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Semihypergroups Characterized by $(\in,\in\vee q_k)$-fuzzy Hyperideals

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Abstract: Generalizing the concepts of $(\in,\in\vee q)$-fuzzy hyperideal, $(\in,\in\vee q)$-fuzzy quasi-hyperideal and $(\in,\in\vee q)$-fuzzy bi-hyperideal, the notions of $(\in,\in\vee q_k)$-fuzzy hyperideal, $(\in,\in\vee q_k)$-fuzzy quasi-hyperideal and $(\in,\in\vee q_k)$-fuzzy bi-hyperideal are defined. Using these notions, different classes of semihypergroups are characterized.

Keywords: Semihypergroups, $(\in,\in\vee q_k)$-fuzzy hyperideals, $(\in,\in\vee q_k)$-fuzzy quasi-hyperideals, $(\in,\in\vee q_k)$-fuzzy bi-hyperideals.

1 Introduction

In 1965, Zadeh introduced the fundamental concept of a fuzzy set in his classic paper [1]. Many authors used this concept to generalize several notions of algebra. Rosenfeld [2] laid the foundations of fuzzy groups. Kuroki introduced the theory of fuzzy semigroups in his papers [3,4]. The monograph by Mordeson et al.[5] deals with the theory of fuzzy semigrpoups and their use in fuzzy coding, fuzzy finite stae machines and fuzzy languages. Fuzziness has a natural place in the field of formal languages. The monograph by Mordeson and Malik [6] deals with the application of fuzzy approach to the concepts of automata and formal languages. In [7] the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset is defined. In [8] the idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined. These two ideas play a vital role in generating some different types of fuzzy subgroups. Using these ideas Bhakat and Das [9,10], gave the concept of (α,β)-fuzzy subgroups, where $\alpha,\beta\in\{\in,q,\in\vee q,\in\wedge q\}$ and $\alpha\neq\in\wedge q$. These fuzzy subgroups are further studied in [11,12]. The concept of $(\in,\in\vee q)$-fuzzy subgroup is a viable generalization of Rosenfeld's fuzzy subgroup. $(\in,\in\vee q)$-fuzzy subrings and ideals are defined in [13]. Davvaz defined $(\in,\in\vee q)$-fuzzy subnearrings and ideals of a nearring in [14]. Jun and Song initiated the study of (α,β)-fuzzy interior ideals of a semigroup in [15]. In [16], Shabir et al. characterized semigroups by the properties of $(\in,\in\vee q)$-fuzzy ideals. Generalizing the concept of a quasi-coincidence of a fuzzy point with a fuzzy subset Jun [17,18] defined $(\in,\in\vee q_k)$-fuzzy subgroups and $(\in,\in\vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras, respectively. In [19] $(\in,\in\vee q_k)$-fuzzy subsemigroup, $(\in,\in\vee q_k)$-fuzzy left (resp. right) ideal, $(\in,\in\vee q_k)$-fuzzy interior ideal, $(\in,\in\vee q_k)$-fuzzy quasi ideal and $(\in,\in\vee q_k)$-fuzzy bi-ideal are defined and different classes of semigroups are characterized in terms of these notions.

Fuzzy sets and hyperstructures introduced by Zahed and Marty, in [1,20] respectively, are now extensively used from both the theoretical point of view and their many applications. A recent book [21] contains a wealth of applications. Via this book, Corsini and Leoreanu presented some of numerous applications of the algebraic hyperstructures. The relations between fuzzy sets and hyperstructures have been already considered by Corsini [22,23,24,25], Davvaz [26], Leoreanu [27], Cristea [28] and others. In [26], Davvaz introduced the concept of fuzzy hyperideals in a semihypergroup. In [29], the concept of (α,β)-fuzzy hyperideals and $(\in,\in\vee q)$-fuzzy hyperideals in semihypergroups is initiated. In [29], using the idea of a quasi-coincedence of a fuzzy point with a fuzzy set, the concept of an (α,β)-fuzzy left (resp. right) hyperideal in semihypergroups is introduced, which is a generalization of the concept of a fuzzy left (resp. right) hyperideal of a semihypergroup and some interesting characterization theorems are obtained. A special attention is given to $(\in,\in\vee q)$-fuzzy left(resp. right) hyperideals. In [30], the notion of (α,β)-fuzzy

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bi-hyperideals and $(\in,\in\vee q)$-fuzzy bi-hyperideals in semihypergroups are introduced and using these notions some classes of semigypergroups are characterized.

In this paper, we generalize the notions of $(\in,\in\vee q)$-fuzzy hyperideal, $(\in,\in\vee q)$-fuzzy quasi-hyperideal and $(\in,\in\vee q)$-fuzzy bi-hyperideal. The notions of $(\in,\in\vee q_k)$-fuzzy hyperideal, $(\in,\in\vee q_k)$-fuzzy quasi-hyperideal and $(\in,\in\vee q_k)$-fuzzy bi-hyperideal are defined. Using these notions, different classes of semihypergroups are characterized.

2 Preliminaries

A hypergroupoid is a non-empty set H equipped with a hyperoperation, that is a map $\circ : H\times H \to P^*(H)$, where $P^*(H)$ denotes the set of all non-empty subsets of H [21]. We shall denote by $x\circ y$, the hyperproduct of elements x,y of H.

A hypergroupoid $(H,\circ)$ is called a semihypergroup if

$$(x\circ y)\circ z = x\circ(y\circ z) \text{ for all } x,y,z \text{ in } H.$$

Throughout this paper H will denote a semihypergroup with hyperoperation $\circ$.

Let A,B be subsets of H. Then the hyperproduct of A and B is defined as:

$$A\circ B = \bigcup_{a\in A,b\in B} a\circ b$$

We shall write $A\circ x$ instead of $A\circ\{x\}$ and $x\circ A$ for $\{x\}\circ A$.

A non-empty subset S of a semihypergroup H is called a subsemihypergroup of H if for all $x,y\in S$, $x\circ y\subseteq S$.

