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On N-Bipolar Soft Continuous Mappings with Application in OMICRON Disease

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Abstract: This paper aims to examine the limitations of an N-bipolar soft mapping and analyze how the N-bipolar soft separation axioms are affected by N-bipolar soft continuous, N-bipolar open, and N-bipolar closed mappings. Ultimately, we propose a mathematical system that utilizes N-bipolar soft mappings to diagnose symptoms of OMICRON disease.

Keywords: N-bipolar soft set, N-bipolar soft continuity, N-bipolar soft mapping, N-bipolar soft separation axiom, Decision making.

Nomenclature	
Abbreviations	
NBS-set	N-bipolar soft set
NBS-sets	N-bipolar soft sets
NBS-subsets	N-bipolar soft subsets
NBS-mapping	N-bipolar soft mapping
NBS-bijection	N-bipolar soft bijection
NBS-continuous mappings	N-bipolar soft continuous
NBS-open	N-bipolar soft open
NBS-closed	N-bipolar soft closed
NBS-separation axioms	N-bipolar soft separation axioms
NBS-homeomorphisms	N-bipolar soft homeomorphisms
NBSTSs	N-bipolar soft topological spaces
NBSTS	N-bipolar soft topological space
NBS-relative topology	N-bipolar soft relative topology
NBS-sub-topology	N-bipolar soft sub-topology
NBS-interior	N-bipolar soft interior
NBS-image	N-bipolar soft image
NBS-inverse image	N-bipolar soft inverse image
NBST$_i$-space	N-bipolar soft T$_i$-space, i=0,1,2,3,4

1 Introduction

N-bipolar soft sets (*NBS*-sets) provide an enhanced framework for bipolar soft set theory, empowering decision-makers to articulate their uncertainties, inconsistencies, and imprecisions throughout the decision-making process. In NBS-set theory, elements in the soft set are assigned to several categories or decision classes based on their positive, negative, or neutral characteristics. This enables the decision-makers to model more complex decision-making situations and take into account the different perspectives and preferences of the stakeholders involved in the decision-making process. *NBS*-sets find extensive application in diverse domains including but not limited to medical diagnosis, engineering, business administration, and other areas. Fatia Fatimah et al. [1] were the first to propose the concept of the *N*-soft set while Heba Mustafa [2] is credited for originating the ideas behind the *NBS*-set. An idea of the *N*-soft mappings and some of their properties with examples and counterexamples are investigated in [3]. They also described a mathematical system design for diagnosing the purpose of the COVID-19 disease. But our work aims to study new properties of *NBS*-continuous mappings and we unveiled an innovative OMICRON diagnostic approach within the framework of *NBS*-mappings. The paper is organized as follows: In Section 1, we review the history of the point, its importance, and related paper. In Section 2, we mention a few key antecedent concepts that are important in this study. Section 3 presents the idea of *N*-bipolar soft continuous mappings notions and describes them in relation to significant theorems and specific features. In Section 4, we analyze the impact of certain *NBS*-separation axioms when applied to *NBS*-continuous,

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NBS-open and *NBS*-closed mappings. In Section 5, this part of medical diagnosis, we showcase the practical implementation of *NBS*-mappings. Finally, we conclude our results in Section 6.

2 Preliminaries

The *NBS*-sets, and *N*-bipolar soft topological space ($NBST_S$) concepts will be delved into more depth in this section. Throughout this work, $2^{\mho}$ is the power set of an initial universe $\mho$. Additionally, S (which is not equal to ϕ) stands for the collection of parameters that are being considered, and $\phi \neq \Upsilon, \check{D}$ are subsets of S. We repeat the following definitions, but in more detail, we refer to [4], [5], [6], [1], [7] and [2] respectively.

Definition 2.1. [2] It can be stated that $(\wp, \Omega, \Upsilon, N)$ is an *N*-bipolar soft set (*NBS*-set) if certain conditions are satisfied. These conditions involve two functions, $\wp : \Upsilon \to 2^{\mho \times R}$ and $\Omega : \neg\Upsilon \to 2^{\mho \times R}$. Additionally, for each $a \in \Upsilon$ and $\mu \in \mho$, there must exist unique pairs $(\mu, ł_a), (\mu, ł_{\neg a}) \in \mho \times R$ such that $(\mu, ł_a) \in \wp(a); (\mu, ł_{\neg a}) \in \Omega(\neg a)$, $ł_a \neq ł_{\neg a}$ and $0 < ł_a + ł_{\neg a} < N-1$, $ł_a$, $ł_{\neg a} \in R$. It will be represented as

$$(\wp, \Omega, \Upsilon, N) = \{(\S, \wp(\S), \Omega(\neg\S), N) : \S \in \Upsilon, \neg\S \in \neg\Upsilon\}.$$

The set of all *NBS*-sets on $\mho$ (briefly $BS^N(\mho, \Upsilon)$).

Definition 2.2. [2] A group of *NBS*-subsets of an *NBS*-set $(\wp, \Omega, S, N)$ is called *N*-bipolar soft topology (*NBST*) on $(\wp, \Omega, S, N)$ (briefly τ_S^N). It is characterized by the fulfillment of the conditions:

(i) $\phi_S^N, \mho_S^N \in \tau_S^N$.

(ii) If $(\wp_j, \Omega_j, S, N) \in \tau_S^N, j \in I$, then $\bigcup_{j \in I} (\wp_j, \Omega_j, S, N) \in \tau_S^N$.

(iii) If $(\wp_j, \Omega_j, S, N) \in \tau_S^N, 1 \leq j \leq n, n \in \mathbb{N}$, then $\bigcap_{1 \leq j \leq n} (\wp_j, \Omega_j, S, N) \in \tau_S^N$.

The $NBST_S$ is denoted by $((\wp, \Omega, S, N), \tau_S^N)$. Each element in τ_S^N is referred to as an *NBS*-open set. In addition, the *NBS*-closed set is the complement of *NBS*-open set.

Definition 2.3. [2] Let $((\wp, \Omega, S, N), \tau_S^N)$ be an $NBST_S$ and $(\wp, \Omega)_1 = (\wp_1, \Omega_1, S, N) \tilde{\subseteq} (\wp, \Omega, S, N)$. Then the collection $\tilde{\tau}^N_{(\wp,\Omega)_1} = \{(\wp, \Omega)_i \cap (\wp_1, \Omega_1, S, N) : (\wp, \Omega)_i \in \tau_S^N\}$ is called *NBS*-relative topology or an *NBS*-sub-topology on $(\wp_1, \Omega_1, S, N)$. The pair $((\wp_1, \Omega_1, S, N), \tilde{\tau}^N_{(\wp,\Omega)_1})$ is called an *NBS*-sub-space of $((\wp, \Omega, S, N), \tau_S^N)$.

Proposition 2.4. [2] For the two *NBS*-sets $(\wp_1, \Omega_1, \Upsilon, N)$ and $(\wp_2, \Omega_2, \Upsilon, N)$ on $\mho$, we get

(1) $((\wp_1, \Omega_1, \Upsilon, N) \cup_{\Im} (\wp_2, \Omega_2, \Upsilon, N))^c = ((\wp_1, \Omega_1, \Upsilon, N))^c \cap_{\Re} ((\wp_2, \Omega_2, \Upsilon, N))^c$,

(2) $((\wp_1, \Omega_1, \Upsilon, N) \cap_{\Re} (\wp_2, \Omega_2, \Upsilon, N))^c = ((\wp_1, \Omega_1, \Upsilon, N))^c \cup_{\Im} ((\wp_2, \Omega_2, \Upsilon, N))^c$,

(3) $(\wp_1, \Omega_1, \Upsilon, N) \cap_{\Re} \mho_S^N = (\wp_1, \Omega_1, \Upsilon, N)$.

Remark 2.5. For the two *NBS*-sets $(\wp_1, \Omega_1, \Upsilon, N)$ and $(\wp_2, \Omega_2, \Upsilon, N)$ on $\mho$, we obtain $(\wp_1, \Omega_1, \Upsilon, N) \tilde{\subseteq} (\wp_2, \Omega_2, \Upsilon, N)$ iff $(\wp_2, \Omega_2, \Upsilon, N)^c \tilde{\subseteq} (\wp_1, \Omega_1, \Upsilon, N)^c$.

Definition 2.6.[2] Suppose $((\wp, \Omega, S, N), \tau_S^N)$ is an $NBST_S$ and $(\wp_1, \Omega_1, S, N)$, $(\wp_2, \Omega_2, S, N)$ are two *NBS*-subsets of $(\wp, \Omega, S, N)$ such that $(\wp_1, \Omega_1, S, N) \tilde{\subseteq} (\wp_2, \Omega_2, S, N)$. Let $(\wp_2, \Omega_2, S, N)$ be an *NBS*-neighborhood of$(\wp_1, \Omega_1, S, N)$, then $(\wp_1, \Omega_1, S, N)$ is an *NBS*-interior of $(\wp_2, \Omega_2, S, N)$. Furthermore, the union of all *NBS*-interior of $(\wp_2, \Omega_2, S, N)$ is referred to as the *NBS*-interior for $(\wp_2, \Omega_2, S, N)$, also symbolized as $(\wp_2, \Omega_2, S, N)^\circ$.

Definition 2.7. [2] Let $((\wp, \Omega, S, N), \tau_S^N)$ be an $NBST_S$ and $(\wp_1, \Omega_1, S, N)$ $\tilde{\subseteq} (\wp, \Omega, S, N)$. The *NBS*-closure for $(\wp_1, \Omega_1, S, N)$ which is denoted by $cl((\wp_1, \Omega_1, S, N))$ or $\overline{(\wp_1, \Omega_1, S, N)}$ is the intersection of all *NBS*-closed superset of $(\wp_1, \Omega_1, S, N)$.

Definition 2.8. [2] For an $NBST_S$ $((\wp, \Omega, S, N), \tau_S^N)$, we have

(1) $\mho_S^N$ and ϕ_S^N are *NBS*-closed sets.

(2) The *NBS*-closed sets are preserved when taking the finite unions of them

(3) The sets resulting from taking arbitrary intersections of *NBS*-closed sets are also *NBS*-closed sets.

Definition 2.9. [8] Let $(\wp, \Omega, S, N)$ be an *NBS*-set over $\mho$ and $\mu \in \mho$. When $\mu \in \wp(\S), \mu \in \Omega(\neg\S)$ for all $\S \in S, \neg\S \in \neg S$, we state that $\mu \in (\wp, \Omega, S, N)$.

Note that if $\mu \notin \wp(\S), \upsilon \notin \Omega(\neg\S)$ for some $\S \in S, \neg\S \in \neg S$, then for every $\mu \in \mho$, $\mu \notin (\wp, \Omega, S, N)$.

Definition 2.10. [8] Let $((\wp, \Omega, S, N), \tau_S^N)$ be an $NBST_S$ over $(\wp, \Omega, S, N)$ and $\mu, \kappa \in (\wp, \Omega, S, N)$ such that $\mu \neq \kappa$.

(1) If $(\wp_1, \Omega_1, S, N)$ and $(\wp_2, \Omega_2, S, N)$ are *NBS*-open subsets of $(\wp, \Omega, S, N)$ such that $\mu \in (\wp_1, \Omega_1, S, N)$ and $\kappa \notin (\wp_1, \Omega_1, S, N)$ or $\kappa \in (\wp_2, \Omega_2, S, N)$ and $\mu \notin (\wp_2, \Omega_2, S, N)$, then $((\wp, \Omega, S, N), \tau_S^N)$ is called an $NBST_0$-space.

(2) If $(\wp_1, \Omega_1, S, N)$ and $(\wp_2, \Omega_2, S, N)$ are *NBS*-open subsets of $(\wp, \Omega, S, N)$ such that $\mu \in (\wp_1, \Omega_1, S, N)$ and $\kappa \notin (\wp_1, \Omega_1, S, N)$ and $\kappa \in (\wp_2, \Omega_2, S, N)$ and $\mu \notin (\wp_2, \Omega_2, S, N)$, then $((\wp, \Omega, S, N), \tau_S^N)$ is called an $NBST_1$-space.

(3) If $(\wp_1, \Omega_1, S, N)$ and $(\wp_2, \Omega_2, S, N)$ are *NBS*-open subsets of $(\wp, \Omega, S, N)$ such that $\mu \in (\wp_1, \Omega_1, S, N)$, $\kappa \in (\wp_2, \Omega_2, S, N)$ and $(\wp_1, \Omega_1, S, N) \cap (\wp_2, \Omega_2, S, N) = \phi$, then $((\wp, \Omega, S, N), \tau_S^N)$ is called an $NBST_2$-space.

