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Simple random walks on wheel graphs

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Abstract: A simple random walk on a graph is defined in which a particle moves from one vertex to any adjacent vertex, each with equal probability. The expected hitting time is the expected number of steps to get from one vertex to another before returning to the starting vertex. In this paper, using the electrical network approach, we provide explicit formulae for expected hitting times for simple random walks on wheel graphs. As a by-product, formulae for expected commute times and expected difference times, and bounds for the expected cover times for simple random walks on wheel graphs are obtained.

Keywords: random walks; expected hitting time; resistance distance; wheel graph

1 Introduction

Let $G=(V(G),E(G))$ be a connected simple graph with vertices labeled as $1,2,\ldots,n$. A simple random walk on G is the Markov chain X_n, $n\geq 0$, that form its current vertex i jumps to one of its d_i neighboring vertices with uniform probability, where d_i denotes the degree of i. The hitting time T_i of the vertex i is the minimum number of steps the random walk takes to reach the vertex, that is, $T_i=\inf\{n\geq 0: X_n=i\}$. The expected hitting time E_iT_j is the expected value of T_j of the vertex j when the walk starts from i. The expected commute time $C(i,j)$ between i and j is the expected number of steps for a random walk starting at i to pass through j and return to i, that is, $C(i,j)=E_iT_j+E_jT_i$. The quantity $E_iT_j-E_jT_i$ is called the expected difference time, and is denoted by $D(i,j)$. The expected cover time $C_i(G)$ is the expected number of steps that it takes a walk that starts at i to visit all vertices of G. The expected cover time $C(G)$ of G is defined as $\max_{i\in V(G)} C_i(G)$.

It is natural to view a graph G as an electrical network with a unit resistor between each pair of nodes interconnected by an edge of G. Then one may define the resistance distance [1] between vertices i and j, denoted by r_{ij}, as the net effective resistance between nodes i and j (were a battery to be connected between i and j) in the corresponding electrical network. As an intrinsic graph metric, resistance distance has been extensively studied. For more information, the readers are referred to [2-5] and references therein.

There exists a strong connection between random walks on graphs and electrical networks. Perhaps Nash-Williams [6] was the first person to established such a connection. Later on, since the appearance of the book of Doyle and Snell [7], more and more attention has been devoted to the relation between effective resistance (resistance distance) and random walks on graphs. Some nice relations, such as the relation between resistance distance and the escape probability, the relation between resistance distance and the expected hitting time, have been established. Here we only introduce the elegant result concerning the expected hitting time, which is paramount in obtaining our main result.

Theorem 1.1. [8] For a simple random walk on G, we have

$$E_iT_j=|E(G)|\,r_{ij}+\frac{1}{2}\sum_{k=1}^{n}d_k(r_{ki}-r_{kj}). \qquad (1)$$

Simple random walks on graphs arise in many models in mathematics and physics and thus it has been studied in a wide variety of contexts. For a general introduction to random walks on graphs, the reader is referred to the survey paper by Lovász [9] and the textbook [10]. In particular, the computation of parameters such as expected hitting times, expected difference times and expected cover times are of significant importance. Sometimes, it is greatly simplified to compute these parameters by the electrical network approach. Thus computing these parameters via electrical network approach becomes more and more popular.

So far, expected hitting times have been computed for some classes of graphs, especially those with some degree of symmetry, such as vertex-transitive graphs [11-13], edge-transitive graphs [14,15], distance-regular graphs [12,16], graphs with cut-vertices [17] and so on. The expected cover time of connected graphs has been extensively studied. Methods of bounding the cover time of graphs have been thoroughly investigated [18-23]. Several bounds on the cover times of particular classes of graphs have been obtained with many positive results [24-27].

In this paper, we consider simple random walks on an important class of graphs, namely, wheel graphs. The wheel graph W_n is a graph that contains a cycle of order $n-1$, and for which every graph vertex in the cycle is connected to one other vertex (which is known as the center). In the present work, first of all, resistance distances between all pairs of vertices of the wheel graph are completely determined. Then according to the relationship between resistance distances and the expected hitting times (Theorem 1.1), explicit formulae for the expected hitting times for simple random walks on wheel graphs are obtained. Finally, simple formulae for expected commute times and expected difference times are derived according to expected hitting times, and lower and upper bounds for the expected cover times are determined.