If a semihypergroup H contains an element e with the property that, for all $x\in H$, $x\in x\circ e$ (resp. $x\in e\circ x$), we say that e is a right (resp. left) identity of H. If $x\circ e=\{x\}$ (resp. $e\circ x=\{x\}$), for all x in H, then e is called scalar right (resp. left) identity in H.

In [31], it is defined that if $A\in P^*(H)$ then A is called,

(i) a right hyperideal in H if

$$x\in A \Longrightarrow x\circ y\subseteq A, \forall y\in H$$

(ii) a left hyperideal in H if

$$x\in A \Longrightarrow y\circ x\subseteq A, \forall y\in H$$

(iii) a hyperideal in H if it is both a left and a right hyperideal in H.

A non-empty subset Q of H is called a quasi-hyperideal if $Q\circ H\cap H\circ Q\subseteq Q$. A non-empty subset B of H is called a generalized bi-hyperideal of H if $B\circ H\circ B\subseteq B$. A non-empty subset B of H is called a bi-hyperideal of H if it is both a subsemihypergroup and a generalized bi-hyperideal of H. A subsemihypergroup A of H is called an interior hyperideal of H if $H\circ A\circ H\subseteq A$. Obviously every one-sided hyperideal of H is a quasi-hyperideal, every quasi-hyperideal is a bi-hyperideal and every bi-hyperideal is a generalized bi-hyperideal but the converse is not true. Also every hyperideal is an interior hyperideal but the converse is not true.

A semihypergroup H is called regular, if for each $a\in H$ there exists $x\in H$ such that $a\in a\circ x\circ a$. A semihypergroup H is called intra-regular, if for every $a\in H$ there exists $x,y\in H$ such that $a\in x\circ a\circ a\circ y$ (see [32]). It is well known that in a regular semihypergroup the concepts of quasi-hyperidel, bi-hyperideal and generalized bi-hyperideal coincide. Also in a regular semihypergroup, every interior hyperideal is a hyperideal.

A fuzzy subset λ of a universe X is a function from X into the unit closed interval $[0,1]$, i.e. $\lambda : X \longrightarrow [0,1]$ (see [1]). A fuzzy subset λ in a universe X of the form

$$\lambda(y) = \begin{cases} t\in(0,1] & \text{if } y=x \\ 0 & \text{if } y\neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set λ in a set X, Pu and Liu [8] gave meaning to the symbol $x_t\alpha\lambda$, where $\alpha\in\{\in,q,\in\vee q,\in\wedge q\}$. A fuzzy point x_t is said to belong to (resp. quasi-coincident with) a fuzzy set λ written as $x_t\in\lambda$ (resp. $x_tq\lambda$) if $\lambda(x)\geq t$ (resp. $\lambda(x)+t>1$), and in this case, $x_t\in\vee q\lambda$ (resp. $x_t\in\wedge q\lambda$) means that $x_t\in\lambda$ or $x_tq\lambda$ (resp. $x_t\in\lambda$ and $x_tq\lambda$). To say that $x_t\overline{\alpha}\lambda$ means that $x_t\alpha\lambda$ does not hold. For any two fuzzy subsets λ and μ of H, $\lambda\leq\mu$ means that, for all $x\in H, \lambda(x)\leq\mu(x)$. The symbols $\lambda\wedge\mu$ and $\lambda\vee\mu$ will mean the following fuzzy subsets of H

$$(\lambda\wedge\mu)(x) = \lambda(x)\wedge\mu(x)$$
$$(\lambda\vee\mu)(x) = \lambda(x)\vee\mu(x)$$

for all $x\in H$.

For any fuzzy subset λ of H and for any $t\in[0,1]$, the set

$$U(\lambda;t) = \{x\in H : \lambda(x)\geq t\}$$

is called a level subset of λ.

For two fuzzy subsets λ and μ of H, define

$$\lambda\circ\mu : H\to[0,1]$$
$$|x \to \begin{cases} \min_{x\in y\circ z}\{\lambda(y),\mu(z)\}, & \text{if } \exists y,z\in H \text{ such that } x\in y\circ z \\ 0 & \text{otherwise.} \end{cases}$$

For a semihypergroup H, the fuzzy subsets "0" and "1" of H are defined as follows:

$0 : H\to[0,1]|x\to 0(x) := 0,$

$1 : H\to[0,1]|x\to 1(x) := 1.$

If $(H,\circ)$ is a semihypergroup and $A\subseteq H$, the characteristic function λ_A of A is a fuzzy subset of H, defined as follows:

$$\lambda_A : H\to[0,1]|x\to\lambda_A(x) = \begin{cases} 1 & \text{if } x\in A, \\ 0 & \text{if } x\notin A. \end{cases}$$

2.1 Definition

(1). [26] A fuzzy subset λ of a semihypergroup H is called a fuzzy subsemihypergroup of H if for every $\alpha \in x \circ y$,

$$\inf_{\alpha \in x \circ y} \{\lambda(\alpha)\} \geq \min\{\lambda(x), \lambda(y)\} \text{ for all } x, y \in H.$$

(2). [26] Let H be a semihypergroup and μ be a fuzzy subset of H. Then μ is called:

i) a fuzzy right hyperideal of H if $\mu(x) \leq \inf_{\alpha \in x \circ y} \{\mu(\alpha)\}$, for every $x, y \in H$;

ii) a fuzzy left hyperideal of H if $\mu(y) \leq \inf_{\alpha \in x \circ y} \{\mu(\alpha)\}$, for every $x, y \in H$;

iii) a fuzzy hyperideal of H (or fuzzy two-sided hyperideal) if it is both a fuzzy left hyperideal and a fuzzy right hyperideal.

