Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110203

Review on Recent Advances in Fractional Differentiation and its Applications

Mohamed Hafez[1], *Faisal Alshowaikh*[2,*], *Betty Wan Niu Voon*[3], *Shawkat Alkhazaleh*[4] *and Hussein Al-Faiz*[3]

[1] Faculty of Engineering and Quantity Surviving, INTI International University Colleges, Nilai, Malaysia
[2] Department of Mathematical Sciences, College of Arts and Science, Ahlia University, 10878, Manama, Kingdom of Bahrain
[3] Department of Civil Engineering, (COE), Universiti Tenaga Nasional, Kajang, Malaysia
[4] Department of Mathematics, Faculty of Sciences, Jadara University, Jordan

Received: 2 Aug. 2024, Revised: 18 Aug. 2024, Accepted: 20 Sep. 2024
Published online: 1 April. 2025

Abstract: This article comprehensively reviews recent developments in fractional differentiation, focusing on its mathematical basis and applications in many fields. Fractional differentiation extends traditional mathematics to include non-integer order derivatives to provide improved complex systems modeling capabilities. The introduction explains the importance of fractional calculus in both theoretical and practical terms. The article presents fractional differentiation, discusses its characteristics, traces its historical development, and provides background information. The article then examines the basic fractional computation operators, including the Riemann-Liouville and Caputo operators, essential to understanding the statistical process. Moreover, we focus on numerical methods for phase differences, describing finite difference schemes and spectral methods that facilitate computational applications. These methods are developed through examples of physical phenomena, such as wave propagation and effects, and signaling functions and systems used in the industrial sector. They determine how differentiation is applied to the relevant industrial context. In addition, the article examines the impact on the ecosystem wom and medicine, especially biomechanics and biomedical imaging. A good-sized emphasis is placed on the rising position of fractional differentiation in device getting to know, showcasing its ability to enhance algorithmic performance. Finally, the object addresses modern challenges and describes future instructions for research, emphasizing the need for further exploration and innovation in this dynamic area. By synthesizing current findings, this review aims to provide a valuable aid for researchers and practitioners interested in fractional differentiation's theoretical and sensible implications. .

Keywords: Fractional differentiation; Fractional Operators; Spectral methods; Fractional Applications

1 Introduction

Since the introduction of classical calculus, investigation in this domain has significantly advanced the fields of science and engineering [1,2,3,4]. However, classical calculus cannot accurately depict the dynamics of a system in several instances [5]. Fractional differentiation offers a mathematical framework for effectively modeling both regular and irregular systems [6]. In recent years, there has been a growing curiosity about fractional calculus in both developed and developing countries, including Singapore and Malaysia, due to its extension of traditional calculus and its potential applications across diverse scientific domains such as control theory, robotics, signal processing, thermodynamics, electromagnetism, viscoelasticity, fluid dynamics, anomalous diffusion, biophysics, and finance [7,8,9]. Prospective advancements in fractional differentiation applications now inspire a prevailing feeling of enthusiasm. Our main goal is to provide a summary of the latest developments in this swiftly evolving domain. These advancements may appeal to a broader audience beyond mathematics, as seen by discussions across several platforms. We recommend that academics, scientists, and students examine the historical evolution of this topic.

This debate's engaging and informative nature, ultimately leading to the fractional differentiation theory, organized this article to make it understandable for readers from diverse scientific disciplines with varying degrees of knowledge.

* Corresponding author e-mail: Falshowaikh@ahlia.edu.bh

2 Fundamentals of Fractional Differentiation

Differential calculus has captured the needs of mathematical analysis and other fields for centuries. After decades, it began to generalize its concepts, ultimately leading to the fractional differentiation theory. Standard definitions and notations differ from writer to writer, but all agree that it allowed us to fill the gap between calculus and other fractional theories. Intuitively, fractional differentiation elevates a function of a fraction in the fractional order of a derivative. By laboriously and precisely satisfying the inequalities, uncertainty, assumptions, and continuous processes required, fractional derivation tries to capture the incremental change in a function over a given period. Once the function to be derived and the value n in dn(tn) are set, universally used notations of translation from integral to fractional calculation can be found.

The theory of fractional differentiation (non-integer order) is essential for a new field of mathematics. It can express the extent of modifying a given function from one place to another. This theory can also be considered a fundamental tool in the theory of fractional equations. It can produce the basis for other fields, such as integral transforms, namely the fractional-order Laplace and Fourier transforms. In addition, research on fractional differentiation is ongoing, particularly in the fields of process identification and system identification [10,11,12,13].

Fractional differentiation is the process of generating non-integer-order derivatives. It has many properties; some of them are similar to those of standard derivatives, whereas others are not. The properties are linearity, continuity, modality, reduction of order, Leibniz, and the chain rule. One of the important problems in standard calculus is solving differential and algebraic equations and boundary value problems. Similarly, the fractional differential and integral equations have similar problems. These problems are still not solved systematically.

Lately, after a vast amount of empirical and natural evidence in applications, the fractional description of the operator became famous for signal processing, treatment, robot control, vibration reduction, and so on. However, the active efforts towards establishing the theory revealed that the fractional derivative has intricate behavior. Different ideas were subsequently merged to cover the complete picture of the new science. It works by revealing a large amount of integral theory embedded into it due to the different impasses faced by the mathematical community, which also have their solutions in the new window [11,14,15].

2.1 Definition and Properties

Fractional calculus provides a variety of definitions for the derivative of real order. The most straightforward definitions are Riemann-Liouville, Weyl, and Caputo types of fractional derivatives. The Riemann-Liouville type of fractional derivative is the non-local representation of the derivative. In this section, we use the Riemann-Liouville type of fractional derivative. Several authors introduced different definitions for the fractional derivatives. For example [5,11,16]:

i. Left and right Riemann-Liouville

$$D_a^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} x(\tau)\, d\tau, \tag{1}$$

$$D_b^\alpha x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (\tau-t)^{n-\alpha-1} x(\tau)\, d\tau, \tag{2}$$

ii. Regularized Liouville

$$D^\alpha x(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left[x(t-\tau) - \sum_{m=0}^{N-1} \frac{(-1^m x^{(m)}(t))}{m!} \tau^m \right] d\tau, \quad N = \lfloor \alpha \rfloor + 1, \tag{3}$$

iii. Atangana-Baleanu Riemann-Liouville type

$$D_a^\alpha x(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t x(\tau) E_\alpha \left(-\alpha \frac{(t-\tau)^\alpha}{1-\alpha} \right) d\tau, \quad 0 < \alpha < 1, \tag{4}$$

iv. Atangana-Baleanu Caputo type

$$D_a^\alpha x(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t \frac{dx(\tau)}{d\tau} E_\alpha \left(-\alpha \frac{(t-\tau)^\alpha}{1-\alpha} \right) d\tau, \quad 0 < \alpha < 1, \tag{5}$$

v. Left and right Caputo

$$D_a^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau)\, d\tau, \tag{6}$$

$$D_b^{\alpha} x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_a^t (\tau - t)^{n-\alpha-1} x^{(n)}(\tau)\, d\tau, \tag{7}$$

vi. Jumarie

$$D_a^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} [x(\tau) - x(0)]\, d\tau, \tag{8}$$

where $\Gamma(\cdot)$ is the known gamma function, E_α is the classic Mittag-Leffler function, $B(\alpha)$ is normalization function satisfies $B(0) = B(1) = 1$, $n \in \mathbb{N}$, and $a, b \in \mathbb{R}$. Riemann –Liouville fractional derivative cycle has been shown in Fig (1).

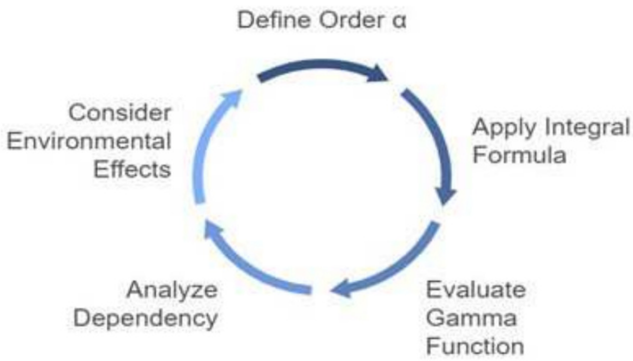

Fig. 1: Riemann –Liouville fractional derivative cycle.