3 *N*-bipolar soft continuous mappings

In this part, our first focus will be on examining the properties of *N*-bipolar soft continuous mappings between two $NBST_{S_S}$. Additionally, some fresh insights into the qualities of *NBS*-continuous, *NBS*-open, and *NBS*-closed mappings are provided.

Definition 3.1. Let $\text{ß}\S^N(\mho, S)$ and $\text{ß}\S^N(\chi, S')$ with characteristics from S and S' be the families of all

N-Bipolar soft sets on $\mho$ and χ respectively. If $p:\mho\to\chi$ is an injective function, and $\eta:S\to S',q:\neg S\to\neg S'$ are two mappings, where $q(\neg\S)=\neg\eta(\S)$ for all $\neg\S\in\neg S$, then an NBS-mapping $\xi_{p\eta q}:\text{ß}\S^N(\mho,S)\to\text{ß}\S^N(\chi,S')$ is defined as: for any NBS-set (Θ,Ω,S,N) in $\text{ß}\S^N(\mho,S)$ the image of (Θ,Ω,S,N) under $\xi_{p\eta q}$, as follows

$$\xi_{p\eta q}(\Theta,\Omega,S,N)=\{\xi_{p\eta q}(\Theta(\alpha)),\xi_{p\eta q}(\Omega(\neg\alpha)),S',N: \alpha\in S',\neg\alpha\in\neg S'\},$$

is an NBS-set in $\text{ß}\S^N(\chi,S')$ given as, for all $\S'\in S'$ and $\neg\S'\in\neg S'$

$$\xi_{p\eta q}(\Theta(\alpha))(\kappa)=\left\{(\kappa,\xi_{p\eta q}(\Theta(\S')(\kappa))):\kappa\in\chi\right\},$$

and

$$\xi_{p\eta q}(\Omega(\neg\alpha))(\kappa)=\left\{(\kappa,\xi_{p\eta q}(\Omega(\neg\S')(\kappa))):\kappa\in\chi\right\},$$

where

$$\xi_{p\eta q}(\Theta(\S')(\kappa))=\begin{cases}\max\{\Theta(\S)(\mu):\S\in\eta^{-1}(\S'),\\ \mu\in p^{-1}(\kappa),\\ \text{if } \eta^{-1}(\S')\cap S\neq\phi,\\ p^{-1}(\kappa)\neq\phi;\\ 0, \qquad\qquad \text{otherwise,}\end{cases}$$

$$\xi_{p\eta q}(\Omega(\neg\S')(\kappa))=\begin{cases}\min\{\Omega(\neg\S)(\mu):\neg\S\in q^{-1}(\neg\S'),\\ \mu\in p^{-1}(\kappa),\\ \text{if } q^{-1}(\neg\S')\cap\neg S\neq\phi,\\ p^{-1}(\kappa)\neq\phi;\\ 0, \qquad\qquad \text{otherwise.}\end{cases}$$

$\xi_{p\eta q}(\Theta,\Omega,S,N)$ is called an NBS-image of (Θ,Ω,S,N) under $\xi_{p\eta q}$.

Definition 3.2. Let $p:\mho\to\chi$ be an injective function, and $\eta:S\to S',q:\neg S\to\neg S'$ be two mappings, where $q(\neg\S)=\neg\eta(\S)$ for all $\neg\S\in\neg S$. We defined a mapping $\xi_{p\eta q}:\text{ß}\S^N(\mho,S)\to\text{ß}\S^N(\chi,S')$ as follows: if (ψ,ω,S',N) is an NBS-set in $\text{ß}\S^N(\chi,S')$, the inverse image of (ψ,ω,S',N) under $\xi^{-1}_{p\eta q}$, written as

$$\xi^{-1}_{p\eta q}(\psi,\omega,S',N)=\{\xi^{-1}_{p\eta q}(\psi(\alpha)),\xi^{-1}_{p\eta q}(\omega(\neg\alpha)),S,N: \alpha\in S,\neg\alpha\in\neg S\},$$

is an NBS-set in $\text{ß}\S^N(\mho,S)$ given as, for all $\S\in S$ and $\neg\S\in\neg S$

$$\xi^{-1}_{p\eta q}(\psi(\alpha))(\S)=\left\{(\mu,\xi^{-1}_{p\eta q}(\psi(\S)(\mu))):\mu\in\mho\right\},$$

and

$$\xi^{-1}_{p\eta q}(\omega(\neg\alpha))(\neg\S)=\left\{(\mu,\xi^{-1}_{p\eta q}(\omega(\neg\S)(\mu))):\mu\in\mho\right\},$$

where

$$\xi^{-1}_{p\eta q}(\psi(\S)(\mu))=\psi p(\mu)\eta(\S),$$
$$\xi^{-1}_{p\eta q}(\omega(\neg\S)(\mu))=\omega p(\mu)q(\neg\S).$$

$\xi^{-1}_{p\eta q}(\psi,\omega,S',N)$ is said to be an NBS-inverse image of (ψ,ω,S',N).

Example 3.3. Let $\mho=\{\mu_1,\mu_2,\mu_3\},\chi=\{\kappa_1,\kappa_2,\kappa_3\},S=\{\S_1,\S_2,\S_3\},\neg S=\{\neg\S_1,\neg\S_2,\neg\S_3\},S'=\{\S'_1,\S'_2\}$ and $\neg S'=\{\neg\S'_1,\neg\S'_2\}$. Define the mapping $p:\mho\to\chi,\eta:S\to S'$ and $q:\neg S\to\neg S$ by

$$p(\mu_1)=\kappa_1\quad p(\mu_2)=\kappa_2\quad p(\mu_3)=\kappa_2$$
$$\eta(\S_1)=\S'_1\quad \eta(\S_2)=\S'_1\quad \eta(\S_3)=\S'_2$$
$$q(\neg\S_1)=\neg\S'_1\quad q(\neg\S_2)=\neg\S'_2\quad q(\neg\S_3)=\neg\S'_2.$$

Take two $5BS$-sets on $\mho$ and χ with parameters from S to S', respectively, as

$$\begin{aligned}(\Theta,\Omega,S,5)=\{&(\langle\S_1,\{(\mu_1,4),(\mu_2,2),(\mu_3,0)\}\rangle,\\ &\langle\neg\S_1,\{(\mu_1,0),(\mu_2,1),(\mu_3,2)\}\rangle),\\ &(\langle\S_2,\{(\mu_1,0),(\mu_2,1),(\mu_3,2)\}\rangle,\\ &\langle\neg\S_2,\{(\mu_1,3),(\mu_2,2),(\mu_3,0)\}\rangle),\\ &(\langle\S_3,\{(\mu_1,3),(\mu_2,1),(\mu_3,0)\}\rangle,\\ &\langle\neg\S_3,\{(\mu_1,1),(\mu_2,3),(\mu_3,4)\}\rangle)\},\end{aligned}$$

and

$$\begin{aligned}(\psi,\omega,S',5)=\{&(\langle\S'_1,\{(\kappa_1,3),(\kappa_2,1),(\kappa_3,2)\}\rangle,\\ &\langle\neg\S'_1,\{(\kappa_1,1),(\kappa_2,2),(\kappa_3,1)\}\rangle),\\ &(\langle\S'_2,\{(\kappa_1,0),(\kappa_2,2),(\kappa_3,4)\}\rangle,\\ &\langle\neg\S'_2,\{(\kappa_1,3),(\kappa_2,2),(\kappa_3,0)\}\rangle)\}.\end{aligned}$$

So, the $5BS$-image of $(\Theta,\Omega,S,5)$ under the $5BS$-mapping $\xi_{p\eta q}:\text{ß}\S^5(\mho,S)\to\text{ß}\S^5(\chi,S')$ is obtained as the following:

Table 1 The tabular form of $\xi_{p\eta q}(\Theta,\Omega,S,5)$

$\xi_{p\eta q}(\Theta,\Omega,S,5)$	$(\S'_1,\neg\S'_1)$	$(\S'_2,\neg\S'_2)$
κ_1	$(4,0)$	$(3,1)$
κ_2	$(2,1)$	$(2,0)$
κ_3	$(0,0)$	$(0,0)$

Therefore, we can write a $5BS$-image of $(\Theta,\Omega,S,5)$ under $\xi_{p\eta q}$ as

$$\begin{aligned}\xi_{p\eta q}(\Theta,\Omega,S,5)&=(\xi_{p\eta q}(\Theta(\alpha)),\xi_{p\eta q}(\Omega(\neg\alpha)),S',5)\\ &=\{(\langle\S'_1,\{(\kappa_1,4),(\kappa_2,2),(\kappa_3,0)\}\rangle,\\ &\quad\langle\neg\S'_1,\{(\kappa_1,0),(\kappa_2,1),(\kappa_3,0)\}\rangle),\\ &\quad(\langle\S'_2,\{(\kappa_1,3),(\kappa_2,2),(\kappa_3,0)\}\rangle,\\ &\quad\langle\neg\S'_2,\{(\kappa_1,1),(\kappa_2,0),(\kappa_3,0)\}\rangle)\}.\end{aligned}$$

Now, let us compute the $5BS$-inverse image of $(\psi,\omega,S',5)$ over χ :

Table 2 The tabular form of $\xi^{-1}_{p\eta q}(\Theta,\Omega,S',5)$

$\xi^{-1}_{p\eta q}(\psi,\omega,S',5)$	$(\S_1,\neg\S_1)$	$(\S_2,\neg\S_2)$	$(\S_3,\neg\S_3)$
μ_1	$(3,1)$	$(3,3)$	$(0,3)$
μ_2	$(1,2)$	$(1,2)$	$(2,2)$
μ_3	$(1,2)$	$(1,2)$	$(2,2)$

Therefore, the $5BS$-inverse image of $(\psi,\omega,S',5)$ is

$$\begin{aligned}\xi^{-1}_{p\eta q}(\psi,t,S',5) &= (\xi^{-1}_{p\eta q}(\psi(\alpha)),\xi^{-1}_{p\eta q}(\omega(\neg\alpha)),S,5)\\ &= \{(\langle §_1,\{(\mu_1,3),(\mu_2,1),(\mu_3,1)\}\rangle,\\ &\quad\langle\neg §_1,\{(\mu_1,1),(\mu_2,2),(\mu_3,2)\}\rangle),\\ &\quad(\langle §_2,\{(\mu_1,3),(\mu_2,1),(\mu_3,1)\}\rangle,\\ &\quad\langle\neg §_2,\{(\mu_1,3),(\mu_2,2),(\mu_3,2)\}\rangle),\\ &\quad(\langle §_3,\{(\mu_1,0),(\mu_2,2),(\mu_3,2)\}\rangle,\\ &\quad\langle\neg §_3,\{(\mu_1,3),(\mu_2,2),(\mu_3,2)\}\rangle)\}.\end{aligned}$$

Definition 3.4. An NBS-mapping $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,S',N),\upsilon^N_{S'})$ for an $NBST_{Ss}$ $((\Theta,\Omega,S,N),\tau^N_S)$ and $((\psi,\omega,S',N),\upsilon^N_{S'})$ is called

(1) NBS-open if $\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)\in\upsilon^N_{S'}$ for each $(\Theta_1,\Omega_1,S,N)\subseteq(\Theta,\Omega,S,N)\in\tau^N_S$.

(2) NBS-closed if $\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)\in\upsilon'^N_{S'}$ for each $(\Theta_1,\Omega_1,S,N)\subseteq(\Theta,\Omega,S,N)\in\tau'^N_S$.

Theorem 3.5. Let $((\Theta_1,\Omega_1,S,N),\tilde{\tau}^N_{(\Theta,\Omega)_1})$ be an NBS-subspace of an $NBST_S$ $((\Theta,\Omega,S,N),\tau^N_S)$ and (Θ_1,Ω_1,S,N) be an NBS-open set in W. If $W^N_S\in\tau^N_S$, then $(\Theta_1,\Omega_1,S,N)\in\tau^N_S$.

Proof. Let (Θ_1,Ω_1,S,N) be an NBS-open set in W. Consequently there exists an NBS-open set $(\psi_1,\omega_1,S,N)\subseteq(\psi,\omega,S,N)$ in $\mho$ where $(\Theta_1,\Omega_1,S,N)=W^N_S\cap(\psi_1,\omega_1,S,N)$. Using the third axiom of the definition of an $NBST_S$, $W^N_S\cap(\psi_1,\omega_1,S,N)\in\tau^N_S$ if $W^N_S\in\tau^N_S$.Therefore, $(\Theta_1,\Omega_1,E,N)\in\tau^N_S$.