(3). [32] Let H be a semihypergroup. A fuzzy subset λ of H is called a fuzzy quasi-hyperideal of H if

$$(\lambda \circ 1) \wedge (1 \circ \lambda) \leq \lambda.$$

(4). A fuzzy subset λ of H is called a generalized bi-hyperideal of H if

$$\inf_{\alpha \in x \circ y \circ z} \lambda(\alpha) \geq \min\{\lambda(x), \lambda(z)\} \text{ for all } x, y, z \in H.$$

(5). [32] A fuzzy subsemihypergroup λ of H is called a fuzzy bi-hyperideal of H if

$$\inf_{\alpha \in x \circ y \circ z} \lambda(\alpha) \geq \min\{\lambda(x), \lambda(z)\} \text{ for all } x, y, z \in H.$$

(6). A fuzzy subsemihypergroup λ of H is called a fuzzy interior hyperideal of H if

$$\inf_{w \in x \circ a \circ y} \lambda(w) \geq \lambda(a) \text{ for all } x, a, y \in H.$$

(7). A fuzzy subset λ of H is called an $(\in, \in \vee q)$-fuzzy subsemihypergroup of H if for all $x, y \in H$ and $t, r \in (0,1]$ the following condition holds

$$x_t \in \lambda, y_r \in \lambda \longrightarrow (z)_{\min\{t,r\}} \in \vee q\lambda, \text{ for each } z \in x \circ y.$$

(8). [29] A fuzzy subset λ of a semihypergroup H is called an $(\in, \in \vee q)$*-fuzzy left* (resp. *right*) *hyperideal* of H if for all $t \in (0,1]$ and for all $x, y \in H$, we have

$y_t \in \lambda \longrightarrow z_t \in \vee q\lambda$(resp. $z_t \in \vee q\lambda$) for each $z \in x \circ y$(resp. for each $z \in y \circ x$).

(10). A fuzzy subset λ of H is called an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H, if for all $x, y, z \in H$ and for all $t, r \in (0,1]$ it satisfies:

(i) $x_t, y_r \in \lambda \longrightarrow (z)_{\min\{t,r\}} \in \vee q\lambda$ for every $z \in x \circ y$.

(ii) $x_t, z_r \in \lambda \longrightarrow (w)_{\min\{t,r\}} \in \vee q\lambda$ for every $w \in x \circ y \circ z$.

(12). A fuzzy subset λ of H is called an $(\in, \in \vee q)$-fuzzy generalized bi-hyperideal of H, if for all $x, y, z \in H$ and for all $t, r \in (0,1]$ the following condition holds

$$x_t, z_r \in \lambda \longrightarrow (w)_{\min\{t,r\}} \in \vee q\lambda \text{ for every } w \in x \circ y \circ z.$$

2.2 Proposition[32]

The following conditions for a semihypergroup H are equivalent:

(1) H is regular.

(2) For every right hyperideal R and left hyperideal L of H, $R \circ L = R \cap L$.

(3) $R(a) \circ L(a) = R(a) \cap L(a)$, for each $a \in H$, where $R(a)$ is the right hyperideal generated by "a" and $L(a)$ is the left hyperideal generated by "a".

2.3 Theorem[32]

The following statments are equivalent for a semihypergroup H:

(1) H is both regular and intra-regular.

(2) $B = B \circ B$ for every bi-hyperideal B of H.

(3) $Q = Q \circ Q$ for every quasi-hyperideal Q of H.

(4) $B_1 \cap B_2 = B_1 \circ B_2 \cap B_2 \circ B_1$ for all bi-hyperideals B_1, B_2 of H.

(5) $R \cap L \subseteq R \circ L \cap L \circ R$ for every right hyperideal R and every left hyperideal L of H.

(6) $R(a) \cap L(a) \subseteq R(a) \circ L(a) \cap L(a) \circ R(a)$ for every $a \in H$.

3 $(\in, \in \vee q_k)$-fuzzy hyperideals

In what follows, let H denote a semihypergroup and k an arbitrary element of $[0,1)$ unless otherwise specified.

Generalizing the concept of $x_t q\lambda$, Jun [17,18] defined $x_t q_k \lambda$, if $\lambda(x) + t + k > 1$ and $x_t \in \vee q_k \lambda$ if $x_t \in \lambda$ or $x_t q_k \lambda$. In this section we generalize the concepts of $(\in, \in \vee q)$-fuzzy hyperideal, $(\in, \in \vee q)$-fuzzy bi-hyperideal, and $(\in, \in \vee q)$-fuzzy quasi hyperideal and define the $(\in, \in \vee q_k)$-fuzzy hyperideal, $(\in, \in \vee q_k)$-fuzzy bi-hyperideal, and $(\in, \in \vee q_k)$-fuzzy quasi hyperideal of a semihypergroup H and study some basic properties.

3.1 Definition

A fuzzy subset λ of H is called an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H if for all $x, y \in H$ and $t, r \in (0,1]$ the following condition holds

$$x_t \in \lambda, y_r \in \lambda \longrightarrow (z)_{\min\{t,r\}} \in \vee q_k \lambda, \text{ for each } z \in x \circ y.$$

3.2 Theorem

Let A be a non-empty subset of H and λ a fuzzy subset in H defined by

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) If A is a subsemihypergroup of H then λ is a $(q,\in\vee q_k)$-fuzzy subsemihypergroup of H.

(2) A is a subsemihypergroup of H if and only if λ is an $(\in,\in\vee q_k)$-fuzzy subsemihypergroup of H.

Proof.(1) Let $x,y\in H$ and $t,r\in(0,1]$ be such that $x_t,y_r q\lambda$. Then $x,y\in A,\lambda(x)+t>1$ and $\lambda(x)+r>1$. Since A is a subsemihypergroup of H, we have $x\circ y\subseteq A$. Thus for every $z\in x\circ y,\lambda(z)\geq\frac{1-k}{2}$. If $\min\{t,r\}\leq\frac{1-k}{2}$, then $\lambda(z)\geq\min\{t,r\}$ and so $(z)_{\min\{t,r\}}\in\lambda$. If $\min\{t,r\}>\frac{1-k}{2}$, then $\lambda(z)+\min\{t,r\}+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$ and so $(z)_{\min\{t,r\}}q_k\lambda$. Therefore $(z)_{\min\{t,r\}}\in\vee q_k\lambda$.