2.1.1 Properties

After integration and differentiation, the fractional derivative may change the character of regular derivatives a few times. Therefore, deriving fractional derivatives is a complicated task, and it is called a non-local operator. Some of the properties of fractional derivatives are provided below. By investigating these properties, it will be understood why the fractional derivative has a wide application area, from applications in the physical world to engineering, mathematics, biology, etc. Moreover, when we derive the numerical methods of these kinds of derivatives, their behavior will also give an idea about the numerical schemes [17]. The fractional derivative is listed as the following Fig. (2).

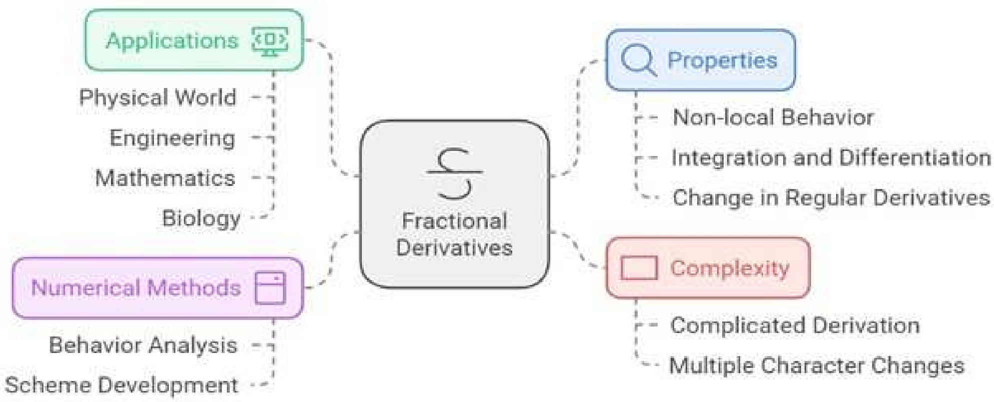

Fig. 2: the fractional derivative list.

2.2 Historical Development

The historical development of a fractional derivative is presented extensively. Many mathematicians made fundamental contributions toward understanding fractional calculus and made it more accessible to various mathematicians. These included mathematicians such as Cesàro, Riemann, Liouville, De Temple, Stieltjes, Hadamard, and Riesz. A few of these overviews are selectively reviewed next. Historically, exceptional cases of fractional differentiation with $\beta = n - 1/\nu$ for $\nu = 1, 2, \ldots$ are presented, and the corresponding fractional integrals were subsequently discovered. The first examples of fractional derivatives appeared in quantum mechanics [18].

In 1832, Liouville resolved question 91 of Legendre's work. One of the earliest versions of function theory was set, and Riemann's monograph on that topic was published in 1890. The main idea was to extend the class of functions, which resulted in the need to introduce three operators. The reader is motivated to explore further these topics, which are quite broad in their own right, primarily because they foster appreciation, not only for the advances over the years in the more abstract foundational areas of fractional calculus but also because they supply insights into the current research in such topics Fig. (3) shows the approaches fractional calculus from different points of view.

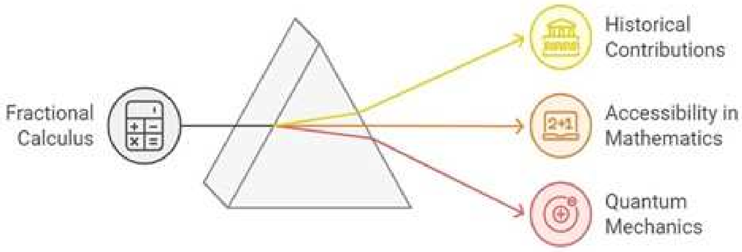

Fig. 3: The fractional calculus.

3 Fractional Calculus Operators

Fractional differentiation is important in defining a class of nonlocal problems involving time. Several fractional calculus operators are in use, and the Riemann-Liouville operator and Caputo selection are more popular.

3.1 Riemann-Liouville Operator

The Riemann-Liouville operator is a fundamental concept for the development of fractional calculus. This subsection presents the Riemann-Liouville operator and some mathematical features, such as its resolved set and spectral representation. The physical meaning of this Riemann-Liouville operator for systems modeling with memory is also presented. Some examples illustrate the results. The Riemann-Liouville operator is also essential to establish the existence and uniqueness of solutions for various fractional differential equations that will be employed to uncover applications in the following sections [19,20]. Fig. (4) shows simple classifications concerning the Riemann-Liouville operator.

The $\alpha-$Riemann-Liouville $(\alpha - RL)$ operator is the integral operator obtained by iterating a given function's $(\alpha-1)-$ times integration. That is, given the function $f:(a,b) \to R$, the $(\alpha-1)^{th}$ antiderivative of f is defined by $f^{(\alpha-1)}(t) := \int_a^t f^{(\alpha-2)}(t)\,dt$ and its $\alpha-$ antiderivative by $f^{(\alpha)}(t) := \int_a^t f^{(\alpha-1)}(t)dt = \int_a^t (t-s) f^{(\alpha)}(s)ds$, where $s \in (a,b)$. This operator is written in the Fourier domain or its Laplace transform counterpart, the so-called integral-multiplicative derivative/integral operator. Its resolved set and the corresponding spectral representation are presented, discussing regularity and decay properties for functions.

3.2 Caputo Operator

The Caputo operator is one of the most widely used definitions for fractional derivatives. It shares specific advantages, mainly because of its principal comparison with the Riemann-Liouville and Grünwald-Letnikov operators. Although

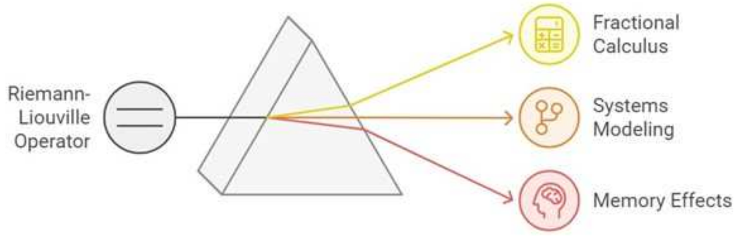

Fig. 4: The Riemann-Liouville operator.

Caputo's operator seems more complicated in the defining expression, it should not hinder its clear mathematical definition and understanding. This operator is defined when the non-integer order α is generally always in the interval $(0,n]$, not necessarily greater than 1, where $n \in \mathbb{N}$.

The original idea of Caputo in using a new operational matrix was restricted only to initial values without relevant significance; nevertheless, these restrictions and many others were later removed, pointing out its central role in defining physical equations corresponding to fractional derivatives [21]. Fig. (5) shows the comparison of Caputo with other fractional derivative operators.

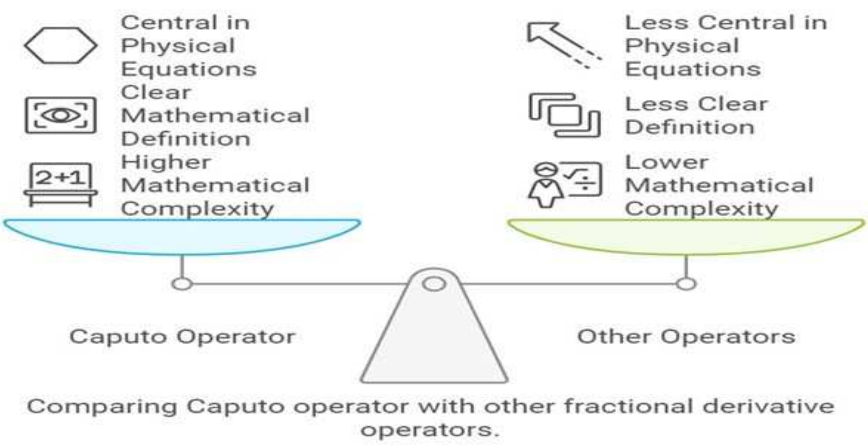

Fig. 5: The comparison of Caputo with other fractional derivative operators.

Because of its immediate use in initial value problems, a common task in engineering that substantially motivates the development of fractional calculus today, the Caputo formulation, in many cases, has become an elegant formulation dealing with the physical meaning of fractional derivatives. In other words, the Caputo operator became the primary preferred player in formulating fractional differential equations because the new coordinate of time has the same dimension, meaning, and measurement as the classical one. This is the reason, in our opinion, for the rapid and recognizable growth of basic research and applications in Fractional Differential Equations from the Caputo point of view. Consequently, rather than focusing on fractional Riemann-Liouville approaches, researchers started looking for 'easy' (from the physical point of view), real, and practical applications of fractional differential equations containing Caputo derivatives of non-integer order. Examples of these 'practical' applications to use the Caputo definition rather than the Riemann-Liouville and others are given in the references. The expositions used in some works, the phrase 'Caputo' differential equation (in the sense of the Riemann-Liouville operator), point out such problems and applications of intrinsic interest in these specific cases [22].