Theorem 3.6. Let $((\Theta_1,\Omega_1,S,N),\tilde{\tau}^N_{(\Theta,\Omega)_1})$ be an NBS-subspace of an $NBST_S$ $((\Theta,\Omega,S,N),\tau^N_S)$ and (Θ_1,Ω_1,S,N) be an NBS-closed set in W. If $W^N_S\in\tau'^N_S$, then $(\Theta_1,\Omega_1,S,N)\in\tau'^N_S$.

Proof. It can be proved directly.

Definition 3.7. An NBS-mapping $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,S',N),\upsilon^N_{S'})$ is NBS-continuous iff $\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\subseteq(\psi,\omega,S,N)\in\tau^N_S$ for every $(\psi_1,\omega_1,S',N)\in\upsilon^N_{S'}$.

Example 3.8. Let $\mho=\{\mu_1,\mu_2,\mu_3\},\chi=\{\kappa_1,\kappa_2,\kappa_3\},S=\{§_1,§_2,§_3\},\neg S=\{\neg§_1,\neg§_2,\neg§_3\},S'=\{§'_1,§'_2\},\neg S'=\{\neg§'_1,\neg§'_2\}$, $\tau^6_S=\{\phi^6_S,\mho^6_S,(\Theta_1,\Omega_1,S,6)\}$, where $(\Theta_1,\Omega_1,S,6)$ is a $6BS$-set on $\mho$, defined as follows:

$$\begin{aligned}(\Theta_1,\Omega_1,S,6) &= \{(\langle §_1,\{(\mu_1,5),(\mu_2,3),(\mu_3,1)\}\rangle,\\ &\quad\langle\neg §_1,\{(\mu_1,0),(\mu_2,2),(\mu_3,2)\}\rangle),\\ &\quad(\langle §_2,\{(\mu_1,0),(\mu_2,2),(\mu_3,3)\}\rangle,\\ &\quad\langle\neg §_2,\{(\mu_1,4),(\mu_2,3),(\mu_3,1)\}\rangle),\\ &\quad(\langle §_3,\{(\mu_1,4),(\mu_2,2),(\mu_3,0)\}\rangle,\\ &\quad\langle\neg §_3,\{(\mu_1,2),(\mu_2,4),(\mu_3,5)\}\rangle)\},\end{aligned}$$

$\upsilon^6_{S'}=\{\phi^6_{S'},\chi^6_{S'},(\psi_1,\omega_1,S',6)\}$, where $(\psi_1,\omega_1,S',5)$ is a $6BS$-set on χ, defined as follows:

$$\begin{aligned}(\psi,\omega,S',6) &= \{(\langle §'_1,\{(\kappa_1,4),(\kappa_2,2),(\kappa_3,3)\}\rangle,\\ &\quad\langle\neg §'_1,\{(\kappa_1,2),(\kappa_2,3),(\kappa_3,2)\}\rangle),\\ &\quad(\langle §'_2,\{(\kappa_1,0),(\kappa_2,3),(\kappa_3,5)\}\rangle,\\ &\quad\langle\neg §'_2,\{(\kappa_1,4),(\kappa_2,2),(\kappa_3,0)\}\rangle)\},\end{aligned}$$

and let $((\Theta,\Omega,S,6),\tau^6_S)$ and $((\psi,\omega,S',6),\upsilon^6_{S'})$ be a $6BST_{Ss}$.

Define the mapping $p:\mho\to\chi,\eta:S\to S'$ and $q:\neg S\to\neg S$ by

$$p(\mu_1)=\kappa_1\quad p(\mu_2)=\kappa_2\quad p(\mu_3)=\kappa_1$$
$$\eta(§_1)=§'_1\quad\eta(§_2)=§'_1\quad\eta(§_3)=§'_2$$
$$q(\neg§_1)=\neg§'_1\quad q(\neg§_2)=\neg§'_2\quad q(\neg§_3)=\neg§'_2.$$

Let $\xi_{p\eta q}:((\Theta,\Omega,S,6),\tau^6_S)\to((\psi,\omega,S',6),\upsilon^6_{S'})$ be a $6BS$-mapping. Then $(\psi_1,\omega_1,S',6)\subseteq(\psi,\omega,S',6)$ is a $6BS$-open in χ and $\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',6)=(\Theta_1,\Omega_1,S,6)\subseteq(\Theta,\Omega,S,6)$ is a $6BS$-open in $\mho$. Therefore, $\xi_{p\eta q}$ is a $6BS$-continuous mapping from $((\Theta,\Omega,S,6),\tau^6_S)$ to $((\psi,\omega,S',6),\upsilon^6_{S'})$.

Definition 3.9. Let $\phi\neq W\subseteq\mho$, then W^N_S denotes the NBS-set $\text{ß}\c{S}^N(W,S)$ over $\mho$ for which $W(§)=W$ and $W(\neg§)=W$ for all $§\in W$ and $\neg§\in\neg W$

Definition 3.10. Let $(\Theta,\Omega,S,N)\in\text{ß}\c{S}^N(\mho,S)$ and $(\psi,\omega,S',N)\in\text{ß}\c{S}^N(\chi,S')$, then $\xi_{p\eta q}:\text{ß}\c{S}^N(\mho,S)\to\text{ß}\c{S}^N(\chi,S')$ is an NBS-mapping and

$$(\Theta_1,\Omega_1,S,N)\tilde{\subseteq}(\Theta,\Omega,S,N)=(\Theta,\Omega)_1\in W\subseteq\mho.$$

An NBS-mapping of $\xi_{p\eta q}|_{\text{ß}\c{S}^N(W,S)}$ from $\text{ß}\c{S}^N(\mho,S)$ to $\text{ß}\c{S}^N(\chi,S')$ is the restriction of $\xi_{p\eta q}$ to $\text{ß}\c{S}^N(W,S)$.

This is defined as $\eta:\Theta\to S',q:\Omega\to\neg S'$ and $p|_W:W\to\chi$, where $p|_W$ is the restriction of p to W.

Proposition 3.11. If $\xi_{p\eta q}:\text{ß}\c{S}^N(\mho,S)\to\text{ß}\c{S}^N(\chi,S')$ is an NBS-mapping and $W\subseteq\mho$, then

$$(\xi_{p\eta q}|_{\text{ß}\c{S}^N(W,S)})^{-1}(\psi,\omega,S',N)=\xi^{-1}_{p\eta q}(\psi,\omega,S',N)\cap W^N_S,$$
$$\text{for all }(\psi,\omega,S',N)\in\text{ß}\c{S}^N(\chi,S').$$

Proof. From the equality $(p|_W)^{-1}(\chi')=p^{-1}(\chi')\cap W$ for all $\chi'\subseteq\chi$, the proof is finished.

Theorem 3.12. If $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,S',N),\upsilon^N_{S'})$ is NBS-continuous, then $\xi_{p\eta q}|_{\text{ß}\c{S}^N(W,S)}:((\Theta_1,\Omega_1,S,N),\tilde{\tau}^N_{(\Theta,\Omega)_1})\to((\psi,\omega,S',N),\upsilon^N_{S'})$ is NBS-continuous for every $(\Theta,\Omega)_1\in W\subseteq\mho$.

Proof. Using Proposition 3.11. with the definition of NBS-relative topology, the proof is finished.

Theorem 3.13. For any $NBST_{Ss}$ $((\Theta,\Omega,S,N),\tau^N_S)$ and $((\psi,\omega,S',N),\upsilon^N_{S'})$, the following are satisfied

(1) Let $\{(W^N_S)_j\}_{j\in I}$ be a family of subsets of $\mho$ with $(W^N_S)_j$'s are NBS-open sets in $\mho$ and $\mho^N_S=\cup_{j\in I}(W^N_S)_j$. Then $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,S',N),\upsilon^N_{S'})$ is NBS-continuous iff $\xi_{p\eta q}|_{\text{ß}\c{S}^N((W^N_E)_j,S)}$:

$((\Theta_j,\Omega_j,S,N),\tilde{\tau}^N_{(\Theta,\Omega)j}) \to ((\psi,\omega,S',N),\upsilon^N_{S'})$ is N-bipolar soft continuous for every $j\in I$.

(2) Let $(W^N_S)_1,(W^N_S)_2,...,(W^N_S)_n$ are an NBS-closed sets in $\mho$ and $\mho^N_S=\cup^n_{i\in I}(W^N_S)_j$, then the NBS-mapping $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,S',N),\upsilon^N_{S'})$ is NBS-continuous iff $\xi_{p\eta q}|_{ßŞ^N((W^N_S)j,S)}: ((\Theta_j,\Omega_j,S,N),\tilde{\tau}^N_{(\Theta,\Omega)j}) \to ((\psi,\omega,S',N),\upsilon^N_{S'})$ is NBS-continuous for each $j=1,2,...,n$.

Proof. (1) $(\Rightarrow)$ It is Theorem 3.12.

$(\Leftarrow)$ For an NBS-open set $(\psi_1,\omega_1,S',N)\subseteq(\psi,\omega,S',N)\in\chi$. If $\xi_{p\eta q}|_{ßŞ^N((W^N_S)_j,S)}$ is NBS-continuous, then $(\xi_{p\eta q}|_{ßŞ^N((W^N_S)_j,S)})^{-1}(\psi_1,\omega_1,S',N)$ is NBS-open set in $(W^N_S)_j$ for all $j\in I$. Using Theorem 3.5, if $(W^N_S)_j\in\mho$ is NBS-open, then $(\xi_{p\eta q}|_{ßŞ^N((W^N_S)_j,S)})^{-1}(\psi_1,\omega_1,S',N)$ is NBS-open set in $\mho$. Therefore,
$\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N) = \xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\cap\mho^N_S = \xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\cap(\cup_{j\in I}(W^N_S)_i) = \cup_{j\in I}(\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\cap(W^N_S)_j) =$
$\cup_{i\in I}(\xi_{p\eta q}|_{ßŞ^N((W^N_S)_j,S)})^{-1}(\psi_1,\omega_1,S',N)$ is NBS-open in $\mho$. Now the proof is complete.

(2) It can be demonstrated similarly.

Lemma 3.14. [2] For an $NBST_S$ $((\Theta,\Omega,S,N),\tau^N_S)$ with $(\Theta_1,\Omega_1,S,N),(\Theta_2,\Omega_2,S,N)$ $\in ßŞ^N(\mho,S)$.

The following are satisfied

(1) (Θ_1,Ω_1,S,N) is NBS-closed set iff $(\Theta_1,\Omega_1,S,N)=\overline{(\Theta_1,\Omega_1,S,N)}$.

(2) $(\Theta_2,\Omega_2,S,N)\tilde{\subseteq}(\Theta_1,\Omega_1,S,N) \Longrightarrow \overline{(\Theta_2,\Omega_2,S,N)}\tilde{\subseteq}\overline{(\Theta_1,\Omega_1,S,N)}$.

(3) (Θ_1,Ω_1,S,N) is an NBS-open set iff $(\Theta_1,\Omega_1,S,N)^\circ=(\Theta_1,\Omega_1,S,N)$.

(4) If $(\Theta_1,\Omega_1,S,N)\tilde{\subseteq}(\Theta_2,\Omega_2,S,N)$, then $(\Theta_1,\Omega_1,S,N)^\circ\tilde{\subseteq}(\Theta_2,\Omega_2,S,N)^\circ$.

Remark 3.15. Let $((\Theta,\Omega,S,N),\tau^N_S)$ be an $NBST_S$ and $(\Theta_1,\Omega_1,S,N)\in ßŞ^N(\mho,S)$. Then

(1) $((\Theta_1,\Omega_1,S,N)^\circ)^c=\overline{((\Theta_1,\Omega_1,S,N)^c)}$.

(2) $(\overline{(\Theta_1,\Omega_1,S,N)})^c=((\Theta_1,\Omega_1,S,N)^c)^\circ$.