Resistance distances in W_n

We start with introducing some more concepts and notations in graph theory terminology. The adjacency matrix $A(G)$ of graph G is a $n\times n$ symmetric matrix with (i,j)-th element equal to 1 if vertices i and j are adjacent and 0 otherwise. Let $D(G)=diag(d_1,d_2,\ldots,d_n)$ be the diagonal matrix of vertex degrees. Then the Laplacian matrix of G is defined as $L(G)=D(G)-A(G)$. Laplacian matrix plays an essential role in the computation of resistance distance. One example is to compute resistance distances in terms of determinate of sub-matrices of the Laplacian matrix. More precisely, let $L(i)$ and $L(i,j)$ denote the matrices obtained from $L(G)$ by deleting its i-th row and column, and by deleting its i-th and j-th rows and columns, respectively. Then it is shown that

Lemma 2.1. [28] Let G be a connected graph on n vertices, $n\geq 3$, and $1\leq i\neq j\leq n$. Then

$$r_{ij}=\det L(i,j)/\det L(i) \qquad (2)$$

Suppose that vertices in W_n are labeled in such a way that the vertices corresponding to the cycle are labeled from 1 to $n-1$ in cyclic order, and the center is labeled as n. In what follows, for convenience, we distinguish resistance distances in W_n into two types, namely, between center and non-center vertices and between pairs of non-center vertices. And then resistance distances in W_n are computed according to different types.

A. Resistance distances between center and non-center vertices

By the symmetry of W_n, it is obvious that resistance distances between center and any non-center vertices are equal. So we may assume that $r_{1n}=r_{2n}=\cdots=r_{n-1,n}=r_0$.

Theorem 2.2.

$$r_0=\frac{1}{\sqrt{5}}\frac{(3+\sqrt{5})^{n-1}-(3-\sqrt{5})^{n-1}}{(3+\sqrt{5})^{n-1}+(3-\sqrt{5})^{n-1}-2^n}. \qquad (3)$$

Proof. Since the Laplacian matrix of W_n is

$$L(W_n)=\begin{bmatrix} 3 & -1 & & & -1 \\ -1 & 3 & -1 & & -1 \\ & \ddots & \ddots & \ddots & \vdots \\ & & -1 & 3 & -1 \\ -1 & -1 & \cdots & -1 & 3 \end{bmatrix}_{n\times n},$$

if we define

$$A_n=\begin{bmatrix} 3 & -1 & & & \\ -1 & 3 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 3 & -1 \\ & & & -1 & 3 \end{bmatrix}_{n\times n},$$

$$B_n=\begin{bmatrix} 3 & -1 & & & -1 \\ -1 & 3 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 3 & -1 \\ -1 & & & -1 & 3 \end{bmatrix}_{n\times n},$$

and let $a_n = \det A_n$, $b_n = \det B_n$, then by Lemma 2.1,

$$r_0 = r_{1n} = \det A_{n-2} / \det B_{n-1} = a_{n-2} / b_{n-1}. \tag{4}$$

Claim 1. For $n \geq 1$,

$$a_n = \frac{1}{\sqrt{5}}[(\frac{3+\sqrt{5}}{2})^{n+1} - (\frac{3-\sqrt{5}}{2})^{n+1}]. \tag{5}$$

$b_1 = 3$, $b_2 = 8$ and for $n \geq 3$,

$$b_n = (\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2. \tag{6}$$

Proof of Claim 1. Frist we prove Eq. (5). It is easy to verify that $a_1 = \det A_1 = 3$ and $a_2 = \det A_2 = 8$. For $n \geq 3$, expanding the determinant of A_n with respect to its first row one may find that $\det A_n = 3\det A_{n-1} - \det A_{n-2}$. Thus the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies the recursion formula $a_n = 3a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 3$ and $a_2 = 8$. Now we solve it. Since the characteristic polynomial of the recurrence is $x^2 - 3x + 1$ with $x_1 = (3-\sqrt{5})/2$ and $x_2 = (3+\sqrt{5})/2$ as its two roots, we may suppose that

$$a_n = [(3-\sqrt{5})/2]^n y_1 + [(3+\sqrt{5})/2]^n y_2.$$

The initial conditions lead to the following linear system

$$\begin{cases} [(3-\sqrt{5})/2]y_1 + [(3-\sqrt{5})/2]y_2 = 3 \\ [(3-\sqrt{5})/2]^2 y_1 + [(3-\sqrt{5})/2]^2 y_2 = 8 \end{cases}.$$

Solving the system for y_1 and y_2 we have $y_1 = (5-3\sqrt{5})/10$ and $y_2 = (5+3\sqrt{5})/10$. Hence a_n is obtained as desired.