4 Numerical Methods for Fractional Differentiation

This section presents numerical methods applicable to fractional differentiation in diverse aspects, which are essential for pursuing accurate results from fractional differential calculus. In previous decades, diverse numerical techniques such as finite difference and Runge Kutta have been developed to approximate integer order derivatives [23,24]. However, due to the rise of fractional calculus, these techniques have been developed to utilize the derivatives of non-integer order. Specifically, finite difference methods are widely reported, which arbitrarily reconstruct the Riemann-Liouville or Caputo definition in a numerical scheme. Other research works in the field have been rearranged to include spectral methods, adaptive time-stepping techniques, and smoothing effects. Before the development, the model of a particular time-fractional Poisson problem is presented. Next, the taxonomy of constructed methods is analyzed from five different sectors, including formulating fractional derivatives in spectral representations, construction of the approximate time-fractional derivatives, accuracy proof, stability, and complexity. Finally, we report the issues from the non-local phenomena in the fractional differential derivatives [25].

Many methods were elaborated for approximating this new derivative, mainly by determining first a suitable representation of the function as a series or integrals on an appropriate domain, such as Chebyshev or Legendre expansion, or by removing discontinuities and applying the inverse Laplace transform. We report many of these approaches synthetically, including a comparison of the most relevant contributions. We then investigate the numerical approximation of the different methods, focusing on possible algorithms for solving the associated negative-index problems. Numerical simulations are reported in a few examples after realistic applications to the Kelvin-Voigt equation and phase-field models in physics and poroelasticity, anyonic model, and linear viscoelastic model in engineering. The potentialities and the limits of this approach, together with the future research lines, are then highlighted. In the present section, we directly move down to applications [19,20,26].

4.1 Finite Difference Schemes

A finite difference scheme is a popular numerical approach to fractional differentiation. It is based on the finite difference method, which is used to approximate the derivative of a function. The choice of the difference stencil depends on the application of fractional derivatives and how online data is available. The choice of a particular finite difference scheme is made to provide a good approximation to the derivative, convergence, and stability in the numerical sense.

A finite difference scheme utilizes real-time data of the functions in the presence of noise to approximate its fractional order derivative. They can be classified as left-sided, right-sided, and central difference schemes based on the direction in which the scheme function advances. These methods also have some general difficulties in approximating derivatives. However, the finite difference method has already been used successfully in physics, engineering, and other fields to approximate time derivatives of functions in the presence of noise. There is no way to control the error incurred accurately by ignoring certain conditions, although this is possible in the latter case. Finite difference approximations of fractional derivatives can provide an approximate value of the derivatives, but in reality, multidimensional data are present, and it is very computationally expensive. The choice of the finite difference scheme depends on the nature of the function, the order of the measure, and the availability of online data. Illustrative examples and theoretical facts show the effectiveness and robustness of the finite difference approximations in real cases [27].

4.2 Spectral Methods

Numerical methods based on integral representations or the Grünwald–Letnikov definition are powerful tools but suffer from poor efficiency and inaccuracy. These drawbacks suggest the benefit of potentially employing different techniques, such as spectral methods [28,29]. Spectral methods have featured as a replacement for other numerical techniques for the computation of derivatives and have shown satisfactory effectiveness for particular fractional operators. Spectral methods present numerous advantageous properties for the equation of partial fractional differential equations, including high computational accuracy, rapid time-to-solution, and facile parallelization. Due to these advantages, interest in spectral methods across various scientific fields has naturally increased [30]. Fig. (6) shows the comparison of numerical methods.

The accuracy of derivatives can be explicitly calibrated using error analysis. However, spectral methods bear several weaknesses and challenges, particularly when accessing broadband problems, arbitrary geometries, or moving interfaces. Students and scholars should bear in mind how to address such obstacles. Given these considerations, the study of fractional characteristic polynomials as tools for fractional spectral approximations in general geometry is emerging as a new frontier. A review focusing on this area would be extremely relevant and timely for those interested in fractional differentiation [31].

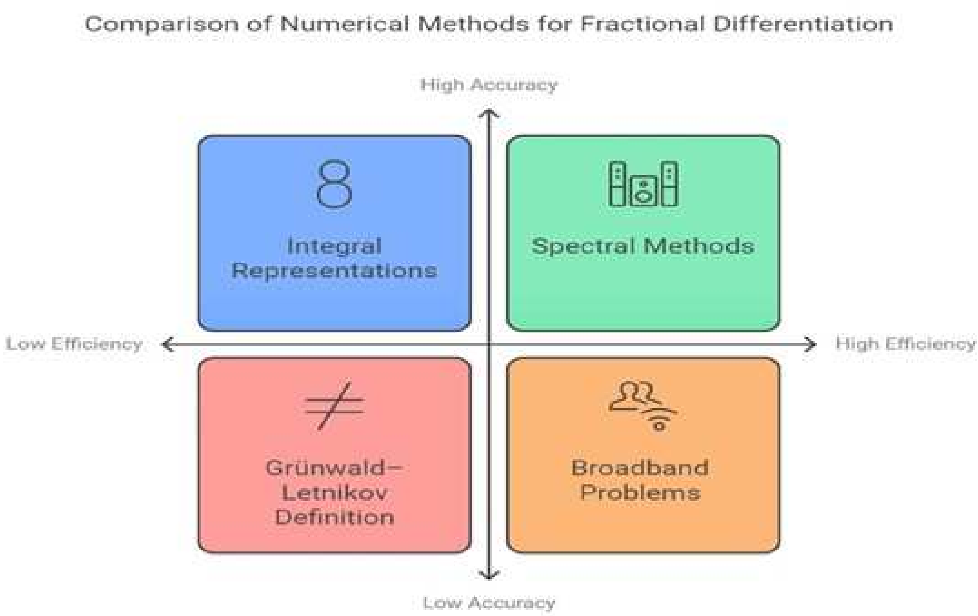

Fig. 6: The comparison of numerical methods.

5 Applications in Physics

Fractional calculus has been widely used to study many advanced features of physical systems. Some of these applications are summarized below. One of the principal results of the development of fractional differential equations is related to the fact that some processes can be described more accurately by a fractional model than by a classical type. Understanding this, it is natural to say that the use of fractional calculus in physics is closely associated with describing phenomena that are impossible to explain by classical methods. Wave propagation: Many have shown that waves have many characteristics that can be represented exclusively by fractional derivatives. The essential features of fractional derivatives are, for instance, a dispersion-like effect, the spatial decay of wave amplitudes, and the overall properties of wave data.

Moreover, applications can be found in Earth's geology, science, electrical, and mechanical systems, and the propagation behavior of most types of waves. For example, fractional partial differential equations describe electromagnetic propagation in plasma and disordered media. In several, the fractional wave equation has been considered to model the dynamics of oscillatory devices and systems, showing promising results for several phenomena. Viscoelastic materials: If a model includes the viscoelastic properties of the material, it is more realistic if it consists of a constitutive equation with a fractional derivative. This need is being observed more and more in recent literature. Viscoelastic constitutive equations with fractional derivatives have been used in studying soil, concrete, human bone, metals, glass, vinyl, tars, cellular materials, polymers, and foam. Fractional calculus can be used to define many physical properties accurately, such as the dynamics of a viscoelastic beam when the Riemann-Liouville operator of fractional theory replaces the Kelvin-Voigt operator. These applications motivated the presentation of all the results and definitions of non-integer derivatives.

In addition, even in non-fractional models, some present the solutions of a model given in terms of special functions. Thus, one can consider the Fourier expansion of the solution, which is also offered in some. The solutions are given in terms of sines and cosines, and the wave's phase velocity is shown to shift with the parameters of the viscoelastic beam, just like in the case of the fractional differential equation. Therefore, these solutions obtained from the fractional differential model were expected a priori by people who originally solved the equation in non-fractional form. These results clearly show the application of physics in studying continuous mechanical systems to benefit advanced studies, such as seeking more efficient solutions in non-destructive tests. They also encourage the reader to pay attention to conducting experiments in our soft matter beam and verify the above properties described by fractional derivatives. These results also encourage conducting additional studies of fractional differentiation by examining such properties as deformation, displacement, and the special velocity of low-frequency or high-frequency wave propagation on the soft matter beams in the non-relativistic case. For example, concerning the recent studies and applications, refer to [32,33,34].