Proof. (1) $\Longrightarrow$ By Lemma 2.21

$$\begin{aligned}((\Theta_1,\Omega_1,S,N)^\circ)^c &= [\cup\{(\Theta_3,\Omega_3,S,N) \\ &\quad : (\Theta_3,\Omega_3,S,N)\in\tau^N_S \text{ is } NBS\text{-open} \\ \text{and } (\Theta_3,\Omega_3,S,N) &\tilde{\subseteq} (\Theta_1,\Omega_1,S,N)\}]^c \\ &= \cap[(\Theta_3,\Omega_3,S,N)^c \\ &\quad : (\Theta_3,\Omega_3,S,N)\in\tau^N_S \text{ is } NBS\text{-open} \\ \text{and } (\Theta_3,\Omega_3,S,N) &\tilde{\subseteq} (\Theta_1,\Omega_1,S,N)] \\ &= \cap[(\Theta_3,\Omega_3,S,N)^c \\ &\quad : (\Theta_3,\Omega_3,S,N)^c\in\tau^N_S \text{ is } NBS\text{-closed} \\ \text{and } (\Theta_1,\Omega_1,S,N)^c &\tilde{\subseteq} (\Theta_3,\Omega_3,S,N)^c] \\ &= \overline{((\Theta_1,\Omega_1,S,N)^c)}.\end{aligned}$$

(2) $\Longrightarrow$ Similar to that of (1).

As in [3] we have the following definition.

Definition 3.16.

(1) The NBS-mapping $\xi_{p\eta q}$ is said to be injective if p,η and q are injective mappings.

(2) The NBS-mapping $\xi_{p\eta q}$ is said to be surjective if p,η and q are surjective mappings.

(3) The NBS-mapping $\xi_{p\eta q}$ is said to be bijective if p,η and q are bijective mappings.

Definition 3.17. Let $\xi_{p\eta q}: ßŞ^N(\mho,S)\to ßŞ^N(\chi,S')$ be an NBS-mapping and (Θ_1,Ω_1,S,N), (Θ_2,Ω_2,S,N) be NBS-sets in $ßŞ^N(\mho,S)$. For $ş'\in S'$, NBS-intersection and union of NBS-images of (Θ_1,Ω_1,S,N) and (Θ_2,Ω_2,S,N) in $ßŞ^N(\mho,S)$ are defined as:

$$\begin{aligned}&(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)\cap\xi_{p\eta q}(\Theta_2,\Omega_2,S,N))(ş')\\ &=\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)(ş')\cap\xi_{p\eta q}(\Theta_2,\Omega_2,S,N)(ş'),\\ &(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)\cup\xi_{p\eta q}(\Theta_2,\Omega_2,S,N))(ş')\\ &=\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)(ş')\cup\xi_{p\eta q}(\Theta_2,\Omega_2,S,N)(ş').\end{aligned}$$

Definition 3.18. Let $\xi_{p\eta q}: ßŞ^N(\mho,S)\to ßŞ^N(\chi,S')$ be an NBS-mapping and (ψ_1,ω_1,S',N), (ψ_2,ω_2,S',N) NBS-sets in $ßŞ^N(\chi,S')$. Then $ş\in S$, NBS-intersection and union of NBS-inverse images of (ψ_1,ω_1,S',N) and (ψ_2,ω_2,S',N) in $ßŞ^N(\chi,S')$ are defined as:

$$\begin{aligned}&(\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\cap\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N))(ş)\\ &=\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)(ş)\cap\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N)(ş),\\ &(\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\cup\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N))(ş)\\ &=\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)(ş)\cup\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N)(ş).\end{aligned}$$

Theorem 3.19. Let $\{(\Theta_i,\Omega_i,S,N)\}_{i\in I}\subseteq ßŞ^N(\mho,S)$ and $\{(\psi_i,\omega_i,S',N)\}_{i\in I}\subseteq ßŞ^N(\chi,S')$. Then for an NBS-mapping $\xi_{p\eta q}: ßŞ^N(\mho,S)\to ßŞ^N(\chi,S')$, the following are true.

(1) If $(\Theta_1,\Omega_1,S,N)\tilde{\subseteq}(\Theta_2,\Omega_2,S,N)$, then $\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)\tilde{\subseteq}\xi_{p\eta q}(\Theta_2,\Omega_2,S,N)$.

(2) If $(\psi_1,\omega_1,S',N)\tilde{\subseteq}(\psi_2,\omega_2,S',N)$, then $\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\tilde{\subseteq}\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N)$.

(3) $\xi_{p\eta q}((\Theta_1,\Omega_1,S,N)\cup(\Theta_2,\Omega_2,S,N)) = \xi_{p\eta q}(\Theta_1,\Omega_1,S,N)\cup\xi_{p\eta q}(\Theta_2,\Omega_2,S,N)$.

In general, $\xi_{p\eta q}(\cup_i(\Theta_i,\Omega_i,S,N))=\cup_i\xi_{p\eta q}(\Theta_i,\Omega_i,S,N)$.

(4) $\xi^{-1}_{p\eta q}((\psi_1,\omega_1,S',N)\cap(\psi_2,\omega_2,S',N)) = \xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\cap\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N)$.

(5) $\xi^{-1}_{p\eta q}((\psi_1,\omega_1,S',N)\cup(\psi_2,\omega_2,S',N)) = \xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\cup\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N)$.

Proof. Proving only $(1)-(3)$, the other proofs adopt a similar approach.

(1) For all $ş'\in S'$ and $\neg ş'\in\neg S'$

$$\xi_{p\eta q}(\Theta_1(ş')(\kappa))=\begin{cases}\max\{\Theta_1(ş)(\mu): ş\in\eta^{-1}(ş'),\mu\in p^{-1}(\kappa),\\ \text{if } \eta^{-1}(ş')\cap S\neq\phi, p^{-1}(\kappa)\neq\phi;\\ 0, \quad\quad \text{otherwise}\end{cases}$$

and

$$\xi_{p\eta q}(\Omega_1(\neg ş')(\kappa))=\begin{cases}\min\{\Omega_1(\neg ş)(\mu):\neg ş\in q^{-1}(\neg ş'),\mu\in p^{-1}(\kappa),\\ \text{if } q^{-1}(\neg ş')\cap\neg S\neq\phi, p^{-1}(\kappa)\neq\phi;\\ 0, \quad\quad \text{otherwise}\end{cases}.$$

We consider the case when $\eta^{-1}(ş') \cap S \neq \phi$ and $q^{-1}(\neg ş') \cap \neg S \neq \phi$ as otherwise it is trivial. Then

$$\xi_{p\eta q}(\Theta_1(ş')(\kappa)) = \max \Theta_1(ş)(\mu) \tilde{\subseteq} \max \Theta_2(ş)(\mu) = \xi_{p\eta q}(\Theta_2(ş')(\kappa)),$$

and

$$\xi_{p\eta q}(\Omega_1(\neg ş')(\kappa)) = \min \Omega_1(\neg ş)(\mu) \tilde{\subseteq} \min \Omega_2(\neg ş)(\mu) = \xi_{p\eta q}(\Omega_2(\neg ş')(\kappa)).$$

This gives (1).

(2) For all $ş \in S$ and $\neg ş \in \neg S$

$$\xi^{-1}_{p\eta q}(\psi_1(\alpha))(ş) = \left\{(\mu, \xi^{-1}_{p\eta q}(\psi_1(ş)(\mu))) : \mu \in \mho\right\},$$

and

$$\xi^{-1}_{p\eta q}(\omega_1(\neg\alpha))(\neg ş) = \left\{(\mu, \xi^{-1}_{p\eta q}(\omega_1(\neg ş)(\mu))) : \mu \in \mho\right\},$$

where

$$\xi^{-1}_{p\eta q}(\psi_1(ş)(\mu)) = \psi_1 p(\mu)\eta(ş) \tilde{\subseteq} \psi_2 p(\mu)\eta(ş) = \xi^{-1}_{p\eta q}(\psi_2(ş)(\mu),$$

and

$$\xi^{-1}_{p\eta q}(\omega_1(\neg ş)(\mu)) = \omega_1 p(\mu) q(\neg ş) \tilde{\subseteq} \omega_2 p(\mu) q(\neg ş) = \xi^{-1}_{p\eta q}(\omega_2(\neg ş)(\mu)).$$

This gives (2).

(3) For all $ş' \in S'$ and $\neg ş' \in \neg S'$, we show that $\xi_{p\eta q}((\Theta_1, \Omega_1, S, N) \cup (\Theta_2, \Omega_2, S, N)) = \xi_{p\eta q}(\Theta_1, \Omega_1, S, N) \cup \xi_{p\eta q}(\Theta_2, \Omega_2, S, N)$.

Consider $\xi_{p\eta q}((\Theta_1, \Omega_1, S, N) \cup (\Theta_2, \Omega_2, S, N))$ $= \xi_{p\eta q}(h, l, S \cup S, \max(N_1, N_2)) =$

$$\xi_{p\eta q}(h(ş')(\kappa)) = \begin{cases} \max\{h(ş)(\mu) : ş \in \eta^{-1}(ş'), \\ \mu \in p^{-1}(\kappa), \\ \text{if } \eta^{-1}(ş') \cap (S \cup S) \neq \phi \\ , p^{-1}(\kappa) \neq \phi; \\ 0, \quad \text{otherwise} \end{cases}$$

and

$$\xi_{p\eta q}(l(\neg ş')(\kappa)) = \begin{cases} \min\{l(\neg ş)(\mu) : \neg ş \in q^{-1}(\neg ş') \\ , \mu \in p^{-1}(\kappa), \\ \text{if } q^{-1}(\neg ş') \cap (\neg S \cup \neg S) \neq \phi, \\ p^{-1}(\kappa) \neq \phi; \\ 0, \quad \text{otherwise} \end{cases},$$

where

$$h(ş)(\mu) = \begin{cases} \Theta_1(ş) & \text{if } ş \in \Upsilon - \check{D} \\ \Theta_2(ş) & \text{if } ş \in \check{D} - \Upsilon \\ (\mu, \mathfrak{t}_ş) \text{ s.t. } \mathfrak{t}_ş = \max(\mathfrak{t}^1_ş, \mathfrak{t}^2_ş), \\ \text{where } (\mu, \mathfrak{t}^1_ş) \in \Theta_1(ş) \text{ and } (\mu, \mathfrak{t}^2_ş) \in \Theta_2(ş), \end{cases},$$

and

$$l(\neg ş)(\mu) = \begin{cases} \Omega_1(\neg ş) & \text{if } \neg ş \in (\neg\Upsilon) - (\neg\check{D}) \\ \Omega_2(\neg ş) & \text{if } \neg ş \in (\neg\check{D}) - (\neg\Upsilon) \\ (\mu, \mathfrak{t}_{\neg ş}) \text{ s.t. } \mathfrak{t}_{\neg ş} = \min(\mathfrak{t}^1_{\neg ş}, \mathfrak{t}^2_{\neg ş}), \\ \text{where } (\mu, \mathfrak{t}^1_{\neg ş}) \in \Omega_1(\neg ş) \\ \text{and } (\mu, \mathfrak{t}^2_{\neg ş}) \in \Omega_2(\neg ş) \end{cases}.$$

We consider the case when $\eta^{-1}(ş') \cap (S \cup S) \neq \phi$ and $q^{-1}(\neg ş') \cap (\neg S \cup \neg S) \neq \phi$ as otherwise it is trivial. Then

$$\xi_{p\eta q}(h(ş')(\kappa)) = \max \begin{cases} \Theta_1(ş) & \text{if } ş \in (\Upsilon - \check{D}) \cap \eta^{-1}(ş') \\ \Theta_2(ş) & \text{if } ş \in (\check{D} - \Upsilon) \cap \eta^{-1}(ş') \\ (\mu, \mathfrak{t}_ş) \text{ s.t. } \mathfrak{t}_ş = \max(\mathfrak{t}^1_ş, \mathfrak{t}^2_ş), \\ \text{where } (\mu, \mathfrak{t}^1_ş) \in \Theta_1(ş) \\ \text{and } (\mu, \mathfrak{t}^2_ş) \in \Theta_2(ş), \end{cases} \tag{1}$$

and

$$\xi_{p\eta q}(l(\neg ş')(\kappa)) = \min \begin{cases} \Omega_1(\neg ş) & \text{if } \neg ş \in ((\neg\Upsilon) - (\neg\check{D})) \cap q^{-1}(\neg ş') \\ \Omega_2(\neg ş) & \text{if } \neg ş \in ((\neg\check{D}) - (\neg\Upsilon)) \cap q^{-1}(\neg ş') \\ (\mu, \mathfrak{t}_{\neg ş}) \text{ s.t. } \mathfrak{t}_{\neg ş} = \min(\mathfrak{t}^1_{\neg ş}, \mathfrak{t}^2_{\neg ş}), \\ \text{where } (\mu, \mathfrak{t}^1_{\neg ş}) \in \Omega_1(\neg ş) \\ \text{and } (\mu, \mathfrak{t}^2_{\neg ş}) \in \Omega_2(\neg ş) \end{cases} \tag{2}$$