(2) Let $x,y\in H$ and $t,r\in(0,1]$ be such that $x_t,y_r\in\lambda$. Then $\lambda(x)\geq t>0$ and $\lambda(y)\geq r>0$. Thus $\lambda(x)\geq\frac{1-k}{2}$ and $\lambda(y)\geq\frac{1-k}{2}$, this implies $x,y\in A$. Since A is a subsemihypergroup of H, we have $x\circ y\subseteq A$. Thus for every $z\in x\circ y,\lambda(z)\geq\frac{1-k}{2}$. If $\min\{t,r\}\leq\frac{1-k}{2}$, then $\lambda(z)\geq\min\{t,r\}$ and so $(z)_{\min\{t,r\}}\in\lambda$. If $\min\{t,r\}>\frac{1-k}{2}$, then $\lambda(z)+\min\{t,r\}+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1$ and so $(z)_{\min\{t,r\}}q_k\lambda$. Therefore $(z)_{\min\{t,r\}}\in\vee q_k\lambda$.

Conversely, assume that λ is a $(\in,\in\vee q_k)$-fuzzy subsemihypergroup of H and $x,y\in A$. Then $\lambda(x)\geq\frac{1-k}{2},\lambda(y)\geq\frac{1-k}{2}$ that is $x_{\frac{1-k}{2}}\in\lambda$ and $y_{\frac{1-k}{2}}\in\lambda$. Now by hypothesis, $z_{\frac{1-k}{2}}\in\vee q\lambda$ for every $z\in x\circ y$. If $z_{\frac{1-k}{2}}\in\lambda$ then $\lambda(z)\geq\frac{1-k}{2}$ and so $z\in A$. If $z_{\frac{1-k}{2}}q_k\lambda$ then $\lambda(z)+\frac{1-k}{2}+k>1$ implies $\lambda(z)>\frac{1-k}{2}$. Thus $z\in A$. Hence $x\circ y\subseteq A$, that is, A is a subsemihypergroup of H.

3.3 Corollary

(1) If a non-empty subset A of H is a subsemihypergroup of H, then the characteristic function of A is a $(q,\in\vee q_k)$-fuzzy subsemihypergroup of H.

(2) A non-empty subset A of H is a subsemihypergroup of H if and only if λ_A is an $(\in,\in\vee q_k)$-fuzzy subsemihypergroup of H.

If we take $k=0$ in Theorem 3.2, then we have the following corollary.

3.4 Corollary

Let A be a subsemihypergroup of H and λ a fuzzy subset in H defined by

$$\lambda(x)=\begin{cases}\geq 0.5 \text{ if } x\in A\\ 0 \quad \text{otherwise.}\end{cases}$$

Then

(1) If A is a subsemihypergroup of H then λ is a $(q,\in\vee q)$-fuzzy subsemihypergroup of H.

(2) A is a subsemihypergroup of H if and only if λ is an $(\in,\in\vee q)$-fuzzy subsemihypergroup of H.

3.5 Theorem

Let λ be a fuzzy subset of H. Then λ is an $(\in,\in\vee q_k)$-fuzzy subsemihypergroup of H if and only if $\inf\limits_{z\in x\circ y}\{\lambda(z)\}\geq\min\{\lambda(x),\lambda(y),\frac{1-k}{2}\}$.

*Proof.*Let λ be an $(\in,\in\vee q_k)$-fuzzy subsemihypergroup of H. On the contrary, assume that there exist $x,y\in H$ such that $\inf\limits_{z\in x\circ y}\{\lambda(z)\}<\min\{\lambda(x),\lambda(y),\frac{1-k}{2}\}$. Then there exists $z\in x\circ y$ such that $\lambda(z)<\min\{\lambda(x),\lambda(y),\frac{1-k}{2}\}$. Choose $t\in(0,1]$ such that $\lambda(z)<t\leq\min\{\lambda(x),\lambda(y),\frac{1-k}{2}\}$. Then $x_t\in\lambda$ and $y_t\in\lambda$ but $\lambda(z)<t$ and $\lambda(z)+t+k<\frac{1-k}{2}+\frac{1-k}{2}+k=1$, so $z_t\overline{\in\vee q_k}\lambda$, which is a contradiction. Hence $\inf\limits_{z\in x\circ y}\{\lambda(z)\}\geq\min\{\lambda(x),\lambda(y),\frac{1-k}{2}\}$.

Conversely, assume that $\inf\limits_{z\in x\circ y}\{\lambda(z)\}\geq\min\{\lambda(x),\lambda(y),\frac{1-k}{2}\}$. Let $x_t\in\lambda$ and $y_r\in\lambda$ for $t,r\in(0,1]$. Then $\lambda(x)\geq t$ and $\lambda(y)\geq r$. Now

$$\inf_{z\in x\circ y}\{\lambda(z)\}\geq\min\{\lambda(x),\lambda(y),\frac{1-k}{2}\}\geq\min\{t,r,\frac{1-k}{2}\}.$$

If $t\wedge r>\frac{1-k}{2}$, then $\inf\limits_{z\in x\circ y}\{\lambda(z)\}\geq\frac{1-k}{2}$. So for every $z\in x\circ y$,

$$\lambda(z)+t\wedge r+k>\frac{1-k}{2}+\frac{1-k}{2}+k=1,$$

which implies that $(z)_{\min\{t,r\}}q_k\lambda$. If $t\wedge r\leq\frac{1-k}{2}$, then $\inf\limits_{z\in x\circ y}\{\lambda(z)\}\geq t\wedge r$. So $(z)_{\min\{t,t\}}\in\lambda$. Thus $(z)_{\min\{t,r\}}\in\vee q_k\lambda$. Therefore λ is an $(\in,\in\vee q_k)$-fuzzy subsemihypergroup of H.

If we take $k=0$ in Theorem 3.5, then we have the following corollary.

3.6 Corollary

Let λ be a fuzzy subset of H. Then λ is an $(\in,\in\vee q)$-fuzzy subsemihypergroup of H if and only if $\inf\limits_{z\in x\circ y}\{\lambda(z)\}\geq\min\{\lambda(x),\lambda(y),0.5\}$.