5.1 Wave Propagation

The wave propagation problems are rich in non-local aspects, which can be revealed via fractional derivatives and integrals. The classical divergence, memory, and unusual behaviors such as self-similarity could be naturally studied. In different media such as porous, viscoelastic, and visco-plastic materials, the complicated wave dynamics could lead to complexity or non-locality in wave equations and hence are well handled by the fractional derivatives or integrals.

The application of fractional order calculus in wave dynamics was introduced in the 1960s in the literature by Mandelbrot, which was included in [35]. The wave equation with fractional differentiation and prime differentiation terms was more capable than the standard hyperbolic equation in narratively capturing the physical phenomena. In waves, we can have diffusion, memory, and attenuation confined to a specific area, as well as compaction and non-locality behavior of the wavefronts. For example, the diffuseness of the wave profile (diffused wave) and the compaction of a wave can be described by the time-Fourier coefficients. So, the primary model of the wave equation should be able to determine the profile of the equipped wave signal. A better mechanism of the physical phenomena is revealed by modeling wave propagation phenomena with wave equations of a particular new order. In the case of the Vermas perturbation model, it is appropriate to choose the order of the spatial elliptic dilation that comes from the parameter of the time derivative. The conclusive view that one gets after seeing the progress in the application of wave equations with fractional derivatives is that the field is rich in various wave dynamics problems being investigated.[36, 37].

There is fractional wave research in acoustics. Electromagnetic pulses and diffusive electromagnetic waves were considered to model the wave equation in one dimension [38]. The effect of fractional derivatives in Keldysh's equation was considered. Results indicate that the fine-structure single-electron level density changes over time scales of just femtoseconds to attoseconds, making it time-dependent. In optics, one can investigate the application of the wave equation with prime partial differentiation and fractional derivatives [39].

5.2 Viscoelasticity

The essence of viscoelasticity can be found in various elastic-visco interactions. When dealing with viscoelastic problems, particularly on nano to micro-scale or what is called a mesoscale, materials are more likely to indicate such behaviors not only because of the phase properties of their system but also due to intrinsic nonlinear constitutive laws in the bond mechanism, their microstructure, and in a multiscale domain. In the last decade, research has started making breakthroughs in unraveling the underlying physics based on fractional calculus, especially in viscoelastic materials. A fractional differentiation is a measure that can describe intermediate characteristics between non-zero integer powers. The need for fractional calculus is based on the properties of most viscoelastic responses under different loading rates and compositions of the viscoelastic materials [40,41].

The size of the non-integer order ranges for the entire empirical domain of materials science. The empirical model for a viscoelastic tip-sample interaction exhibits the ankle phenomenon due to the time-dependent Poisson's ratio. Several studies on atomic/molecular bonding behavior and quasi-van der Waals forces in nanoasperity contacts under various loading and environmental conditions show the fractal nature of scaling nano-microcontacts. These two representative and distinct research lines will likely profit from exploring a fractal-like nature to clarify their characteristics in viscoelastic materials. The potential of this approach is expected to provide tools for materials design and attaining optimum performance of materials in various applications, including automotive, bioengineering, construction industries, and oil drilling practices. This section reviews recent research trends in viscoelasticity from the viewpoint of the commutative nature in showing the applicability of fractional derivatives in engineering. The following section starts with the basics of fractional calculus, followed by representing force-displacement materials in spherical and cylindrical geometries [42].

Fractals, geometries, scratches, and cracks are fractional due to their homogeneity and self-similarity. Since cracks, tips, and scratches in material sciences and biophysics are studied as physical phenomena, a one-dimensional fractional approach combined with the molecular dynamics method was recently introduced. In viscoelastic phenomena and constitutive behavior, the characteristics of stress singularities from micron to nano skew contact become essential for the mesoscale regime where the wavelength becomes effective within the penetration depth. Overall, the recent trend of research in viscoelasticity should be considered in empirical materials design and general engineering, particularly nowadays, theranostics. The primary focus of research in viscoelasticity has consequently been on a fractional approach as a mathematical tool, specifically in modeling as one way to interpret physical phenomena in materials research mathematically. It is expected that research inspired by advances in this approach will soon develop into other areas of science, including engineering, nano, biophysics, and theranostics [43].

6 Applications in Engineering

Applications in Engineering. Engineering systems are becoming increasingly sophisticated, complex, and unrevealing and require sophisticated mathematical skills for modeling, simulation, and overall system analysis. Due to various outstanding features, such as memory and heredity, fractional calculus is an excellent instrument for modeling perfect or transparent system dynamics. It has been found that simulated analysis incorporating fractional differentiation can improve system behavior or demand. A more accurate system response and an appropriate demonstration of system properties can be achieved by utilizing these active approaches. In electromagnetics, the staff could pinpoint the target velocity duration embedded in the skyline of the absorption rate in twin-ply surgical fabric. In this manner, they showed the possible applications of the additional differential to deal with inherent lengths and the presumed variety with tools for surgical clothing design from a mathematically interesting standpoint. Several significant advancements have been made utilizing these dynamic terms in mathematics, science, and engineering. Conduct the control systems of vessels that include fractional-differential buoyancy, where the convergence theorem and computational approach are made and examined [44]. Fig. (7) shows the advancing Engineering with Fractional calculus

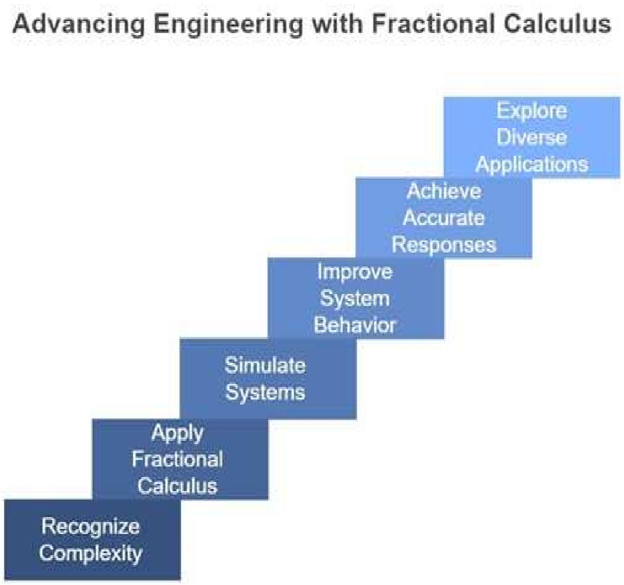

Fig. 7: The advancing engineering with fractional calculus.

Moreover, in related areas of science and engineering, present research work could lead to results including integrating fractional differential terms in hydrology, which could be utilized, for example, in designing environmental features. Additionally, the gradual entry of homodyne quantification could increase range significantly. A methodological tactic of gradual landslides and gullies that may be advanced, incorporating fractional derivatives, could also be advanced. Ultimately, each model, simulation, and range in such regions must carefully evaluate and analyze this newly developing approach to include all potential distortions. For a fresh appearance, relevant new points, a suggested shift, and consequently designed Vale emphasizes the avoidance of dieseling. Additional research and development have been carried out in engineering divisions using fractional derivatives [45].

6.1 Signal Processing

Proof of the usefulness of fractional differentiation may be outlined in many applications. Convincingly, fractional calculus offers a new perspective for signal analytic formulations, given the ability to deal with properties such as memory, especially in a non-local context. In contrast to integer derivatives, one prominent advantage of fractional differentiation for both filtering and denoising is that it can select specific signal characteristics that contribute to high spatial corners, leaving stillness or a smooth background behind intact. This is why fractional differentiation is preferable to integer-based high-pass differentiation for signal smoothing and denoising. The parameter α and threshold value must be optimized in a trial-and-error procedure [46]. Fig. (8) shows the application of fractional Differentiation in the signal process.