Next, for the non-trivial case, using Definition 3.17 and for $ş' \in S'$ and $\neg ş' \in \neg S'$, we have

$$\begin{aligned} &(\xi_{p\eta q}((\Theta_1, \Omega_1, S, N) \cup (\Theta_2, \Omega_2, S, N))) \\ &= \xi_{p\eta q}(\Theta_1, \Omega_1, S, N) \cup \xi_{p\eta q}(\Theta_2, \Omega_2, S, N) \\ &= \xi_{p\eta q}(\Theta_1(ş')(\kappa)) = \max \Theta_1(ş)(\mu) \cup \max \Theta_2(ş)(\mu) \\ &= \xi_{p\eta q}(\Theta_2(ş')(\kappa)), \end{aligned}$$

and

$$\xi_{p\eta q}(\Omega_1(\neg ş')(\kappa)) = \min \Omega_1(\neg ş)(\mu) \cup \min \Omega_2(\neg ş)(\mu) = \xi_{p\eta q}(\Omega_2(\neg ş')(\kappa))$$

$$= \xi_{p\eta q}(h(ş')(\kappa)) = \max \begin{cases} Theta_1(ş) & \text{if } ş \in (\Upsilon - \check{D}) \cap \eta^{-1}(ş') \\ \Theta_2(ş) & \text{if } ş \in (\check{D} - \Upsilon) \cap \eta^{-1}(ş') \\ (\mu, \mathfrak{t}_ş) \text{ s.t. } \mathfrak{t}_ş = \max(\mathfrak{t}^1_ş, \mathfrak{t}^2_ş), \\ \text{where } (\mu, \mathfrak{t}^1_ş) \in \Theta_1(ş) \text{ and } (\mu, \mathfrak{t}^2_ş) \in \Theta_2(ş), \end{cases} \tag{3}$$

and

$$= \xi_{p\eta q}(l(\neg ş')(\kappa)) = \min \begin{cases} \Omega_1(\neg ş) & \text{if } \neg ş \in ((\neg\Upsilon) - (\neg\check{D})) \cap q^{-1}(\neg ş') \\ \Omega_2(\neg ş) & \text{if } \neg ş \in ((\neg\check{D}) - (\neg\Upsilon)) \cap q^{-1}(\neg ş') \\ (\mu, \mathfrak{t}_{\neg ş}) \text{ s.t. } \mathfrak{t}_{\neg ş} = \min(\mathfrak{t}^1_{\neg ş}, \mathfrak{t}^2_{\neg ş}), \\ \text{where } (\mu, \mathfrak{t}^1_{\neg ş}) \in \Omega_1(\neg ş) \text{ and } (\mu, \mathfrak{t}^2_{\neg ş}) \in \Omega_2(\neg ş) \end{cases} \tag{4}$$

From Equations $(1-4)$, we have (3).

Theorem 3.20. For an *NBS*-mapping $\xi_{p\eta q} : ßŞ^N(\mho, S) \to ßŞ^N(\chi, S')$, the following are true.

(1) $\xi^{-1}_{p\eta q}((\psi_1, \omega_1, S', N)^c) = (\xi^{-1}_{p\eta q}(\psi_1, \omega_1, S', N))^c$ for every $(\psi_1, \omega_1, S', N) \in ßŞ^N(\chi, S')$.

(2) $\xi_{p\eta q}(\xi^{-1}_{p\eta q}(\psi_1, \omega_1, S', N)) \tilde{\subseteq} (\psi_1, \omega_1, S', N)$ for every $(\psi_1, \omega_1, S', N) \in ßŞ^N(\chi, S')$. If $\xi_{p\eta q}$ is surjective, the equality is satisfied.

(3) $(\Theta_1, \Omega_1, S, N) \tilde{\subseteq} \xi^{-1}_{p\eta q}(\xi_{p\eta q}(\Theta_1, \Omega_1, S, N))$ for every $(\Theta_1, \Omega_1, S, N) \in ßŞ^N(\mho, S)$. If $\xi_{p\eta q}$ is injective, the equality is satisfied.

Proof. We have proven (1). The remaining proofs adhere to analogous approaches.

(1) We will first prove $\xi_{p\eta q}^{-1}(\psi_1^c)=\xi_{p\eta q}^{-1}((\psi_1)^c)_{\eta^{-1}(S')}$ and $\xi_{p\eta q}^{-1}(\omega_1^c)=\xi_{p\eta q}^{-1}((\omega_1)^c)_{q^{-1}(\neg S')}$. For every $\text{ş}\in S$ and $\neg\text{ş}\in\neg S$, we have

$$\xi_{p\eta q}^{-1}((\psi_1)^c)_{\eta^{-1}(S')}(\text{ş})(\mu)=\begin{cases}\mho-\xi_{p\eta q}^{-1}(\psi_1)(\text{ş})(\mu),\\ \eta(\text{ş})\in S',\mu\in\mho,\\ \mho,\eta(\text{ş})\notin S'\end{cases}$$

$$=\begin{cases}\mho-(\mu,\xi_{p\eta q}^{-1}(\psi_1(\text{ş})(\mu)))\\ =\mho-\psi_1 p(\mu)\eta(\text{ş}),\eta(\text{ş})\in S'\\ \mho,\eta(\text{ş})\notin S',\end{cases}$$

and

$$\xi_{p\eta q}^{-1}((\omega_1)^c)_{q^{-1}(\neg S')}(\neg\text{ş})(\mu)=\begin{cases}\mho-\xi_{p\eta q}^{-1}(\omega_1)(\neg\text{ş})(\mu),\\ q(\neg\text{ş})\in S',\mu\in\mho,\\ \mho,q(\neg\text{ş})\notin S'\end{cases}$$

$$=\begin{cases}\mho-(\mu,\xi_{p\eta q}^{-1}(\omega_1(\neg\text{ş})(\mu)))\\ =\mho-\omega_1 p(\mu)q(\neg\text{ş}),\\ q(\neg\text{ş})\in S',\\ \mho,q(\neg\text{ş})\notin S'.\end{cases}$$

On the other side, for every $\text{ş}\in S$ and $\neg\text{ş}\in\neg S$,

$$\xi_{p\eta q}^{-1}((\psi_1)^c)(\text{ş})(\mu)=\begin{cases}\xi_{p\eta q}^{-1}(\chi-(\psi_1)(\text{ş})(\mu)),\\ \eta(\text{ş})\in S',\mu\in\mho,\\ \mho,\eta(\text{ş})\notin S'\end{cases}$$

$$=\begin{cases}\mho-(\mu,\xi_{p\eta q}^{-1}(\psi_1(\text{ş})(\mu)))\\ =\mho-\psi_1 p(\mu)\eta(\text{ş}),\eta(\text{ş})\in S'\\ \mho,\eta(\text{ş})\notin S'\end{cases},$$

and

$$\xi_{p\eta q}^{-1}((\omega_1)^c)(\neg\text{ş})(\mu)=\begin{cases}\xi_{p\eta q}^{-1}(\chi-(\omega_1)(\neg\text{ş})(\mu)),\\ q(\neg\text{ş})\in S',\mu\in\mho,\\ \mho,q(\neg\text{ş})\notin S'\end{cases}$$

$$=\begin{cases}\mho-(\mu,\xi_{p\eta q}^{-1}(\omega_1(\neg\text{ş})(\mu)))\\ =\mho-\omega_1 p(\mu)q(\neg\text{ş}),q(\neg\text{ş})\in S',\\ \mho,q(\neg\text{ş})\notin S'.\end{cases}$$

Consequently, $\xi_{p\eta q}^{-1}((\psi_1)^c)(\text{ş})(\mu)=\xi_{p\eta q}^{-1}((\psi_1)^c)_{\eta^{-1}(S')}(\text{ş})(\mu)$ and $\xi_{p\eta q}^{-1}((\omega_1)^c)(\neg\text{ş})(\mu)=\xi_{p\eta q}^{-1}((\omega_1)^c)_{q^{-1}(\neg S')}(\neg\text{ş})(\mu)$. Hence, $\xi_{p\eta q}^{-1}((\psi_1,\omega_1,S',N)^c)=\xi_{p\eta q}^{-1}((\psi_1^c,\omega_1^c,S',N))=\xi_{p\eta q}^{-1}(((\psi_1^c)_{\eta^{-1}(S')},(\omega_1^c)_{q^{-1}(\neg S')},S',N))=(\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N))^c$.

The proof is complete.

Theorem 3.21. For an *NBS*-mapping $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau_S^N)\to((\psi,\omega,S',N),\upsilon_{S'}^N)$, the following conditions are equal to each other

(1) $\xi_{p\eta q}$ is *NBS*-continuous;

(2) $\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N)\in\tau_S'^N,\forall(\psi_1,\omega_1,S',N)\in\upsilon_{S'}'^N$;

(3) $\overline{\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N)}\tilde{\subseteq}\xi_{p\eta q}^{-1}\left(\overline{(\psi_1,\omega_1,S',N)}\right)$, $\forall(\psi_1,\omega_1,S',N)\in\text{ẞ}\text{Ş}^N(\chi,S')$;

(4) $\xi_{p\eta q}\left(\overline{(\Theta_1,\Omega_1,S,N)}\right)\tilde{\subseteq}\overline{\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)}$, $\forall(\Theta_1,\Omega_1,S,N)\in\text{ẞ}\text{Ş}^N(\mho,S)$;

(5) $\xi_{p\eta q}^{-1}((\psi_1,\omega_1,S',N)^\circ)\tilde{\subseteq}\left(\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N)\right)^\circ$, $\forall(\psi_1,\omega_1,S',N)\in\text{ẞ}\text{Ş}^N(\chi,S')$.

Proof. (1) $\Rightarrow$ (2) Let $(\psi_1,\omega_1,S',N)\in\upsilon_{S'}'^N$. We will show that $\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N)\in\tau_S'^N$. Since $\xi_{p\eta q}$ is *NBS*-continuous, there exits $(\Theta_1,\Omega_1,S,N)\in\tau_S'^N$ such that $\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)\subseteq(\psi_1,\omega_1,S',N)$. Then $\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N)\in\tau_S'^N$.

(2) $\Rightarrow$ (1) Let $(\psi_2,\omega_2,S',N)\in\upsilon_{S'}'^N$. Then, $(\psi_1,\omega_1,S',N)\in\upsilon_{S'}'^N$ is an *NBS*-open set such that $(\psi_1,\omega_1,S',N)\tilde{\subseteq}(\psi_2,\omega_2,S',N)$. By (2) $\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N)\in\tau_S'^N$ and $\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N)\tilde{\subseteq}\xi_{p\eta q}^{-1}(\psi_2,\omega_2,S',N)$. This shows that $\xi_{p\eta q}^{-1}(\psi_2,\omega_2,S',N)\in\tau_S'^N$. Therefore, we have $\xi_{p\eta q}$ is *NBS*-continuous for every $(\psi_1,\omega_1,S',N)\in\upsilon_{S'}^N$.

(2) $\Rightarrow$ (3) Let (ψ_1,ω_1,S',N) be an *NBS*-set on (ψ,ω,S',N). Then $(\psi_1,\omega_1,S',N)\tilde{\subseteq}\overline{(\psi_1,\omega_1,S',N)}$. Therefore, we have $\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N)\tilde{\subseteq}\xi_{p\eta q}^{-1}\left(\overline{(\psi_1,\omega_1,S',N)}\right)$ and so, by using (2), we obtain that $\overline{\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N)}\tilde{\subseteq}\overline{\xi_{p\eta q}^{-1}\left(\overline{(\psi_1,\omega_1,S',N)}\right)}$ $\tilde{\subseteq}\xi_{p\eta q}^{-1}\left(\overline{(\psi_1,\omega_1,S',N)}\right)$. This shows $\overline{\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N)}\tilde{\subseteq}\xi_{p\eta q}^{-1}\left(\overline{(\psi_1,\omega_1,S',N)}\right)$.

(2) $\Rightarrow$ (4) Let (Θ_1,Ω_1,S,N) be an *NBS*-set on (Θ,Ω,S,N). Since $(\Theta_1,\Omega_1,S,N)\tilde{\subseteq}$ $\xi_{p\eta q}^{-1}(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))\tilde{\subseteq}\xi_{p\eta q}^{-1}\left(\overline{\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)}\right)\in\tau_S'^N$, we have $\overline{(\psi_1,\omega_1,S',N)}\tilde{\subseteq}$ $\xi_{p\eta q}^{-1}\left(\overline{\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)}\right)$. By Theorem 3.19 and Theorem 3.20, we get $\xi_{p\eta q}\left(\overline{(\Theta_1,\Omega_1,S,N)}\right)$ $\tilde{\subseteq}\overline{\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)}$.