3.7 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q_k)$-fuzzy subsemihypergroup of H if and only if $U(\lambda;t)(\neq\emptyset)$ is a subsemihypergroup of H for all $t\in(0,\frac{1-k}{2}]$.

*Proof.*Let λ be an $(\in,\in\vee q_k)$-fuzzy subsemihypergroup of H and $x,y\in U(\lambda;t)$ for some $t\in(0,\frac{1-k}{2}]$. Then $\lambda(x)\geq t$ and $\lambda(y)\geq t$. It follows from Theorem 3.5 that $\inf\limits_{z\in x\circ y}\{\lambda(z)\}\geq\min\{\lambda(x),\lambda(y),\frac{1-k}{2}\}\geq\min\{t,\frac{1-k}{2}\}=t$. Thus for every $z\in x\circ y$,

$\lambda(z) \geq t$ and so $z \in U(\lambda;t)$, that is $x \circ y \subseteq U(\lambda;t)$. Hence $U(\lambda;t)$ is a subsemihypergroup of H.

Conversely, assume that $U(\lambda;t)(\neq \emptyset)$ is a subsemihypergroup of H for all $t \in (0, \frac{1-k}{2}]$. Suppose that there exist $x, y \in H$ such that

$$\inf_{z \in x \circ y} \{\lambda(z)\} < \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}.$$

Thus there exists $z \in x \circ y$ such that $\lambda(z) < \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$. Choose $t \in (0, \frac{1-k}{2}]$ such that $\lambda(z) < t < \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$. Then $x, y \in U(\lambda;t)$ but $z \notin U(\lambda;t)$ i.e. $x \circ y \nsubseteq U(\lambda;t)$, which contradicts our hypothesis.

Hence $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$ and so λ is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H.

3.8 Corollary

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H if and only if $U(\lambda;t)(\neq \emptyset)$ is a subsemihypergroup of H for all $t \in (0, 0.5]$.

3.9 Definition

A fuzzy subset λ of H is an $(\in, \in \vee q_k)$-fuzzy left (right) hyperideal of H if for all $x, y \in H$ and $t \in (0, 1]$ the following condition holds;

$y_t \in \lambda \longrightarrow z_t \in \vee q_k \lambda$ for every $z \in x \circ y$ ($y_t \in \lambda \longrightarrow z_t \in \vee q_k \lambda$ for every $z \in y \circ x$).

3.10 Theorem

Let L be a subset of H and λ a fuzzy subset in H defined by

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in L \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) If L is a left (resp. right) hyperideal of H then λ is a $(q, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H.

(2) L is a left (resp. right) hyperideal of H if and only if λ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H.

Proof. The proof is similar to the proof of Theorem 3.2.

3.11 Corollary

(1) If a non-empty subset L of H is a left (resp. right) hyperideal of H, then the characteristic function of L is a $(q, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H.

(2) A non-empty subset L of H is a left (resp. right) hyperideal of H if and only if λ_L is an $(\in, \in \vee q_k)$-fuzzy sleft (resp. right) hyperideal of H.

If we take $k = 0$ in Theorem 3.10, then we have the following corollary.

3.12 Corollary

Let L be a left (resp. right) hyperideal of H and λ a fuzzy subset in H defined by

$$\lambda(x) = \begin{cases} \geq \frac{1}{2} & \text{if } x \in L \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) If L is a left (resp. right) hyperideal of H then λ is a $(q, \in \vee q)$-fuzzy left (resp. right) hyperideal of H.

(2) L is a left (resp. right) hyperideal of H if and only if λ is an $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal of H.

3.13 Theorem

A fuzzy subset λ of H is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H if and only if $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(y), \frac{1-k}{2}\}$ (resp. $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \frac{1-k}{2}\}$).

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H. On the contrary, assume that there exist $x, y \in H$ such that $\inf_{z \in x \circ y} \{\lambda(z)\} < \min\{\lambda(y), \frac{1-k}{2}\}$. Then there exists $z \in x \circ y$ such that $\lambda(z) < \min\{\lambda(y), \frac{1-k}{2}\}$. Choose $t \in (0, 1]$ such that $\lambda(z) < t \leq \min\{\lambda(y), \frac{1-k}{2}\}$. Then $y_t \in \lambda$ but $\lambda(z) < t$ and $\lambda(z) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $z_t \overline{\in \vee q_k} \lambda$, which is a contradiction. Hence $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(y), \frac{1-k}{2}\}$.

Conversely, assume that $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(y), \frac{1-k}{2}\}$. Let $x, y \in H$ and $t \in (0, 1]$ be such that $y_t \in \lambda$. Then $\lambda(y) \geq t$. Thus $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(y), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}$. If $t > \frac{1-k}{2}$, then $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \frac{1-k}{2}$. So for every $z \in x \circ y, \lambda(z) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, which imlpies that $z_t q_k \lambda$. If $t \leq \frac{1-k}{2}$, then $\inf_{z \in x \circ y} \{\lambda(z)\} \geq t$. So for every $z \in x \circ y, z_t \in \lambda$. Thus $z_t \in \vee q_k \lambda$ for every $z \in x \circ y$. Therefore λ is an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H.

3.14 Corollary

A fuzzy subset λ of H is an $(\in, \in \vee q_k)$-fuzzy hyperideal of H if and only if $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(y), \frac{1-k}{2}\}$ and $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \frac{1-k}{2}\}$.

If we take $k = 0$ in Theorem 3.13, then we have the following corollary.

3.15 Corollary

A fuzzy subset λ of H is an $(\in,\in\vee q)$-fuzzy left (resp. right) hyperideal of H if and only if $\inf\limits_{z\in x\circ y}\{\lambda(z)\}\geq\min\{\lambda(y),0.5\}$ *(resp.* $\inf\limits_{z\in x\circ y}\{\lambda(z)\}\geq\min\{\lambda(x),0.5\}$).