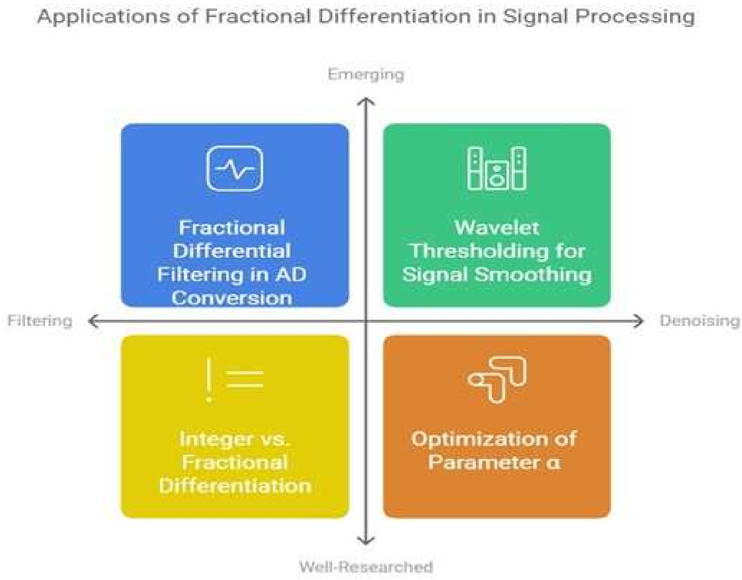

Fig. 8: The application of fractional Differentiation in the signal process.

In fractional differential filtering and signal smoothing, noise characteristics are essential in interfering with the filtering and denoising outcomes. A method has been practically applied in AD conversion to detect based on the fractional derivatives-based local maxima editor approach applied to demodulate the time-derivative photoplethysmogram (PPG) for data transmission. A principal design of variability ranges for embedding the payload into JPG steganography employed fractional differential intensity values multiplication, error correction coding, and adjustable windows variance for hiding the secrets. In addition, signal smoothing facilitates this variation detection. Accordingly, this study coalesces a joint fractional derivative-based method with the wavelet thresholding technique for filtering and denoising smooth and jiggle signals commonly appearing, for example, in biomedical or physiological data, including voice audio. Notably, to our knowledge, the opportunity to apply fractional differential filtering methods in these two unrelated fields has not been thoroughly researched academically. A new perspective study of the signal will be introduced for application, in which helpful future work related to existing and novel fractional differentiation applications will be addressed [47,48].

6.2 Control Systems

Within the different applications of fractional calculus, the design and analysis of control systems have been major topics of interest among scientists and engineers in the past decades. The introduction of fractional order differentiation in the design of control systems marked a turning point in control engineering. It has been shown that fractional controllers may provide better performance than traditional integer-order controllers. The mathematical background of fractional control systems lies mainly in the Laplace transform, as the theory is also based on fractional-order differentiation [15].

The fractional order of the system indicates the amount of 'memory' the system has and is conceptualized as the permissible delay between the relationship of a degree of the input and the degree of the output indicated by α, a non-integer. This possibility of delaying the relation between input and output is the key concept that shows how the fractional control system effectively handles and controls the dynamics of practical processes that are difficult to manage with conventional control systems. For instance, in fluid dynamics and viscoelastic materials, the fractional controller is used to enhance the stability and robustness of the feedback mechanism. As an example of a viscoelastic process in the steel rolling elastic system, external disturbances may greatly affect the system in terms of the roll gap size. The fractional control system, by introducing the right combination of fractional gains, reveals not only a source of stability but also a source of robustness in the emergence of the system. Although the laws of fractional calculus have been found in literature and old books, they have long been unseen by engineers and practitioners [49]. Fig. (9) shows the fractional control system: theory vs. practice.

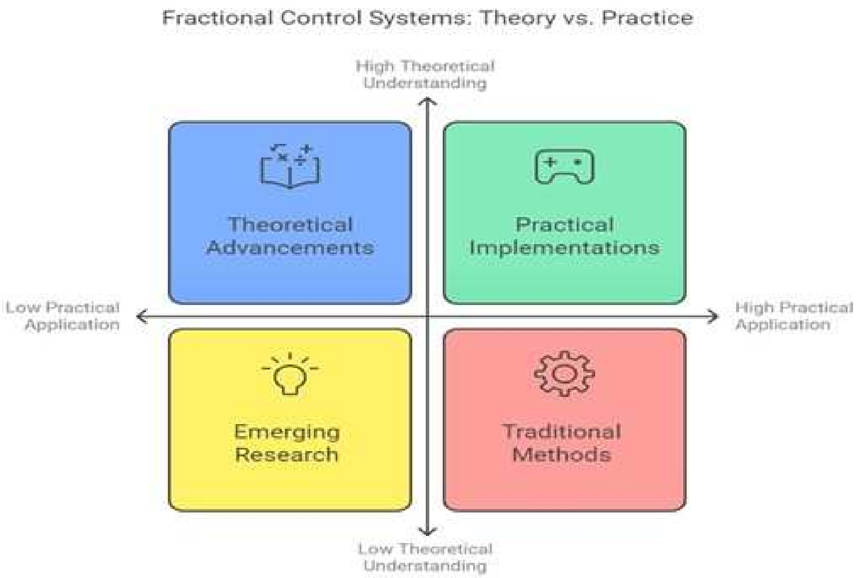

Fig. 9: The fractional control system: theory vs. practice.

Today, researchers in many sciences and engineering disciplines are beginning to understand how to adapt the system model to modern science and its mechanisms. The central issue in controller design is whether we can use fractional calculus to find an optimal controller and understand how an optimal fractional-order controller can be designed. Whenever a new application field obtains a remarkable analogy with the fractional calculus model, it indicates the importance and capability of this area. An in-depth study on control is necessary. However, the practical application of the fractional controller to real-world process control remains challenging. With the growing topic of fractional calculus, research, and technology, recent advances show that fractional controllers are starting to be practically implemented in real-world control systems. Consequently, the demand for many current works to be done in practical implementations is increasing. This means different questions should be answered, and further work should be conducted. For example, should we reconsider the design procedure? Is only the current design of fractional controllers applicable, or should some points be considered? In this broad scope, our motivation is to investigate the recent advances in the fractional control system [50].

Other review papers in this field have dealt with both controllers in the same paper. However, although both share some conceptualization, the design of controllers using fractional order calculus in terms of the optimal or nonlinear controller concept may not be the same as the design of fractional optimal derivatives and integrators. In addition, the impulsive variant of these frameworks also does not seem to be listed in the review papers. Many reviews are available in fractional calculus, but the focus and application in engineering are not identical and may differ significantly. Thus, the field of fractional control has received enough attention to warrant this review [10].

7 Applications in Biology and Medicine

The modeling and analyses in biology and medicine have attracted much attention over the past few decades. Many real systems in these disciplines display non-singular memory defined by differential and integro-differential equations of fractional order. Biomechanics involving the macroscopic and microscopic levels of biological processes, such as the cell life cycle, respiratory, nervous, hematological disorders, and the immune system, have drawn interest and show relevant aspects when modeling these processes. In recent years, several groups have reported whisker dynamics and biomechanical and biomechatronic designs that appear to model this system. Biomedical imaging is also an area where the significance and advantages of fractional derivatives over integer derivatives are exploited. It is noted that fractional calculus has a wide range of potential applications in describing human and animal dynamics and in simultaneously modeling connective tissue mechanics, controls, actuators, and signal sharing in biotransport [51].

Biological, physiological, and medical processes present basic properties such as long memory, nonlinearity, and stability, indicating that fractional models can better describe these properties than integer-order models. A more flexible representation method will be obtained by modeling the time series data with fractional models, which are more suitable for biological systems that undergo gradual changes and renovations. In biology, systems exhibit a broad spectrum of motion complexity, including linear and nonlinear, smooth and non-smooth, stiff and compliant, strong and weak coupling, and human inspiration. In biological dynamic systems, several findings raise a level of comfort with the power and applicability of the described approach. Several recent studies are still in progress examining mathematical modeling, simulation, and application of fractional calculus to various medical topics and problems. The security of fractional calculus in biology and the field of medicine is promising. Implementing a program devoted to this subject would further promote applied research and create new professions in information sciences and health care [52].

7.1 Biomechanics

Biomechanics focuses on understanding the mechanical aspects of a biological system. Biological tissues, or more practically according to mechanics, structures, and organs, possess complex mechanical properties originating from living organisms. The properties of living tissues and organisms, for example, non-linearities in their deformation and force relationships, may not be described adequately with the help of traditional models based on integer order calculus. In parallel with more theoretical contexts, being close to clinical applications in the field, theoretical models using fractional calculus deliver an ability to describe complex behavior and analyze an organism's response or tissue's functional properties. The development of these bio-oriented topics and new findings indicate the benefits of using fractional calculus in this field. The reasoning is evident from the following example [53].