(4) $\Rightarrow$ (5) If (ψ_1,ω_1,S',N) is an *NBS*-set over (ψ,ω,S',N), then $\xi_{p\eta q}^{-1}((\psi_1,\omega_1,S',N)^c)$ is an *NBS*-set on (Θ,Ω,S,N). From (4), Theorem 3.19(2) and Theorem 3.14(6),

$\xi_{p\eta q}(\overline{\xi_{p\eta q}^{-1}((\psi_1,\omega_1,S',N)^c)})\tilde{\subseteq}\overline{\xi_{p\eta q}(\xi_{p\eta q}^{-1}((\psi_1,\omega_1,S',N)^c))}$ $\tilde{\subseteq}\overline{(\psi_1,\omega_1,S',N)}=((\psi_1,\omega_1,S',N)^\circ)^c$.

Therefore, we have $\overline{\xi_{p\eta q}^{-1}((\psi_1,\omega_1,S',N)^c)}\tilde{\subseteq}\xi_{p\eta q}^{-1}(((\psi_1,\omega_1,S',N)^\circ)^c)=(\xi_{p\eta q}^{-1}((\psi_1,\omega_1,S',N)^\circ))^c$.

Since $\overline{\xi_{p\eta q}^{-1}((\psi_1,\omega_1,S',N)^c)}=\overline{(\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N))^c}=((\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N))^\circ)^c$, by Remark 2.18 we obtain that $\xi_{p\eta q}^{-1}((\psi_1,\omega_1,S',N)^\circ)\tilde{\subseteq}(\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N))^\circ$.

(5) $\Leftrightarrow$ (3) This follows from Theorem 3.20(1) and Theorem 3.14(6).

Theorem 3.22. If $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau_S^N)\to((\psi,\omega,S',N),\upsilon_{S'}^N)$ is an *NBS*-mapping, then the following conditions are equal to each other

(1) $\xi_{p\eta q}$ is *NBS*-open;

(2) $\xi_{p\eta q}((\Theta_1,\Omega_1,S,N)^\circ)\tilde{\subseteq}(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))^\circ$, $\forall(\Theta_1,\Omega_1,S,N)\in\text{ẞ}\text{Ş}^N(\mho,S)$;

(3) $\left(\xi_{p\eta q}^{-1}(\psi_1,\omega_1,S',N)\right)^\circ\tilde{\subseteq}\xi_{p\eta q}^{-1}((\psi_1,\omega_1,S',N)^\circ)$, $\forall(\psi_1,\omega_1,S',N)\in\text{ẞ}\text{Ş}^N(\chi,S')$.

(4) $\xi^{-1}_{p\eta q}\left(\overline{(\psi_1,\omega_1,S',N)}\right)\tilde{\subseteq}(\overline{\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)})$, $\forall(\psi_1,\omega_1,S',N)\in \text{ßŞ}^N(\chi,S')$.

Proof. (1) $\Rightarrow$ (2) Let (Θ_1,Ω_1,S,N) be an *NBS*-set on (Θ,Ω,S,N).

Then $(\Theta_1,\Omega_1,S,N)^\circ\tilde{\subseteq}(\Theta_1,\Omega_1,S,N)$. By using (1), we have $\xi_{p\eta q}((\Theta_1,\Omega_1,S,N)^\circ)\tilde{\subseteq}(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))^\circ$.

(2) $\Rightarrow$ (3) Let (ψ_1,ω_1,S',N) be an *NBS*-set on (ψ,ω,S',N). Then $\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)$ is an *NBS*-set on (Θ,Ω,S,N). By (2), $\xi_{p\eta q}((\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N))^\circ)\tilde{\subseteq}(\xi_{p\eta q}(\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)))^\circ$ $\tilde{\subseteq}(\psi_1,\omega_1,S',N)^\circ$. Therefore, we have $(\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N))^\circ\tilde{\subseteq}\xi^{-1}_{p\eta q}((\psi_1,\omega_1,S',N)^\circ)$.

(4) $\Leftrightarrow$ (3) These follow from Theorem 3.20(1) and Theorem 3.14(6).

(3) $\Rightarrow$ (1) Let (Θ_1,Ω_1,S,N) be an *NBS*-open set in (Θ,Ω,S,N).

Then for $\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)\in \text{ßŞ}^N(\chi,S')$, by (3) $(\xi^{-1}_{p\eta q}(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)))^\circ\tilde{\subseteq}$ $\xi^{-1}_{p\eta q}((\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))^\circ)$. Also, since $(\Theta_1,\Omega_1,S,N)=(\Theta_1,\Omega_1,S,N)^\circ$, $(\Theta_1,\Omega_1,E,N)\tilde{\subseteq}(\xi^{-1}_{p\eta q}(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)))^\circ$ $\tilde{\subseteq}\xi^{-1}_{p\eta q}((\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))^\circ)$ and so $\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)\tilde{\subseteq}(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))^\circ$. This shows that $\xi_{p\eta q}$ is *NBS*-open.

Theorem 3.23. Let $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,S',N),\upsilon^N_{S'})$ be an *NBS*-bijection. Then $\xi_{p\eta q}$ is *NBS*-continuous iff $(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))^\circ\tilde{\subseteq}$ $\xi_{p\eta q}((\Theta_1,\Omega_1,S,N)^\circ)$ for every $(\Theta_1,\Omega_1,S,N)\in \text{ßŞ}^N(\ddot{U},S)$.

Proof. ($\Rightarrow$) Let $(\Theta_1,\Omega_1,S,N)\in \text{ßŞ}^N(\mho,S)$. Then for $\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)\in \text{ßŞ}^N(\chi,S')$, $(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))^\circ\tilde{\subseteq}\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)$ and so $\xi^{-1}_{p\eta q}((\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))^\circ)\tilde{\subseteq}$ $\xi^{-1}_{p\eta q}(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))$. Since $\xi_{p\eta q}$ is injective and *NBS*-continuous, $\xi^{-1}_{p\eta q}((\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))^\circ)\tilde{\subseteq}(\Theta_1,\Omega_1,S,N)^\circ$. Again since $\xi_{p\eta q}$ is surjective, $(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))^\circ\tilde{\subseteq}\xi_{p\eta q}((\Theta_1,\Omega_1,S,N)^\circ)$ as claimed.

($\Leftarrow$) Let (ψ_1,ω_1,S',N) be an *NBS*-open set in χ. Then since $\xi_{p\eta q}$ is surjective, $(\psi_1,\omega_1,S',N)=(\psi_1,\omega_1,S',N)^\circ=(\xi_{p\eta q}(\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)))^\circ$. By using the hypothesis, $(\psi_1,\omega_1,S',N)\tilde{\subseteq}\xi_{p\eta q}((\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N))^\circ)$. Since $\xi_{p\eta q}$ is injective, $\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\tilde{\subseteq}(\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N))^\circ$. This shows that $\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)$ is *NBS*-open set in $\mho$.

Theorem 3.24. An *NBS*-mapping $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,S',N),\upsilon^N_{S'})$ is *NBS*-closed iff $\overline{\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)}\tilde{\subseteq}\xi_{p\eta q}\left(\overline{(\Theta_1,\Omega_1,S,N)}\right)$, $\forall(\Theta_1,\Omega_1,S,N)\in \text{ßŞ}^N(\mho,S)$.

Proof. Obvious.

Theorem 3.25. Let $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,S',N),\upsilon^N_{S'})$ be an *NBS*-bijection. Then $\xi_{p\eta q}$ is *NBS*-closed iff $\xi^{-1}_{p\eta q}\left(\overline{(\psi_1,\omega_1,S',N)}\right)\tilde{\subseteq}$ $(\overline{\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)})$, $\forall(\psi_1,\omega_1,S',N)\in \text{ßŞ}^N(\chi,S')$.

Proof. It is similar to that of Theorem 3.23.

Definition 3.26. An *NBS*-mapping $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,E',N),\upsilon^N_{S'})$ is called *NBS*-homeomorphism if $\xi_{p\eta q}$ is *NBS*-continuous, *NBS*-open, surjective and injective.

The next theorem will be obtained.

Theorem 3.27. If $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,S',N),\upsilon^N_{S'})$ is an *NBS*-mapping, then the following conditions are equal to each other

(1) $\xi_{p\eta q}$ is *NBS*-homeomorphism;

(2) $\xi_{p\eta q}((\Theta_1,\Omega_1,S,N)^\circ)=(\xi_{p\eta q}(\Theta_1,\Omega_1,S,N))^\circ,\forall(\Theta_1,\Omega_1,S,N)\in \text{ßŞ}^N(\mho,S)$;

(3) $(\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N))^\circ=\xi^{-1}_{p\eta q}((\psi_1,\omega_1,S',N)^\circ),\forall(\psi_1,\omega_1,S',N)\in \text{ßŞ}^N(\chi,S')$.

(4) $\xi^{-1}_{p\eta q}\overline{(\psi_1,\omega_1,S',N)}=\overline{\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)},\forall(\psi_1,\omega_1,S',N)\in \text{ßŞ}^N(\chi,S')$.

(5) $\xi_{p\eta q}\overline{(\Theta_1,\Omega_1,S,N)}=\overline{\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)},\forall(\Theta_1,\Omega_1,S,N)\in \text{ßŞ}^N(\mho,S)$.

4 N-bipolar soft mappings and separation axioms

In this part, we delve into the examination of various separation axioms that have been explored in [8] under *NBS*-continuous, *NBS*-open, and *NBS*-closed mappings. Furthermore, novel characterizations are provided for them.

Theorem 4.1. If $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,S',N),\upsilon^N_{S'})$ is *NBS*-continuous injection and $((\psi,\omega,S',N),\upsilon^N_{S'})$ is $NBST_0$, then $((\Theta,\Omega,S,N),\tau^N_S)$ is $NBST_0$-space.

Proof. Suppose that $((\psi,\omega,S',N),\upsilon^N_{S'})$ is $NBST_0$. For any distinct points μ_1 and μ_2 in (Θ,Ω,S,N), there exists *NBS*-open sets (ψ_1,ω_1,S',N), (ψ_2,ω_2,S',N) in (ψ,ω,S',N) such that $p(\mu_1)\in(\psi_1,\omega_1,S',N),p(\mu_2)\notin(\psi_1,\omega_1,S',N)$ or $p(\mu_1)\notin(\psi_2,\omega_2,S',N),p(\mu_2)\in(\psi_2,\omega_2,S',N)$. Since $\xi_{p\eta q}$ is *NBS*-continuous, $\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)$ and $\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N)$ are *NBS*-open sets in (ψ,ω,S',N).

Furthermore, it is apparent that $\mu_1\in\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N),\mu_2\notin(\psi_1,\omega_1,S',N)$ or $\mu_1\notin\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N),\mu_2\in\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N)$. This shows that $((\Theta,\Omega,S,N),\tau^N_S)$ is $NBST_0$.

Theorem 4.2. If $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\to((\psi,\omega,S',N),\upsilon^N_{S'})$ is $N-$bipolar soft continuous injection and

$((\psi,\omega,S',N),\upsilon^N_{S'})$ is $NBST_1$, then $((\Theta,\Omega,S,N),\tau^N_S)$ is $NBST_1$-space.

Proof. Similar to Theorem 4.1.

Theorem 4.3. If $\xi_{p\eta q}:((\Theta,\Omega,S,N),\tau^N_S)\rightarrow((\psi,\omega,S',N),\upsilon^N_{S'})$ is NBS-continuous injection and $((\psi,\omega,S',N),\upsilon^N_{S'})$ is $NBST_2$, then $((\Theta,\Omega,S,N),\tau^N_S)$ is $NBST_2$-space.

Proof. For $\mu_1,\ \mu_2\in(\Theta,\Omega,S,N)$ with $\mu_1\neq\mu_2$, there exist disjoint NBS-open sets (ψ_1,ω_1,S',N) and (ψ_2,ω_2,S',N) in (ψ,ω,S',N) where $p(\mu_1)\in(\psi_1,\omega_1,S',N), p(\mu_2)\in(\psi_2,\omega_2,S',N)$. Since $\xi_{p\eta q}$ is NBS-continuous, $\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)$ and $\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N)$ are NBS-open in (Θ,Ω,S,N) containing μ_1 and μ_2 respectively. Moreover, it is clear that $\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\cap\xi^{-1}_{p\eta q}(\psi_2,\omega_2,S',N)=\phi$. This shows that $((\Theta,\Omega,S,N),\tau^N_S)$ is $NBST_2$.