For b_n , it is straightforward to see that $b_1 = 3$ and $b_2 = 8$. For $n \geq 3$, we expand the determinant of B_n with respect to its first row to obtain that $\det B_n = 3\det A_{n-1} - 2\det A_{n-2} - 2$, which means that $b_n = 3a_{n-1} - 2a_{n-2} - 2$. Hence Eq. (6) may be derived from Eq. (5) and Claim 1 is proved.

Since W_n has at least four vertices, by substituting Eqs. (5) and (6) into Eq. (4), we could obtain r_0.

B. Resistance distances between pairs of non-center vertices.

For any two non-center vertices i and j , one may easily find that the resistance distance between them depends only on the distance between them. So we may assume that $r_{ij} = r_k$ whenever i and j are at distance k .

Before computing r_k, we introduce two results that will be used later. The first one is the famous Foster's (first) formula.

Lemma 2.3. [17] For G an n-vertex connected graph with edge set $E(G)$,

$$\sum_{ij \in E(G)} r_{ij} = n-1.$$

The second one is a sum rule on resistance distances.

Lemma 2.4. [30] Let i and j be vertices of a connected graph G . Then

$$d_i r_{ij} + \sum_{k:k \sim i} (r_{ik} - r_{jk}) = 2, \tag{7}$$

where $k \sim i$ means k is adjacent to i.

Now we are ready to compute r_k as given in the following result.

Theorem 2.5. For $1 \leq k \leq \lfloor (n-1)/2 \rfloor$,

$$r_k = \frac{1}{\sqrt{5}}\{(\frac{3+\sqrt{5}}{2})^k - (\frac{3-\sqrt{5}}{2})^k - \frac{[(\frac{3+\sqrt{5}}{2})^k + (\frac{3-\sqrt{5}}{2})^k - 2][(3+\sqrt{5})^{n-1} - (3-\sqrt{5})^{n-1}]}{(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1} - 2^n}\} \tag{8}$$

Proof. Firstly we show that the assertion holds for r_1. By Foster's formula, we have

$$\sum_{ij \in E(W_n)} r_{ij} = \sum_{i=1}^{n-2} r_{i,i+1} + r_{n-1,1} + \sum_{i=1}^{n-1} r_{in} = (n-1)(r_0 + r_1) = n-1$$

that is, $r_0 + r_1 = 1$. Hence by Eq. (3), we have

$$r_1 = 1 - r_0 = 1 - \frac{1}{\sqrt{5}} \frac{(3+\sqrt{5})^{n-1} - (3-\sqrt{5})^{n-1}}{(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1} - 2^n}. \tag{9}$$

Now we show that the assertion also holds for both r_2 and r_3. Apply Lemma 2.4 to pairs of vertices {1,2} and {2,3}, we have

$$\begin{cases} 3r_{1,2}+r_{1,2}-r_{2,2}+r_{1,n-1}-r_{2,n-1}+r_{1,n}-r_{2,n}=2 \\ 3r_{1,3}+r_{1,2}-r_{3,2}+r_{1,n-1}-r_{3,n-1}+r_{1,n}-r_{3,n}=2 \end{cases},$$

that is, $5r_1-r_2=2$ and $3r_2+r_1-r_3=2$. These indicate that $r_2=5r_1-2$ and $r_3=16r_1-8$. By substituting r_1 into the above equations we can obtain r_2 and r_3 as desired.