For many years, models used to describe the human body's movement and the body's response to a workload have not been able to precisely transfer information about the tissues in the human body, particularly from the point of view of mechanical features. The main reason was the limited possibilities of standard methods used to analyze real-life signal behavior. In biomechanics, progress will help in developing non-integer order mathematical models. Such approaches have been demonstrated when analyzing patient walk following lower extremity surgery. By observing the initial period of recovery, rehabilitation, and after returning to everyday physical activity, fractional and multi-fractional models were offered to describe the time parameters of the particular act [54]. Fig. (10) shows the biomechanics complexity.

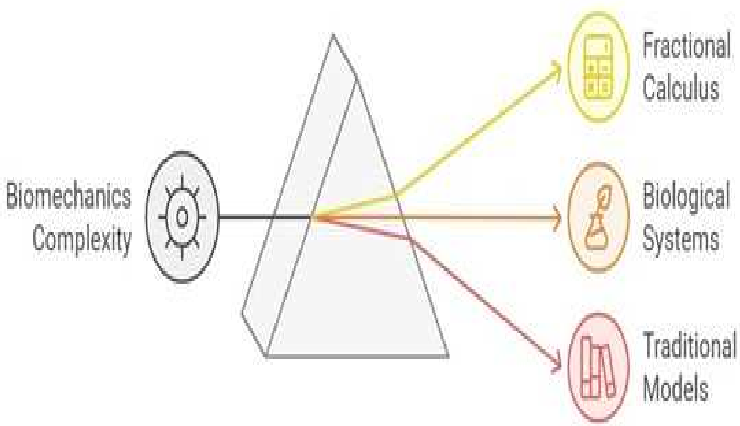

Fig. 10: The biomechanics complexity.

New findings show that the fractional calculus approach is successfully applied in describing the adaptation of physical parameters with loads. Since the modeling solutions are measured for the human knee joint, the fractional differential equations comprehensively and accurately describe viscoelastic properties. That is vital for a deeper understanding of adaptability in sports and rehabilitation science. Simulations show more convincing results when the fractional analogs of the classical model are implemented. New research in mechanics is still in progress, and both approaches using the fractional time derivative and the normal equation of integral order are currently combined. We believe these new models and improved description methods will offer new knowledge about the behavior of biological structures and the mechanical properties of masses. It would also address under-researched areas of mechanics. In the case of biomechanics, mathematical models can be even more accurate by using a more advanced mathematical description in the form of fractional calculus based on generalizations of fractional equations [33,36,55].

7.2 Biomedical Imaging

Fractional differentiation techniques are strong candidates for biomedical imaging applications because they deliver enhanced medical images, thus fostering improved accuracy in medical diagnostics. In that context, some benefits of using fractional calculus in imaging can be listed as follows [56]:

i. Fractional models enhance the contrast in the acquired medical images, which can support better visualization of small anatomic structures.
ii. Fractional models can also promote optimization of post-image acquisition operations, which paves the way for enhanced image-based diagnosis.
iii. Images processed based on the fractional models could be efficient regarding the amount of generated imaging data, paving the way for reduced imaging costs.

Currently, the fraction of ongoing work that adapts fractional calculus in biomedical imaging is directed toward exploring applications. Reports on fractional models typically involve magnetic resonance imaging, although some reports on ultrasound imaging have also been proposed. No research directly concerns integrating fractional calculus with magnetic resonance imaging and ultrasound in clinical practice. The study aims to draw attention to the potential and outstanding performance of fractional differentiation operators in biomedical imaging applications, addressing such a concern. In general, research on the application of fractional differentiation in medical imaging would be crucial in directing the capabilities of fractional calculus to healthcare technology. Biomedical imaging applications can illustrate the significance of such an approach [47]. Fig. (11) shows medical imaging enhancement with fractional calculus.

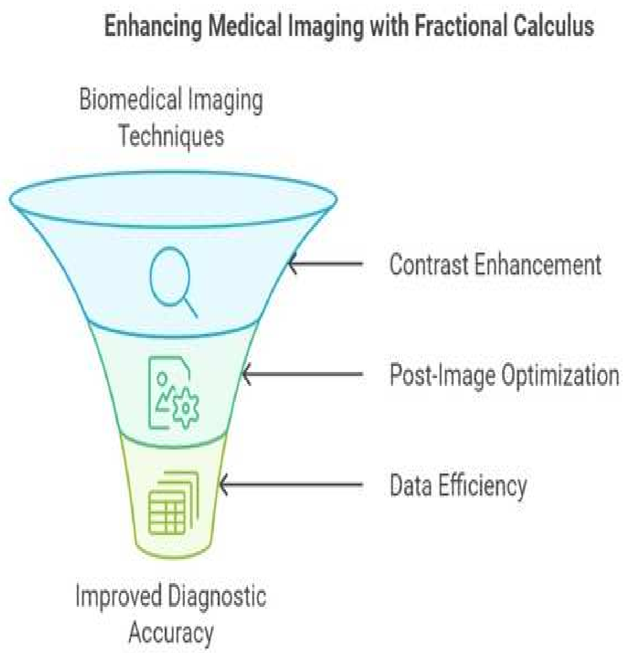

Fig. 11: The enhancement of medical imaging with fractional calculus

8 Fractional Differentiation in Machine Learning

Machine learning has become a crucial tool for decision-making and prediction tasks in intelligent systems. However, machine learning methods face challenges in capturing the complex relationships in available data. Fractional models have the flexibility to capture higher-order dependency, long-range memory, and complex system dynamics, which can benefit machine learning systems. The performance of machine learning systems has been improved by applying fractional derivatives and integrals or developing new fractional machine learning models. Currently, two popular variants of fractional machine learning methods can achieve state-of-the-art results in applications of traffic flow prediction and image recognition or classification [14,57].

Research in improved fractional learning or extensions of fractional learning for machine learning is still in progress, and the use of fractional differentiation in machine learning can improve classifiers' performance in psychiatric and disease diagnosis problems, intelligent robots and intelligent devices, information processing, and other artificial intelligence or intelligent systems domains. In conclusion, as society and human life are undergoing significant transformations through information technology and machine learning, integrating mathematical methods from fractional differentiation into the framework of machine learning can lead to more efficient and effective solutions to complex real-life problems. Mathematically, hard-to-solve problems can be reformulated or approximated in a new way, which can lead to new algorithms and methods that can enhance the predictive power of machine learning models and lead to novel findings or machine learning methodologies in various scientific and application areas [57].

9 Challenges and Future Directions

Challenges Confronting Fractional Differentiation: The theoretical development of fractional differentiation is very complex because of the non-local characteristic of the operator. Numerical instability is the most noticeable problem in practical applications. Lack of convergence and, more specifically, an arbitrary choice of the order of the derivative and a numerical methodology or criteria for enforcing the desired characteristic are the significant constraints in deploying the fractional differentiation operator. Stability and convergence issues are still not fully addressed. Various types of fractional operators generally lead to different kinds of results, and there is no standardized methodology available, allowing one to choose a priori in the sense that various types of results can be obtained depending on the selected approach [32].

Formulating a fixed-point theorem and/or algorithm is necessary to solve nonlinear fractional differential and integral equations by using various types of contraction mappings. The area of fractional differentiation and social science seems ripe for advancement, and with developmental advances in computer technology and supportive mathematical theory, problems in other domains of science may be set out better. There is no doubt that, so far, there has been increasing research interest in fractional differentiation and its applications, with recent novel mathematical theory because of advancing various domains of science such as engineering, biophysics, and social systems. In the last 20 years, remarkable innovative trends in fractional calculus have seen ongoing growth, and advancements are cropping up [58].

10 Conclusion

This introductory essay is devoted to recent advances in the theory and application of fractional calculus and fractional differential equations. In the above-leveled Sections, we attempted to overview some recent achievements in fractional differentiation, discuss the results of some selected applications, and illustrate new problems that fractional differentiated models have initiated. Fractional differentiation is an emerging subject in the last decade. Recent research has clarified that fractional differentiation is the mother of many nonsmooth features. Fractional differentiation has been linked jointly to several popular image-and-text-inpainting techniques.