Theorem 4.4. If $\xi_{p\eta q}$ is NBS-open function from an $NBST_0$-space $((\Theta,\Omega,S,N),\tau^N_S)$ onto an $NBST_S$ $((\psi,\omega,S',N),\upsilon^N_{S'})$, then $((\psi,\omega,S',N),\upsilon^N_{S'})$ is $NBST_0$-space.

Proof. Let $\kappa_1,\ \kappa_2\in(\psi,\omega,S',N)$ with $\kappa_1\neq\kappa_2$. Since p is surjective, there exist $\mu_1,\ \mu_2\in(\Theta,\Omega,S,N)$ with $\mu_1\neq\mu_2$ such that $p(\mu_1)=\kappa_1$ and $p(\mu_2)=\kappa_2$. Again since $((\Theta,\Omega,S,N),\tau^N_S)$ is $NBST_0$-space, there exists NBS-open sets (Θ_1,Ω_1,S,N),$(\Theta_2,\Omega_2,S,N)\in\mho$ such that $\mu_1\in(\Theta_1,\Omega_1,S,N),\ \mu_2\notin(\Theta_1,\Omega_1,S,N)$ or $\mu_1\notin(\Theta_2,\Omega_2,S,N),\mu_2\in(\Theta_2,\Omega_2,S,N)$. Then $\xi_{p\eta q}$ (Θ_1,Ω_1,S,N) and $\xi_{p\eta q}$ (Θ_2,Ω_2,S,N) are NBS-open sets in (ψ,ω,S',N). Because $\xi_{p\eta q}$ is NBS-open.

Furthermore, it is clear that $\kappa_1\in\xi_{p\eta q}(\Theta_1,\Omega_1,S,N),\kappa_2\notin\xi_{p\eta q}$ (Θ_1,Ω_1,S,N) or $\kappa_1\notin\xi_{p\eta q}(\Theta_2,\Omega_2,S,N),\kappa_2\in\xi_{p\eta q}(\Theta_2,\Omega_2,S,N)$. This shows that $((\psi,\omega,S',N),\upsilon^N_{S'})$ is $NBST_0$-space.

Theorem 4.5. If $\xi_{p\eta q}$ is NBS-open function from an $NBST_1$-space $((\Theta,\Omega,S,N),\tau^N_S)$ onto an $NBST_S$ $((\psi,\omega,S',N),\upsilon^N_{S'})$, then $((\psi,\omega,S',N),\upsilon^N_{S'})$ is $NBST_1$-space.

Proof. Similar to Theorem 4.4.

Theorem 4.6. If an NBS-open function $\xi_{p\eta q}$ from an $NBST_2$-space $((\Theta,\Omega,S,N),\tau^N_S)$ onto an $NBST_S$ $((\psi,\omega,S',N),\upsilon^N_{S'})$ is injective, then $((\psi,\omega,S',N),\upsilon^N_{S'})$ is $NBST_2$-space.

Proof. The proof is clear and direct.

Definition 4.7. Let $((\Theta,\Omega,S,N),\tau^N_S)$ be an $NBST_S$ over (Θ,Ω,S,N), (Θ_1,Ω_1,S,N) be an NBS-closed set in (Θ,Ω,S,N) and $\mu\in(\Theta,\Omega,S,N)$ such that $\mu\notin(\Theta_1,\Omega_1,S,N)$. If there exist NBS-open sets (Θ_2,Ω_2,S,N) and (Θ_3,Ω_3,S,N) such that $\mu\in(\Theta_2,\Omega_2,S,N)$, $(\Theta_1,\Omega_1,S,N)\tilde{\subseteq}$ (Θ_3,Ω_3,S,N) and $(\Theta_2,\Omega_2,S,N)\cap(\Theta_3,\Omega_3,S,N)=\phi$, then $((\Theta,\Omega,S,N),\tau^N_S)$ is called an NBS-regular space. If $((\Theta,\Omega,S,N),\tau^N_S)$ is NBS-regular and $NBST_1$-space, then it is $NBST_3$-space.

Theorem 4.8. If $\xi_{p\eta q}$ is NBS-continuous and NBS-open bijection from an NBS-regular space $((\Theta,\Omega,S,N),\tau^N_S)$ to an $NBST_S-$ $((\psi,\omega,S',N),\upsilon^N_{S'})$, then $((\psi,\omega,S',N),\upsilon^N_{S'})$ is NBS-regular.

Proof. Let $\kappa\in(\psi,\omega,S',N)$ and $\kappa\notin(\psi,\omega,S',N)\in\upsilon'^N_{S'}$. Since p is surjective, there exists $\mu\in(\Theta,\Omega,S,N)$ with $p(\mu)=\kappa$. Since $\xi_{p\eta q}$ is NBS-continuous, $\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\in\tau'^N_S$ and $\mu\notin\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)$. By NBS-regularity of $((\Theta,\Omega,S,N),\tau^N_S)$, there exist disjoint NBS-open sets (Θ_1,Ω_1,S,N) and (Θ_2,Ω_2,S,N) such that $\mu\in(\Theta_1,\Omega_1,S,N)$, $\xi^{-1}_{p\eta q}(\psi_1,\omega_1,S',N)\tilde{\subseteq}$ (Θ_2,Ω_2,S,N). Thus, we obtain disjoint NBS-open sets $\xi_{p\eta q}(\Theta_1,\Omega_1,S,N)$ and $\xi_{p\eta q}(\Theta_2,\Omega_2,S,N)$ such that $\kappa\in\xi_{p\eta q}$ (Θ_1,Ω_1,S,N) and $(\psi_1,\omega_1,S',N)\tilde{\subseteq}\xi_{p\eta q}$ (Θ_2,Ω_2,S,N). Because $\xi_{p\eta q}$ is bijective and NBS-open. Thus, $((\psi,\omega,S',N),\upsilon^N_{S'})$ is NBS-regular.

Corollary 4.9. If $\xi_{p\eta q}$ is NBS-continuous and NBS-open bijection from an $NBST_3$-space $((\Theta,\Omega,S,N),\tau^N_S)$ to an $NBST_S$ $((\psi,\omega,S',N),\upsilon^N_{S'})$, then $((\psi,\omega,S',N),\upsilon^N_{S'})$ is $NBST_3$-space.

Definition 4.10. Let $((\Theta,\Omega,S,N),\tau^N_S)$ be an $NBST_S$ over (Θ,Ω,S,N). (Θ_1,Ω_1,S,N), $(\Theta_2,\Omega_2,S,N)\in(\Theta,\Omega,S,N)$ are NBS-closed sets where $(\Theta_1,\Omega_1,S,N)\cap(\Theta_2,\Omega_2,S,N)=\phi$. If there exist NBS-open sets (Θ_3,Ω_3,S,N) and (Θ_4,Ω_4,S,N) such that $(\Theta_1,\Omega_1,S,N)\tilde{\subseteq}(\Theta_3,\Omega_3,S,N)$, $(\Theta_2,\Omega_2,S,N)\tilde{\subseteq}(\Theta_4,\Omega_4,S,N)$ and $(\Theta_3,\Omega_3,S,N)\cap(\Theta_4,\Omega_4,S,N)=\phi$, then

$((\Theta,\Omega,S,N),\tau^N_S)$ is called an NBS-normal space. If $((\Theta,\Omega,S,N),\tau^N_S)$ is NBS-normal and $NBST_1$-space, then it is an $NBST_4$-space.

Theorem 4.11. If $\xi_{p\eta q}$ is NBS-continuous and NBS-open bijection from an NBS-normal space $((\Theta,\Omega,S,N),\tau^N_S)$ to an $NBST_S$ $((\psi,\omega,S',N),\upsilon^N_{S'})$, then $((\psi,\omega,S',N),\upsilon^N_{S'})$ is NBS-normal.

Proof. Similar to that of Theorem 4.8.

Corollary 4.12. If $\xi_{p\eta q}$ is NBS-continuous and NBS-open bijection from an $NBST_4$-space $((\Theta,\Omega,S,N),\tau^N_S)$ to an $NBST_S$ $((\psi,\omega,S',N),\upsilon^N_{S'})$, then $((\psi,\omega,E',N),\upsilon^N_{S'})$ is $NBST_4$-space.

Theorem 4.13. $((\Theta,\Omega,S,N),\tau^N_S)$ is NBS-regular space iff for every $\mu\in(\Theta,\Omega,S,N)$ and every NBS-open set (Θ_1,Ω_1,S,N) with $\mu\in(\Theta_1,\Omega_1,S,N)$, there exists an NBS-open set (ϖ,σ,S,N) such that $\mu\in(\varpi,\sigma,S,N)\tilde{\subseteq}\overline{(\varpi,\sigma,S,N)}$
$\tilde{\subseteq}(\Theta_1,\Omega_1,S,N)$.

Proof. Let $((\Theta,\Omega,S,N),\tau^N_S)$ is NBS-regular, (Θ_1,Ω_1,S,N) is NBS-open in (Θ,Ω,S,N) and $\mu\in(\Theta_1,\Omega_1,S,N)$. Then $\mu\notin(\Theta_1,\Omega_1,S,N)^c$ and $(\Theta_1,\Omega_1,S,N)^c$ is an NBS-closed set. Therefore, NBS-open disjoint sets (ϖ,σ,S,N) and (Θ_2,Ω_2,S,N) can be found with $\mu\in(\varpi,\sigma,S,N)$ and $(\Theta_1,\Omega_1,S,N)^c\tilde{\subseteq}(\Theta_2,\Omega_2,E,N)$. Then $(\Theta_2,\Omega_2,S,N)^c$ is NBS-closed set containing (ϖ,σ,S,N) and contained in (Θ_1,Ω_1,S,N). It means that $\mu\in(\varpi,\sigma,S,N)\tilde{\subseteq}\overline{(\varpi,\sigma,E,N)}\tilde{\subseteq}(\Theta_1,\Omega_1,S,N)$. To prove the opposite direction, let $\mu\notin(\Theta_2,\Omega_2,S,N)$ which is the NBS-closed set. Suppose there is an NBS-open set (ϖ,σ,S,N) such that $\mu \in (\varpi,\sigma,S,N)\tilde{\subseteq} \overline{(\varpi,\sigma,S,N)}\tilde{\subseteq}(\Theta_2,\Omega_2,S,N)^c$. The NBS-open sets (ϖ,σ,S,N) and $(\varpi,\sigma,S,N)^c$ are disjoint NBS-open sets that contain μ and (Θ_2,Ω_2,S,N), respectively.

Theorem 4.14. $((\Theta,\Omega,S,N),\tau_S^N)$ is NBS-normal space iff for every NBS-closed set (Θ_2,Ω_2,S,N) and every NBS-open set (Θ_1,Ω_1,S,N) with $(\Theta_2,\Omega_2,S,N)\tilde{\subseteq}(\Theta_1,\Omega_1,S,N)$, there exists an NBS-open set (ϖ,σ,S,N) such that $(\Theta_2,\Omega_2,S,N)\tilde{\subseteq}$

$(\varpi,\sigma,S,N)\tilde{\subseteq} \overline{(\varpi,\sigma,S,N)}\tilde{\subseteq} (\Theta_1,\Omega_1,S,N)$.

Proof. The argument presented in this proof remains consistent but with one key modification. Replacing the point μ by an NBS-set (Θ_2,Ω_2,S,N) in its place.

5 N-bipolar soft mappings in medical diagnosis

The new approach we propose involves utilizing NBS-mappings to establish a relationship between diseases and their symptoms. By employing this methodology, we aim to improve disease diagnosis. In this approach, diseases are characterized by a set of symptoms. These symptoms vary in their intensity and can be graded on an N-bipolar scale, indicating both positive and negative evaluations. By mapping the relationship between diseases and symptoms on this scale, we can represent the complex nature of disease symptoms more accurately.

By utilizing soft mappings, we can capture the gradual transition of symptoms from positive to negative values. This allows for a more nuanced understanding of how symptoms may manifest in different diseases. Additionally, the use of N-bipolar scales allows for the incorporation of uncertainty and ambiguity in symptom evaluation.

To apply this approach to disease diagnosis, we can develop a database that stores the mappings between diseases and their symptoms. This database can be populated through expert knowledge or by analyzing medical records. When a patient presents with a set of symptoms, we can compare their symptom profile with the established mappings to identify potential diseases.

By incorporating the concept of NBS-mappings, our approach provides a more comprehensive representation of the relationship between diseases and symptoms. This can lead to more accurate and personalized disease diagnoses, ultimately improving patient care and outcomes.