For $k\ge 4$, we claim that r_k satisfies the following recursion formula

$$r_k=4r_{k-1}-4r_{k-2}+r_{k-3} \tag{10}$$

To prove this, we apply Lemma 2.4 to pairs of vertices $\{1,k-2\}$ and $\{1,k-1\}$ to obtain

$$\begin{cases} 3r_{1,k-2}+r_{1,2}-r_{k-2,2}+r_{1,n-1}-r_{k-2,n-1}+r_{1,n}-r_{k-2,n}=2 \\ 3r_{1,k-1}+r_{1,2}-r_{k-1,2}+r_{1,n-1}-r_{k-1,n-1}+r_{1,n}-r_{k-1,n}=2 \end{cases}$$

that is,

$$\begin{cases} 3r_{k-2}+r_1-r_{k-3}+r_1-r_{k-1}=2 \\ 3r_{k-1}+r_1-r_{k-2}+r_1-r_k=2 \end{cases}.$$

This linear system yields $r_k=4r_{k-1}-4r_{k-2}+r_{k-3}$ as claimed.

To solve the recursion relation, first note that the characteristic polynomial of the recurrence is x^3-4x^2+4x-1 whose roots are 1, $(3+\sqrt{5})/2$ and $(3-\sqrt{5})/2$. And then the initial conditions r_1, r_2 and r_3 lead to the following linear system

$$\begin{cases} x_1+[(3+\sqrt{5})/2]x_2+[(3-\sqrt{5})/2]x_3=r_1 \\ x_1+[(3+\sqrt{5})/2]^2x_2+[(3-\sqrt{5})/2]^2x_3=5r_1-2 \\ x_1+[(3+\sqrt{5})/2]^3x_2+[(3-\sqrt{5})/2]^3x_3=16r_1-8 \end{cases}$$

Solving the linear system for x_1, x_2 and x_3, we have $x_1=2-2r_1$ $\quad x_2=r_1-1+\sqrt{5}/5$ and $x_3=r_1-1-\sqrt{5}/5$. Hence

$$r_k=2-2r_1+\left(\frac{3+\sqrt{5}}{2}\right)^k\left(r_1-1+\frac{\sqrt{5}}{5}\right)+\left(\frac{3-\sqrt{5}}{2}\right)^k\left(r_1-1-\frac{\sqrt{5}}{5}\right)$$

and the desired result is obtained by substituting r_1 into the above equation.

In conclusion, resistances distances between all pairs of vertices in W_n can be computed as follows.

Theorem 2.6. In W_n, for $1\le i\le n-1$,

$$r_{in}=r_{ni}=\frac{1}{\sqrt{5}}\frac{(3+\sqrt{5})^{n-1}-(3-\sqrt{5})^{n-1}}{(3+\sqrt{5})^{n-1}+(3-\sqrt{5})^{n-1}-2^n}, \tag{11}$$

for $1\le i<j\le n-1$

$$r_{ij}=r_{ji}=\frac{1}{\sqrt{5}}\left\{\left(\frac{3+\sqrt{5}}{2}\right)^k-\left(\frac{3-\sqrt{5}}{2}\right)^k-\frac{\left[\left(\frac{3+\sqrt{5}}{2}\right)^k+\left(\frac{3-\sqrt{5}}{2}\right)^k-2\right]\left[(3+\sqrt{5})^{n-1}-(3-\sqrt{5})^{n-1}\right]}{(3+\sqrt{5})^{n-1}+(3-\sqrt{5})^{n-1}-2^n}\right\} \tag{12}$$

where $k=\min\{j-i,n-1-j+i\}$.

Expected hitting times, expected commute times, expected difference times and expected cover times for simple random walks on W_n

In this section, firstly explicit formulae for expected hitting times for simple random walks on W_n will be derived. Secondly, expected commute times and expected difference times are also computed according to expected hitting times. Finally, bounds for expected cover times are determined.

The following result by Palacios [19] is crucial in obtaining our main result.

Lemma 3.1. [19] Let G be a connected graph with edge set $E(G)$. Then $\sum_{j:j\sim i}E_iT_j=2|E(G)|-d_i$.

Theorem 3.2. For simple random walks on W_n, we have

(1) for $1\le i\le n-1$,

$$E_iT_n=3 \tag{13}$$

$$E_nT_i=\frac{4(n-1)}{\sqrt{5}}\frac{(3+\sqrt{5})^{n-1}+(3-\sqrt{5})^{n-1}}{(3+\sqrt{5})^{n-1}+(3-\sqrt{5})^{n-1}-2^n}-3 \tag{14}$$

(2) for $1\le i<j\le n-1$,

$$E_iT_j=E_jT_i=\frac{2(n-1)}{\sqrt{5}}\left\{\left(\frac{3+\sqrt{5}}{2}\right)^k-\left(\frac{3-\sqrt{5}}{2}\right)^k-\frac{\left[\left(\frac{3+\sqrt{5}}{2}\right)^k+\left(\frac{3-\sqrt{5}}{2}\right)^k-2\right]\left[(3+\sqrt{5})^{n-1}-(3-\sqrt{5})^{n-1}\right]}{(3+\sqrt{5})^{n-1}+(3-\sqrt{5})^{n-1}-2^n}\right\}, \tag{15}$$

where $k=\min\{j-i,n-1+i-j\}$.