In physics, we have discussed the loss and dynamical characterization of a fractional differentiator in a cavity that agrees over a wide band. In elasticity, atomistic simulations have been replaced by fractional differential equations to capture the non-local and memory features present in the domain. Importantly, fractional models otherwise follow standard methods. For instance, fractional cables have been deployed in the deployment of trans-Pacific cables. In biochemistry, fractional differential equations have recently been deployed to solve nonlinear systems of fractional differential equations that involve several temperature reactions of the body. In a biological application, a fractional derivative machine has been used to diagnose and prognosis dengue infection early. In epidemiology, fractional systems' hidden dynamics have been unveiled using fractional differential equations. In developing autonomous drones, an intelligent king has been proposed utilizing a fractional derivative. In power systems, fractional differential equations have been applied to model the behavior of both inductive and capacitive loads. A fractional differential equation has been used for memory power distribution and optimization in biomedical research, radar research, and adaptive LDO and ADC critical dimension quality control. Art, of course, can never be left out and has been created by acclaim using some simple equations with fractional differentiation. A concurrent engineering application proposes an intelligent frame based on gradient-based fractional adaptive boosting.

In optimization theory and control systems, fractional differential equations of the Liouville-Caputo sense have been used to control the synchronous behavior of globally coupled piecewise linear systems. It has also been used to optimize Markov processes. A fractional calculus model has been proposed in hydraulics to unify the results [8,59]. However, it should be remembered that fractional calculus is delicate and that carrying out some computations is extremely difficult in theory and numeric, although there is a large body of literature and software. However, this seems to be its strength, and complex analysis is the focus of many current research articles and texts. The limited research trends and avenues today are only a few, and we hope that exciting new applications are created by the readers or software creators themselves. At the same time, new disciplines such as image and text-in-painting and control engineering for special types of systems such as power-law or bi-fractional differentially coupled systems, such as the ring of Kuramoto models and the Fokker-Planck fractional differentially coupled system, are beginning to emerge [60,61,62,63,64].

Fractional calculus is still in its infancy, and fast-track research development would enhance research opportunities. For instance, in the case of fractional tensors, the fundamental theorem of fractional tensors needs to be explored for further research. Numerical approximation schemes are particularly applicable for fractional derivatives, and finding relatively low-order, simple approximations for accurate integration of fractional functions is a challenge for research scientists. The functional and complex analytical properties of local and non-local fractional operators are early research into fractional calculus. These challenges, we hope, combined with interdisciplinary collaboration, will attract novice researchers to the beautifully entangled web of fractional calculus to address the complex demands of modern science.

References

[1] A. R. Ali, W. S. Abbasi, B. Bibi, H. Rahman, S. Ul-Islam, A. H. Majeed and I. Ahmad, Spacing effects on flows around two square cylinders in staggered arrangement via LBM, *Sci. Rep.*, **14**, 18049 (2024).

[2] A. H. Majeed, D. Liu, A. R. Ali, H. Alotaibi, Z. J. Yin, and R. H. Yi, Numerical simulations of energy storage performance in a close configuration: A Galerkin finite element-based computation, *Alexandria Engineering Journal*, **104**, 56-65 (2024).

[3] M. Abdelhakem, M. Fawzy, M. El-Kady, and H. Moussa, Legendre polynomials' second derivative Tau method for solving Lane-Emden and Ricatti equations, *Appl. Math. Inf. Sci.*, **17**(3), 437-445 (2023).

[4] M. Gamal, M. El-Kady, and M. Abdelhakem, Solving real-life bvps via the second derivative Chebyshev pseudo-Galerkin method, *Int. J. Mod. Phys. C*, **35**(7), 2450089 (2024).

[5] G. S. Teodoro, J. A. T. Machado and E. C. de Oliveira, A review of definitions of fractional derivatives and other operators, *J. Comput. Phys.*, **388**, 195–208 (2019).

[6] M. Abdelhakem, Shifted Legendre Fractional Pseudo-spectral Integration Matrices for Solving Fractional Volterra Integro-Differential Equations and Abel's Integral Equations, *Fractals*, **31**(10), 2340190 (2023).

[7] M. Abdelhakem, H. Moussa, D. Baleanu, and M. El-kady, Shifted Chebyshev schemes for solving fractional optimal control problems, *Fractals*, **25**(15), 2143–2150 (2019).

[8] W. Gao, J. L. G. Guirao, M. Abdel-Aty and W. Xi, An independent set degree condition for fractional critical deleted graphs, *Discrete Contin. Dyn. Syst. S*, **12**, 877–886 (2019).

[9] A. S. Alshomrani, M. Z. Ullah and D. Baleanu, A new approach on the modelling, chaos control and synchronization of a fractional biological oscillator, *Adv. Differ. Equ.*, **2021**, 63 (2021).

[10] Z. Da, V. W. Fang, and W. Lin, Fractional Trading, *The Review of Financial Studies*, **38**(3), 623–660 (2025).

[11] K. Lan, Equivalence of higher order linear Riemann-Liouville fractional differential and integral equations, *P. Am. Math. Soc.*, **148**(12), 5225-5234 (2020).

[12] Y. Zhou, Basic theory of fractional differential equations, World Scientific, Singapore (2014).

[13] A. Atangana, Mathematical model of survival of fractional calculus, critics and their impact: How singular is our world?, *Adv. Differ. Equ.*, **2021**, 403 (2021).

[14] R. Walasek, and J. Gajda, Fractional differentiation and its use in machine learning, *International Journal of Advances in Engineering Sciences and Applied Mathematics*, **13**(2), 270-277 (2021).

[15] J. A. Conejero, J. Franceschi, and E. Picó-Marco, Fractional vs. ordinary control systems: what does the fractional derivative provide?, *Mathematics*, **10**(15), 2719 (2022).

[16] B. B. Delgado, and J. E. Macías-Díaz, On the general solutions of some non-homogeneous Div-Curl systems with Riemann–Liouville and Caputo fractional derivatives, *Fractal Fract.*, **5**(3), 117 (2021).

[17] B. Ghanbari, and A. Atangana, Some new edge detecting techniques based on fractional derivatives with non-local and non-singular kernels, *Adv. Differ. Equ.*, **2020**, 435 (2020).

[18] M. Abu-Shady, and M. K. A. Kaabar, A generalized definition of the fractional derivative with applications, *Mathematical Problems in Engineering*, **2021**, 9444803 (2021).

[19] H. Srivastava, and P. O. Mohammed, Existence and uniqueness of a class of uncertain Liouville-Caputo fractional difference equations, *Journal of King Saud University-Science*, **33**(6), 101497 (2021).