To set up this mathematical system, we can define a set of linguistic variables for each symptom and assign numerical values that can be readily associated with numerical representations, such that

No holds for ”0”,
Rare holds for ”1”,
Mild holds for ”2”,
Sometimes holds for ”3”,
Common holds for ”4”.

Also, in light of the symptoms given in Table 3, the doctor’s opinion and the website’s information https://www.who.int/news-room/fact-sheets/detail/coronavirus-disease-(covid-19), we classify the symptoms that the patient has as low significance, middle significance, high significance and very high significance. Therefore, we create a $5BS$-mapping to document the relationship between the disease and its symptoms as follows

Table 3 Comparison of symptoms

Symptoms	Cold	Influenza	Covid-19	Omicron
Fatigue	Sometimes	Common	Common	Common
Fever	Common	Common	Common	Common
Cough	Common	Common	Common	Common
Diarrhea	Mild	Mild	Sometimes	Common
Taste loss or Smell	Rare	Rare	Sometimes	Rare
Throat ofSore	Common	Sometimes	Sometimes	Common
Breath Shortness	Rare	Sometimes	Common	Common
Watery eyes or Itchy	No	No	Rare	Sometimes
Painsand Bodyaches	Sometimes	Common	Sometimes	Sometimes

Next, we can define an $N-$bipolar soft mapping that takes these symptom values as inputs and computes a value indicating the likelihood of the patient having **OMICRON**. This mapping can be designed based on the doctor’s expertise, statistical analysis, or machine learning algorithms. The mapping can take into account the symptoms and their severity levels to assign a likelihood value.

To determine the patient’s status utilizing an NBS-mappings-based algorithm.

Step 1 : Categorize the patient’s symptoms into categories of very high significance, high significance, middle significance, or low significance.

Step 2 : Construct a $5BS$-set $(\wp,\Omega,S,5)$ based on the patient’s symptoms.

Step 3 : Find the $5BS$-image of $(\wp,\Omega,S,N)$ under the $5BS$-mapping $\xi_{p\eta q} : ßȘ^5(\mho,S) \to ßȘ^5(\chi,S')$. This involves applying a mapping function to the $5BS$-set to obtain a transformed set.

Step 4 : Calculate the score $(\wp,S,5)$.

Step 5 : Calculate the score $(\Omega,\neg S,5)$.

Step 6 : Calculate the bipolar score $(\wp,\Omega,S,5) = (\wp,S,5) - (\Omega,\neg S,5)$ which could involve evaluating the significance levels and conditions in the $5BS$-set.

Step 7 : Decide the patient’s condition by utilizing the information obtained from steps 1,2, and 3. This may involve making a diagnosis based on the calculated scores and determining the severity or type of condition the patient may have.

Let $\mho = \chi = \{F, FA, \text{ß}, \text{Ç}, \text{ŢŞ}, P\text{ß}, \text{ŦĦ}, \check{D}, \hat{W}\tilde{I}\}$, $S = \{V\text{Ħ}, \text{Ħ}, M, \acute{L}\breve{O}W\}$ and $S' = \{\text{Ə}\}$ where

$$F = \text{Fever}, FA = \text{Fatigue}, \text{ß} = \text{ Breath shortness},$$
$$\text{Ç} = \text{Cough}, \ \text{ŢŞ} = \text{Taste loss or Smell},$$
$$P\text{ß} = \text{Pains and Body aches}, \text{ŦĦ} = \text{Throat of Sore},$$
$$\check{D} = \text{Diarrhea}, \hat{W}\tilde{I} = \text{Watery eyes or Itchy},$$

and

$$V\text{Ħ} = \text{Very high significance}, \text{Ħ} = \text{High significance},$$
$$M = \text{Middle significance}, \ \acute{L}\breve{O}W = \text{Low significance},$$
$$\text{Ə} = \text{Disease}.$$

Thus, we define a $5BS$-mapping $\xi_{p\eta q} : \text{ßŞ}^5(\mho, S) \to \text{ßŞ}^5(\chi, S')$ by the mappings $p : \mho \to \chi, \eta : S \to S'$ and $q : \neg S \to \neg S'$ with

$$p(\mu) = \mu$$
$$\eta(V\text{Ħ}) = \text{Ə} \quad \eta(\text{Ħ}) = \text{Ə} \quad \eta(M) = \text{Ə} \quad \eta(\acute{L}\breve{O}W) = \text{Ə},$$
$$q(\neg V\text{Ħ}) = \neg\text{Ə} \quad q(\neg\text{Ħ}) = \neg\text{Ə} \quad q(\neg M) = \neg\text{Ə} \quad q(\neg\acute{L}\breve{O}W) = \neg\text{Ə},$$

for all $\mu \in \mho$.

Now, depending on the patient's symptoms, we create $5BS$-sets $(\wp, \Omega, S, 5)$ such that the patient has the following symptoms:

- Symptom A: Very high significance
- Symptom B: High significance
- Symptom C: Middle significance
- Symptom D: Low significance.

According to the given grading system, we assign the following grades:

- Symptom A: 4
- Symptom B: 3
- Symptom C: 2
- Symptom D: 1.

We then calculate the diagnostic score, which is defined as

$$\text{Score}(\wp, S, 5) = \sum_{\alpha \in S, \kappa \in \chi} \xi_{p\eta q}(\wp(\alpha))(\text{Ə})(\kappa),$$

$$\text{Score}(\Omega, \neg S, 5) = \sum_{\neg\alpha \in \neg S, \kappa \in \chi} \xi_{p\eta q}(\Omega(\neg\alpha))(\text{Ə})(\kappa),$$

$$\begin{aligned} \text{Score}(\wp, \Omega, S, 5) &= (\wp, S, 5) - (\Omega, \neg S, 5) \\ &= \sum_{\alpha \in S, \kappa \in \chi} \xi_{p\eta q}(\wp(\alpha))(\text{Ə})(\kappa) \\ &\quad - \sum_{\neg\alpha \in \neg S, \kappa \in \chi} \xi_{p\eta q}(\Omega(\neg\alpha))(\text{Ə})(\kappa) \end{aligned}$$

Based on the given deductions, we can conclude the following:

- If the score $(\wp, \Omega, S, 5)$ is less than or equal to 12, the patient is suffering from a COLD.
- If the score $(\wp, \Omega, S, 5)$ is greater than 12 and less than or equal to 16, the patient is suffering from INFLUENZA.
- If the score $(\wp, \Omega, S, 5)$ is greater than 16 and less than or equal to 22, the patient is suffering from COVID-19.
- If the score $(\wp, \Omega, S, 5)$ is greater than 22, the patient is suffering from **OMICRON**.

Therefore, given the aforementioned analysis, we are able to suggest an algorithm that relies on N-bipolar soft mappings as previously explained. To exemplify how this approach is employed, we will consider a case using the symptoms presented in Table 3. In this case, we will classify the patient's symptoms in the following manner:

$$V\text{Ħ} = \{F, FA, \text{ß}, P\text{ß}, \text{Ç}\}, \text{Ħ} = \{\text{ŦĦ}, \check{D}\},$$
$$M = \{\text{ŢŞ}\}, \acute{L}\breve{O}W = \{\hat{W}\tilde{I}\}.$$

Then we find a $5BS$-set $(\wp, \Omega, S, 5)$ such that

$$\begin{aligned} (\wp, \Omega, S, 5) = \Bigg\{ & \Bigg(\left\langle \begin{matrix} V\text{Ħ}, \{(F,4), (FA,4), (\text{ß},4), (\text{Ç},4), \\ (\text{ŢŞ},0), (P\text{ß},4), (\text{ŦĦ},0), (\check{D},0), (\hat{W}\tilde{I},0)\} \end{matrix} \right\rangle, \\ & \left\langle \begin{matrix} \neg V\text{Ħ}, \{(F,0), (FA,0), (\text{ß},0), (\text{Ç},0), \\ (\text{ŢŞ},4), (P\text{ß},0), (\text{ŦĦ},4), (\check{D},4), (\hat{W}\tilde{I},4)\} \end{matrix} \right\rangle \Bigg), \\ & \Bigg(\left\langle \begin{matrix} \text{Ħ}, \{(F,0), (FA,0), (\text{ß},0), (\text{Ç},0), \\ (\text{ŢŞ},0), (P\text{ß},0), (\text{ŦĦ},3), (\check{D},3), (\hat{W}\tilde{I},0)\} \end{matrix} \right\rangle, \\ & \left\langle \begin{matrix} \neg\text{Ħ}, \{(F,4), (FA,4), (\text{ß},4), (\text{Ç},1), \\ (\text{ŢŞ},4), (P\text{ß},4), (\text{ŦĦ},1), (\check{D},1), (\hat{W}\tilde{I},4)\} \end{matrix} \right\rangle \Bigg), \\ & \Bigg(\left\langle \begin{matrix} M, \{(F,0), (FA,0), (\text{ß},0), (\text{Ç},0), \\ (\text{ŢŞ},2), (P\text{ß},0), (\text{ŦĦ},0), (\check{D},0), (\hat{W}\tilde{I},0)\} \end{matrix} \right\rangle, \\ & \left\langle \begin{matrix} \neg M, \{(F,4), (FA,4), (\text{ß},4), (\text{Ç},4), \\ (\text{ŢŞ},0), (P\text{ß},4), (\text{ŦĦ},4), (\check{D},4), (\hat{W}\tilde{I},4)\} \end{matrix} \right\rangle \Bigg), \\ & \Bigg(\left\langle \begin{matrix} \acute{L}\breve{O}W, \{(F,0), (FA,0), (\text{ß},0), (\text{Ç},0), \\ (\text{ŢŞ},0), (P\text{ß},0), (\text{ŦĦ},0), (\check{D},0), (\hat{W}\tilde{I},1)\} \end{matrix} \right\rangle, \\ & \left\langle \begin{matrix} \neg\acute{L}\breve{O}W, \{(F,4), (FA,4), (\text{ß},4), (\text{Ç},4), \\ (\text{ŢŞ},4), (P\text{ß},4), (\text{ŦĦ},4), (\check{D},4), (\hat{W}\tilde{I},3)\} \end{matrix} \right\rangle \Bigg) \Bigg\}, \end{aligned}$$

and embed the 5−bipolar soft image of $(\wp, \Omega, S, N)$ under the 5−bipolar soft mapping $\xi_{p\eta q} : \text{ßŞ}^5(\mho, S) \to \text{ßŞ}^5(\chi, S')$ is obtained as the following:

$$\begin{aligned} \xi_{p\eta q}(\wp, \Omega, S, 5) &= (\xi_{p\eta q}(\wp(\alpha)), \xi_{p\eta q}(\Omega(\neg\alpha)), S', 5) \\ &= \Bigg\{ \Bigg(\left\langle \begin{matrix} \text{Ə}, \{(F,4), (FA,4), (\text{ß},4), (P\text{ß},4), (\text{Ç},4), \\ (\text{ŦĦ},3), (\check{D},3), (\text{ŢŞ},2), (\hat{W}\tilde{I},1)\} \end{matrix} \right\rangle, \\ &\quad , \left\langle \begin{matrix} \neg\text{Ə}, \{(F,0), (FA,0), (\text{ß},0), (P\text{ß},0), (\text{Ç},0), \\ (\text{ŦĦ},1), (\check{D},1), (\text{ŢŞ},0), (\hat{W}\tilde{I},3)\} \end{matrix} \right\rangle \Bigg) \Bigg\}. \end{aligned}$$

Thus, from the fact that the score

$$\begin{aligned} \text{Score}(\wp, \Omega, S, 5) &= (\wp, S, 5) - (\Omega, \neg S, 5) \\ &= \sum_{\alpha \in S, \kappa \in \chi} \xi_{p\eta q}(\wp(\alpha))(\text{Ə})(\kappa) \\ &\quad - \sum_{\neg\alpha \in \neg S, \kappa \in \chi} \xi_{p\eta q}(\Omega(\neg\alpha))(\text{Ə})(\kappa) \\ &= 29 - 5 = 24, \end{aligned}$$

it follows that the patient is suffering from OMICRON.

6 Conclusion

In the present study, we carried out a comprehensive examination of NBS-mappings and explored the distinct characteristics of NBS-continuous, NBS-closed, and NBS-open mappings within the realm of $NBST_{Ss}$. Our analysis resulted in new characterizations for these mappings and allowed us to investigate their preservation capabilities. We expect that the discoveries made in this study will lay the groundwork for future implementations of *NBS*-mappings within the field of soft sets theory. Additionally, we introduced a novel **OMICRON** diagnostic model within the framework of *NBS*-mappings.

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