized Successive Approximation-Padé Approximants Method.

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