[20] H. M. Srivastava, A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics, *Symmetry*, **13**(12), 2294 (2021).
[21] R. B. Albadarneh, I. Batiha, A. K. Alomari, and N. Tahat, Numerical approach for approximating the Caputo fractional-order derivative operator, *AIMS Mathematics*, **6**(11), 12743-12756 (2021).
[22] O. Brandibur, R. Garrappa, and E. Kaslik, Stability of systems of fractional-order differential equations with Caputo derivatives, *Mathematics*, **9**(8), 914 (2021).
[23] M. T. Hoang, and M. Ehrhardt, A second-order nonstandard finite difference method for a general Rosenzweig–MacArthur predator–prey model, *J. Comput. Appl. Math.*, **444**, 11572 (2024).
[24] A. J. Ali, A.F. Abbas, and M. Abdelhakem, Comparative Analysis of Adams-Bashforth-Moulton and Runge-Kutta Methods for Solving Ordinary Differential Equations Using MATLAB, *Mathematical Modelling of Engineering Problems*, **11**(3), 641-647 (2024).
[25] M. D'Elia, Q. Du, C. Glusa, M. Gunzburger, X. Tian, and Z. Zhou, Numerical methods for nonlocal and fractional Models, *Acta. Numerica.*, **29**, 1-124 (2020).
[26] H. M. Srivastava, Fractional-order derivatives and integrals: Introductory overview and recent developments, *Kyungpook Mathematical Journal*, **60**(1), 73-116 (2020).
[27] D. Baleanu, S. Zibaei, M. Namjoo, and A. Jajarmi, A nonstandard finite difference scheme for the modeling and nonidentical synchronization of a novel fractional chaotic system, *Adv. Differ. Equ.*, **2021**, 308 (2021).
[28] M Fawzy, H Moussa, D Baleanu, M El-Kady, and M Abdelhakem, Legendre derivatives direct residual spectral method for solving some types of ordinary differential equations, *Math. Sci. Lett.*, **11**(3), 103-108, (2022).
[29] D. Abdelhamied, M. Abdelhakem, M. El-Kady, and Y. H. Youssri, Adapted Shifted ChebyshevU Operational Matrix of Derivatives: Two Algorithms for Solving Even-Order BVPs, *Appl. Math. Inf. Sci.*, **17**(3), 505-511 (2023).
[30] H. M. Mobarak, E. M. Abo-Eldahab, R. Adel, and M. Abdelhakem, MHD 3D nanofluid flow over nonlinearly stretching/shrinking sheet with nonlinear thermal radiation: Novel approximation via Chebyshev polynomials' derivative pseudo-Galerkin method, *Alex. Eng. J.*, **102**, 119-131 (2024).
[31] S. Harizanov, R. Lazarov, and S. Margenov, A survey on numerical methods for spectral space-fractional diffusion problems, *Fractional Calculus and Applied Analysis*, **23**(6), 1605-1646 (2020).
[32] G. Failla, and M. Zingales, Advanced materials modelling via fractional calculus: challenges and perspectives, *Philosophical Transactions of the Royal Society A*, **378**(2172), 20200050 (2020).
[33] E. Viera-Martin, J. F. Gómez-Aguilar, J. E. Solís-Pérez, J. A. Hernández-Pérez, and R. F. Escobar-Jiménez, Artificial neural networks: a practical review of applications involving fractional calculus, *The European Physical Journal Special Topics*, **231**(10), 2059-2095 (2022).
[34] S. Arora, T. Mathur, S. Agarwal, K. Tiwari, and P. Gupta, Applications of fractional calculus in computer vision: a survey, *Neurocomputing*, **489**, 407-428 (2022).
[35] A. Rocco, and B. J. West, Fractional calculus and the evolution of fractal phenomena, *Physica A*, **265**, 535-546 (1999).
[36] L. Guo, Y. Chen, S. Shi, and B. J. West. Renormalization group and fractional calculus methods in a complex world: A review, *Fract. Calc. Appl. Anal.*, **24**, 5-53 (2021).
[37] W. Chen, H. G. Sun and X. Li, Fractional derivative modeling in mechanics and engineering, Springer (2022).
[38] F. Iafrate, and E. Orsingher, On the fractional wave equation, *Mathematics*, **8**(6), 874 (2020).
[39] G. Barbero, L. R. Evangelista, R. S. Zola, E. K. Lenzi, and A. M. Scarfone, A brief review of fractional calculus as a tool for applications in physics: Adsorption phenomena and electrical impedance in complex fluids, *Fractal Fract.*, **8**(7), 369 (2024).
[40] L-J Shen, Fractional derivative models for viscoelastic materials at finite deformations, *International Journal of Solids and Structures*, **190**, 226-237 (2020).
[41] X. Yu, and Y. Yin, Operator Kernel Functions in Operational Calculus and Applications in Fractals with Fractional Operators, *Fractal Fract.*, **7**(10), 755 (2023).
[42] W. Ding, S. Patnaik, S. Sidhardh, and F. Semperlotti, Applications of distributed-order fractional operators: A review, *Entropy*, **23**(1), 110 (2021).
[43] B. Yu, X. Zheng, P. Zhang, and L. Zhang, Computing solution landscape of nonlinear space-fractional problems via fast approximation algorithm, *Journal of Computational Physics*, **468**(1), 111513 (2022).
[44] L. C. Vieira, R. S. Costa, and D. Valério, An overview of mathematical modelling in cancer research: fractional calculus as modelling tool, *Fractal Fract.*, **7**(8), 595 (2023).
[45] N. Su, Fractional Calculus for Hydrology, Soil Science and Geomechanics: An Introduction to Applications, (1st ed.). CRC Press, Boca Raton (2020).
[46] G. Huang, H. Y. Qin, Q. Chen, Z. Shi, S. Jiang, and C. Huang, Research on Application of Fractional Calculus Operator in Image Underlying Processing, *Fractal Fract.*, **8**(1), 37 (2024).
[47] A. Gómez-Echavarría, J. P. Ugarte, and C. Tobón, The fractional Fourier transform as a biomedical signal and image processing tool: A review. *Biocybernetics and Biomedical Engineering*, **40**(3), 1081-1093 (2020).
[48] C. I. Muresan, I. Birs, C. Ionescu, E. H. Dulf, and R. De Keyser, A review of recent developments in autotuning methods for fractional-order controllers, *Fractal Fract.*, **6**(1), 37 (2022).
[49] H. Jahanshahi, J. M. Munoz-Pacheco, S. Bekiros, and N. D. Alotaibi, A fractional-order SIRD model with time-dependent memory indexes for encompassing the multi-fractional characteristics of the COVID-19, *Chaos, Solitons & Fractals*, **143**, 110632 (2021).

[50] A. Tepljakov, B. B. Alagoz, C. Yeroglu, E. A. Gonzalez, S. H. Hosseinnia, and E. Petlenkov, Towards industrialization of FOPID controllers: A survey on milestones of fractional-order control and pathways for future developments, *IEEE Access*, **9**, 21016-21042 (2021).
[51] M. Arfan, M. M. Lashin, P. Sunthrayuth, K. Shah, A. Ullah, K. Iskakova, M. R. Gorji, and T. Abdeljawad, On nonlinear dynamics of COVID-19 disease model corresponding to nonsingular fractional order derivative, *Med. Biol. Eng. Comput.*, **60**(11), 3169-3185 (2022).
[52] E. H. Dulf, D. C. Vodnar, A. Danku, C. I. Muresan, and O. Crisan, Fractional-order models for biochemical processes, *Fractal Fract.*, **4**(2), 12 (2020).
[53] C. A. Valentim, J. A. Rabi, and S. A. David, Fractional mathematical oncology: On the potential of non-integer order calculus applied to interdisciplinary models, *Biosystems*, **204**, 104377 (2021).
[54] M. Olinski, and P. Marciniak, Development of a Simplified Human Body Model for Movement Simulations, *Appl. Sci.*, **15**(13), 1011 (2025).
[55] A. P. Singh, and K. Bingi, Applications of fractional-order calculus in robotics, *Fractal Fract.*, **8**(7), 403 (2024).
[56] R. W. Ibrahim, H. A. Jalab, F. K. Karim, E. Alabdulkreem, and M. N. Ayub, A medical image enhancement based on generalized class of fractional partial differential equations, *Quantitative Imaging in Medicine and Surgery*, **12**, 1 (2021).
[57] H. Niu, Y. Q. Chen, and B. J. West, Why do big data and machine learning entail the fractional dynamics?, *Entropy*, **23**(3), 297 (2021).
[58] E. İlhan, and O. Kıymaz, A generalization of truncated M-fractional derivative and applications to fractional differential equations, *Applied Mathematics and Nonlinear Sciences*, **5**(1), 171-188 (2020).
[59] A. Abdel-Aty, M. M. A. Khater, R. A. M. Attia, M. Abdel-Aty, and H. Eleuch, On the new explicit solutions of the fractional nonlinear space-time nuclear model, *Fractals*, **28**(08), 2040035 (2020).
[60] X.-J. Yang, M. Abdel-Aty, and C. Cattani, A New General Fractional-Order Derivative with Rabotnov Fractional-Exponential Kernel Applied to Model the Anomalous Heat Transfer, *Thermal Science*, **23**(3), 1677–1681 (2019).
[61] G. M. Ismail, H. R. Abdl-Rahim, A. Abdel-Aty, W. Alharbi, and M. Abdel-Aty, An analytical solution for fractional oscillator in a resisting medium, *Chaos, Solitons and Fractals*, **130**, 109395 (2020).
[62] A. H. Bhrawy, M. A. Zaky, and M. Abdel-Aty, A fast and precise numerical algorithm for a class of variable-order fractional differential equations, *Proceedings of the Romanian Academy Series A - Mathematics Physics Technical Sciences Information Science*, **18**(1), pp. 17–24 (2017).
[63] M. A. Noor, K. I. Noor, S. Iftikhar, and M. U. Awan, Fractal Integral Inequalities for Harmonic Convex Functions, *Appl. Math. Inf. Sci.*, **12**(4), 831-839 (2018).
[64] R. Uthayakumar, and A. Gowrisankar, Generalized Fractal Dimensions in Image Thresholding Technique, *Inf. Sci. Lett. Vol.*, **3**(3), 125-134 (2014).