3.16 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q_k)$-fuzzy left (resp. right) hyperideal of H if and only if $U(\lambda;t)(\neq\emptyset)$ is a left (resp. right) hyperideal of H for all $t\in(0,\frac{1-k}{2}]$.

Proof. The proof is similar to the proof of Theorem 3.7.

3.17 Corollary

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q)$-fuzzy left (resp. right) hyperideal of H if and only if $U(\lambda;t)(\neq\emptyset)$ is a left (resp. right) hyperideal of H for all $t\in(0,0.5]$.

3.18 Theorem

If λ *is an $(\in,\in\vee q_k)$-fuzzy left hyperideal and* μ is an *$(\in,\in\vee q_k)$-fuzzy right hyperideal of H* then $\lambda\circ\mu$ is an *$(\in,\in\vee q_k)$-fuzzy two-sided hyperideal of H.*

Proof. Let $x,y\in H$. Then

$$\begin{aligned}(\lambda\circ\mu)(y)\wedge\frac{1-k}{2}&=\left(\bigvee_{y\in p\circ q}\{\lambda(p)\wedge\mu(q)\}\right)\wedge\frac{1-k}{2}\\&=\bigvee_{y\in p\circ q}\left\{\lambda(p)\wedge\mu(q)\wedge\frac{1-k}{2}\right\}\\&=\bigvee_{y\in p\circ q}\left\{\lambda(p)\wedge\frac{1-k}{2}\wedge\mu(q)\right\}.\end{aligned}$$

(If $y\in p\circ q$, then $x\circ y\subseteq x\circ(p\circ q)=(x\circ p)\circ q$. Now for each $z\in x\circ y$, there exists $a\in x\circ p$ such that $z\in a\circ q$. Since λ is an $(\in,\in\vee q_k)$-fuzzy left hyperideal, therefore by Theorem 3.13, we have $\inf\limits_{a\in x\circ p}\{\lambda(a)\}\geq\min\{\lambda(p),\frac{1-k}{2}\}$ that is $\lambda(a)\geq\min\{\lambda(p),\frac{1-k}{2}\}$.)

Thus

$$\begin{aligned}(\lambda\circ\mu)(y)\wedge\frac{1-k}{2}&=\bigvee_{y\in p\circ q}\left\{\lambda(p)\wedge\frac{1-k}{2}\wedge\mu(q)\right\}\\&\leq\bigvee_{z\in a\circ q}\{\lambda(a)\wedge\mu(q)\}\text{ because }\lambda(a)\geq\min\{\lambda(p),\frac{1-k}{2}\}\\&=\bigvee_{z\in c\circ d}\{\lambda(c)\wedge\mu(d)\}\\&=(\lambda\circ\mu)(z),\text{ for every }z\in x\circ y\subseteq a\circ q.\end{aligned}$$

So

$$\min\left\{(\lambda\circ\mu)(y),\frac{1-k}{2}\right\}\leq\inf_{z\in x\circ y}\{(\lambda\circ\mu)(z)\}.$$

Similarly we can show that $\inf\limits_{z\in x\circ y}\{(\lambda\circ\mu)(z)\}\geq\min\left\{(\lambda\circ\mu)(x),\frac{1-k}{2}\right\}$. Thus $\lambda\circ\mu$ is an $(\in,\in\vee q_k)$-fuzzy two-sided hyperideal of H.

Next we show that if λ and μ are $(\in,\in\vee q_k)$-fuzzy hyperideals of a semihypergroup H, then $\lambda\circ\mu\nleq\lambda\wedge\mu$.

3.19 Example

Consider the semihypergroup $H=\{a,b,c,d\}$ with the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a,b\}$	a
d	a	a	$\{a,b\}$	$\{a,b\}$

One can easily check that $\{a\},\{a,b\},\{a,b,c\},\{a,b,d\}$ and $\{a,b,c,d\}$ are all hyperideals of H.

Define fuzzy sets λ,μ of H by

$$\lambda(a)=0.7,\quad\lambda(b)=0.3,\quad\lambda(c)=0.4,\quad\lambda(d)=0,$$
$$\mu(a)=0.8,\quad\mu(b)=0.3,\quad\mu(c)=0.4,\quad\mu(d)=0.2.$$

Then we have

$$U(\lambda;t)=\begin{cases}\{a,b,c\}&\text{if }0<t\leq0.3,\\\{a,c\}&\text{if }0.3<t\leq0.4,\\\{a\}&\text{if }0.4<t\leq0.7,\\\emptyset&\text{if }0.7<t\leq1.\end{cases}$$

$$U(\mu;t)=\begin{cases}\{a,b,c,d\}&\text{if }0<t\leq0.2,\\\{a,b,c\}&\text{if }0.2<t\leq0.3,\\\{a,c\}&\text{if }0.3<t\leq0.4,\\\{a\}&\text{if }0.4<t\leq0.8,\\\emptyset&\text{if }0.8<t\leq1.\end{cases}$$

Thus by Theorem 3.16, λ,μ are $(\in,\in\vee q_k)$-fuzzy hyperideals of H with $k=0.4$.

Now

$$\begin{aligned}(\lambda\circ\mu)(b)&=\bigvee_{b\in x\circ y}\{\lambda(x)\wedge\mu(y)\}\\&=\bigvee\{0.4,0,0\}\\&=0.4\nleq(\lambda\wedge\mu)(b)=0.3.\end{aligned}$$

Hence $\lambda\circ\mu\nleq\lambda\wedge\mu$ in general.

3.20 Lemma

The intersection of any family of $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideals of H is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H.

*Proof.*Let $\{\lambda_i\}_{i\in I}$ be a family of $(\in, \in \vee q_k)$-fuzzy left hyperideals of H and $x, y \in H$. Then $\inf\limits_{z\in x\circ y}\{(\bigwedge\limits_{i\in I}\lambda_i)(z)\} = \bigwedge\limits_{i\in I}\{\inf\limits_{z\in x\circ y}\{\lambda_i\}(z)\}$.