Proof. (i) For any two non-center vertices i and j, since $\sum_{k=1}^{n} d_k r_{ik} = \sum_{k=1}^{n} d_k r_{jk}$, by Theorem 1.1, we know that $E_i T_n = E_j T_n$. On the other hand, by Lemma 3.1, we have $\sum_{i=1}^{n-1} E_i T_n = 2\times 2(n-1) - d_n = 3(n-1)$, and thus for each i, $E_i T_n = 3$. Since for $1 \le i \le n-1$,

$$E_i T_n + E_n T_i = 2|E(W_n)| r_{in} = 4(n-1) r_0,$$

$$E_n T_i = 4(n-1) r_0 - 3 = \frac{4(n-1)}{\sqrt{5}} \frac{(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1}}{(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1} - 2^n} - 3.$$

(ii) For $1 \le i < j \le n-1$, since

$\sum_{k=1}^{n} d_k r_{ik} = \sum_{k=1}^{n} d_k r_{jk}$, by Theorem 1.1 we have $E_i T_j = E_j T_i = |E(W_n)| r_{ij} = 2(n-1) r_k$, where $k = \min\{j-i, n-1+i-j\}$. Hence Eq. (15) is an immediately consequence of Eq. (12).

Now we compute the expected commute times and expected difference times for simple random walks on W_n according to expected hitting times.

Theorem 3.3. For simple random walks on W_n, we have

(1) for $1 \le i \le n-1$,

$$C(i,n) = C(n,i) = \frac{4(n-1)}{\sqrt{5}} \frac{(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1}}{(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1} - 2^n} \tag{16}$$

for $1 \le i < j \le n-1$,

$$C(i,j) = \frac{4(n-1)}{\sqrt{5}} \{ (\frac{3+\sqrt{5}}{2})^k - (\frac{3-\sqrt{5}}{2})^k - \frac{[(\frac{3+\sqrt{5}}{2})^k + (\frac{3-\sqrt{5}}{2})^k - 2][(3+\sqrt{5})^{n-1} - (3-\sqrt{5})^{n-1}]}{(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1} - 2^n} \} \tag{17}$$

(2) for $1 \le i \le n-1$,

$$D(n,i) = -D(i,n) = \frac{4(n-1)}{\sqrt{5}} \frac{(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1}}{(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1} - 2^n} - 6, \tag{18}$$

for $1 \le i < j \le n-1$,

$$D(i,j) = D(j,i) = 0. \tag{19}$$

Proof. Bearing in mind that for any two vertices i and J, $C(i,j) = C(j,i) = E_i T_j + E_j T_i$ and $D(i,j) = E_i T_j - E_j T_i$, we may obtain Theorem 3.3 immediately from Theorem 3.2.

In the end of this section, we discuss bounds for the expected cover times for simple random walks on W_n. Using the electrical approach, Chandra et al. [19] proved very useful bounds in terms of R, the electrical resistance of the graph, defined as the maximum effective resistance between any pair of vertices:

Theorem 3.3. [19] Let $R = \max_{i,j} r_{ij}$ be the electrical resistance of a graph G on n vertices with edge set $E(G)$. Then $|E(G)| R \le C(G) \le (2+o(1)) |E(G)| R \log n$. For the wheel graph W_n, we have

Theorem 3.4.

$n-1 < C(W_n) < 4(2+o(1))(n-1)\log n$.

Proof. Since it has been shown in the proof of Theorem 2.5 that $r_0 + r_1 = 1$, either $r_0 \ge 1/2$ or $r_1 \ge 1/2$. Hence $R > 1/2$. On the other hand, since for any two vertices i and j, $r_{ij} \le r_{in} + r_{nj} \le 2r_0 < 2$, we have $R < 2$. Thus Theorem 3.4 is proved by Theorem 3.3.

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