(Since each λ_i is an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H, so $\inf\limits_{z\in x\circ y}\{\lambda_i\}(z) \geq \min\left\{\lambda_i(y), \frac{1-k}{2}\right\}$ for all $i \in I$.)

Thus

$$\begin{aligned}\inf_{z\in x\circ y}\{\left(\bigwedge_{i\in I}\lambda_i\right)(z)\} &= \bigwedge_{i\in I}\{\inf_{z\in x\circ y}\{\lambda_i\}(z)\} \\ &\geq \bigwedge_{i\in I}\left\{\lambda_i(y)\wedge\frac{1-k}{2}\right\} \\ &= \left(\bigwedge_{i\in I}\lambda_i(y)\right)\wedge\frac{1-k}{2} \\ &= \left(\bigwedge_{i\in I}\lambda_i\right)(y)\wedge\frac{1-k}{2}.\end{aligned}$$

Hence $\bigwedge\limits_{i\in I}\lambda_i$ is an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H.

3.21 Lemma

The union of any family of $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideals of H is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H.

*Proof.*Let $\{\lambda_i\}_{i\in I}$ be a family of $(\in, \in \vee q_k)$-fuzzy left hyperideals of H and $x, y \in H$. Then $\inf\limits_{z\in x\circ y}\{(\bigvee\limits_{i\in I}\lambda_i)(z)\} = \bigvee\limits_{i\in I}\{\inf\limits_{z\in x\circ y}\{\lambda_i\}(z)\}$.

(Since each λ_i is an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H,

so $\inf\limits_{z\in x\circ y}\{\lambda_i\}(z) \geq \min\left\{\lambda_i(y), \frac{1-k}{2}\right\}$ for all $i \in I$.)

Thus

$$\begin{aligned}\inf_{z\in x\circ y}\left\{\left(\bigvee_{i\in I}\lambda_i\right)\right\}(z) &= \bigvee_{i\in I}\{\inf_{z\in x\circ y}\{\lambda_i\}(z)\} \\ &\geq \bigvee_{i\in I}\left\{\lambda_i(y)\wedge\frac{1-k}{2}\right\} \\ &= \left(\bigvee_{i\in I}\lambda_i(y)\right)\wedge\frac{1-k}{2} \\ &= \left(\bigvee_{i\in I}\lambda_i\right)(y)\wedge\frac{1-k}{2}.\end{aligned}$$

Hence $\bigvee\limits_{i\in I}\lambda_i$ is an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H.

3.22 Definition

An $(\in, \in \vee q_k)$-fuzzy subsemihypergroup λ of H is called an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H if for all $x, y, z \in H$ and $t, r \in (0, 1]$ the following condition holds;

$x_t \in \lambda$ and $z_t \in \lambda \longrightarrow (w)_{\min\{t,r\}} \in \vee q_k\lambda$ for every $w \in x\circ y\circ z$.

3.23 Theorem

Let B be a non-empty subset of H and λ be a fuzzy subset in H defined by

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) If B is a bi-hyperideal of H then λ is a $(q, \in \vee q_k)$-fuzzy bi-hyperideal of H.

(2) B is a bi-hyperideal of H if and only if λ is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H.

*Proof.*The proof is similar to the proof of Theorem 3.2.

3.24 Corollary

(1) *If a non-empty subset B of a semihypergroup H is a bi-hyperideal of H then the characteristic function of B is a $(q, \in \vee q_k)$-fuzzy bi-hyperideal of H.*

(2) *A non-empty subset B of a semihypergroup H is a bi-hyperideal of H if and only if λ_B is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H.*

3.25 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H if and only if it satisfies the following conditions,

(1) $\inf\limits_{z\in x\circ y}\{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$ *for all* $x, y \in H$.

(2) $\inf\limits_{w\in x\circ y\circ z}\{\lambda(w)\} \geq \min\{\lambda(x), \lambda(z), \frac{1-k}{2}\}$ *for all* $x, y, z \in H$.

*Proof.*The proof is similar to the proof of Theorem 3.5.

If we take $k = 0$ in Thoerem 3.25, then we get the following Corollary.

3.26 Corollary

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H if and only if it satisfies the following conditions,

(1) $\inf\limits_{z\in x\circ y}\{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), 0.5\}$ *for all* $x, y \in H$,

(2) $\inf\limits_{w\in x\circ y\circ z}\{\lambda(w)\} \geq \min\{\lambda(x), \lambda(z), 0.5\}$ *for all* $x, y, z \in H$.

3.27 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q_k)$-fuzzy bi-hyperideal of H if and only if $U(\lambda;t)(\neq\emptyset)$ is a bi-hyperideal of H for all $t\in(0,\frac{1-k}{2}]$.

*Proof.*The proof is similar to the proof of Theorem 3.7.

If we take $k=0$ in Thoerem 3.27, then we get the following Corollary.

3.28 Corollary

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q)$-fuzzy bi-hyperideal of H if and only if $U(\lambda;t)(\neq\emptyset)$ is a bi-hyperideal of H for all $t\in(0,0.5]$.

3.29 Definition

A fuzzy subset λ of a semihypergroup H is called an $(\in,\in\vee q_k)$-fuzzy generalized bi-hyperideal of H if for all $x,y,z\in H$ and $t,r\in(0,1]$ the following condition holds;

$x_t\in\lambda$ and $z_t\in\lambda\longrightarrow(w)_{\min\{t,r\}}\in\vee q_k\lambda$ for every $w\in x\circ y\circ z$.

3.30 Theorem

Let B be a non-empty subset of H and λ be a fuzzy subset in H such that

$$\lambda(x)=\begin{cases}\geq\frac{1-k}{2} & \text{if } x\in B\\ 0 & \text{otherwise.}\end{cases}$$

Then

(1) If B is a generalized bi-hyperideal of H then λ is a $(q,\in\vee q_k)$-fuzzy generalized bi-hyperideal of H.

(2) B is a generalized bi-hyperideal of H if and only if λ is a $(\in,\in\vee q_k)$-fuzzy generalized bi-hyperideal of H.

*Proof.*The proof is similar to the proof of Theorem 3.2.

3.31 Corollary

(1) If a non-empty subset B of a semihypergroup H is a generalized bi-hyperideal of H then the characteristic function of B is a $(q,\in\vee q_k)$-fuzzy generalized bi-hyperideal of H.

(2) A non-empty subset B of a semihypergroup H is a generalized bi-hyperideal of H if and only if λ_B is a $(\in,\in\vee q_k)$-fuzzy generalized bi-hyperideal of H.

If we take $k=0$ in Thoerem 3.30, then we get the following Corollary.

3.32 Corollary

Let B be a non-empty subset of H and λ be a fuzzy subset in H such that

$$\lambda(x)=\begin{cases}\geq 0.5 & \text{if } x\in B\\ 0 & \text{otherwise.}\end{cases}$$

Then

(1) If B is a generalized bi-hyperideal of H then λ is a $(q,\in\vee q)$-fuzzy generalized bi-hyperideal of H.

(2) B is a generalized bi-hyperideal of H if and only if λ is a $(\in,\in\vee q)$-fuzzy generalized bi-hyperideal of H.

3.33 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q_k)$-fuzzy generalized bi-hyperideal of H if and only if it satisfies the following condition,

$$\inf_{w\in x\circ y\circ z}\{\lambda(w)\}\geq\min\{\lambda(x),\lambda(z),\frac{1-k}{2}\}\ \text{for all } x,y,z\in H.$$

*Proof.*The proof is similar to the proof of Theorem 3.5.

If we take $k=0$ in Thoerem 3.33, then we get the following Corollary.

3.34 Corollary

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q)$-fuzzy generalized bi-hyperideal of H if and only if it satisfies the following condition,

$$\inf_{w\in x\circ y\circ z}\{\lambda(w)\}\geq\min\{\lambda(x),\lambda(z),0.5\}\ \text{for all } x,y,z\in H.$$

3.35 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q_k)$-fuzzy generalized bi-hyperideal of H if and only if $U(\lambda;t)(\neq\emptyset)$ is a generalized bi-hyperideal of H for all $t\in(0,\frac{1-k}{2}]$.

*Proof.*The proof is similar to the proof of Theorem 3.7.

If we take $k=0$ in Thoerem 3.35, then we get the following Corollary.

3.36 Corollary

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q)$-fuzzy generalized bi-hyperideal of H if and only if $U(\lambda;t)(\neq\emptyset)$ is a generalized bi-hyperideal of H for all $t\in(0,0.5]$.

It is clear that every $(\in,\in\vee q_k)$-fuzzy bi-hyperideal of a semihypergroup H is an $(\in,\in\vee q_k)$-fuzzy generalized bi-hyperideal of H. The next example shows that the $(\in,\in\vee q_k)$-fuzzy generalized bi-hyperideal of H is not necessarily an $(\in,\in\vee q_k)$-fuzzy bi-hyperideal of H.

3.37 Example

Consider the semihypergroup $H = \{a,b,c,d\}$ with the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a,b\}$	a
d	a	a	$\{a,b\}$	$\{a,b\}$

One can easily check that $\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\}$ and $\{a,b,c,d\}$ are all generalized bi-hyperideals of H and $\{a\},\{a,b\},\{a,b,c\},\{a,b,d\}$ and $\{a,b,c,d\}$ are all bi-hyperideals of H.

Define a fuzzy subset λ of H by

$$\lambda(a) = 0.8, \quad \lambda(b) = 0, \quad \lambda(c) = 0.4, \quad \lambda(d) = 0.$$

Then, we have

$$U(\lambda;t) = \begin{cases} \{a,c\} & \text{if } 0 < t \leq 0.4, \\ \{a\} & \text{if } 0.4 < t \leq 0.8, \\ \emptyset & \text{if } 0.8 < t \leq 1. \end{cases}$$

Thus by Theorem 3.35, λ is an $(\in,\in \vee q_k)$-fuzzy generalized bi-hyperideal of H for every $k \in [0,1)$ but λ is not an $(\in,\in \vee q_k)$-fuzzy bi-hyperideal of H, because $U(\lambda;0.4) = \{a,c\}$ is a generalized bi-hyperideal of H but not a bi-hyperideal of H.

3.38 Lemma

Every $(\in,\in \vee q_k)$-fuzzy generalized bi-hyperideal of a regular semihypergroup H is an $(\in,\in \vee q_k)$-fuzzy bi-hyperideal of H.

*Proof.*Let λ be an $(\in,\in \vee q_k)$-fuzzy generalized bi-hyperideal of H and $a,b \in H$. Then there exists $x \in H$ such that $b \in b \circ x \circ b$. Thus we have $a \circ b \subseteq a \circ (b \circ x \circ b) = a \circ (b \circ x) \circ b$. Thus

$$\inf_{z \in a \circ b} \{\lambda(z)\} \geq \inf_{z \in a \circ u \circ b} \{\lambda(z)\} \geq \min\{\lambda(a), \lambda(b), \frac{1-k}{2}\}.$$

This shows that λ is an $(\in,\in \vee q_k)$-fuzzy subsemihypergroup of H and so λ is an $(\in,\in \vee q_k)$-fuzzy bi-hyperideal of H.

3.39 Definition

A fuzzy subset λ of a semihypergroup H is called an $(\in,\in \vee q_k)$-fuzzy interior hyperideal of H if for all $x,y,a \in H$ and $t,r \in (0,1]$ the following conditions hold;

(1) $x_t \in \lambda$ and $y_r \in \lambda \longrightarrow (z)_{\min\{t,r\}} \in \vee q_k \lambda$ for every $z \in x \circ y$,

(2) $a_t \in \lambda \longrightarrow w_t \in \vee q_k \lambda$ for every $w \in x \circ a \circ y$.

3.40 Theorem

Let A be a non-empty subset of H and λ be a fuzzy subset in H such that

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) If A is an interior hyperideal of H then λ is a $(q,\in \vee q_k)$-fuzzy interoir hyperideal of H.

(2) A is an interior hyperideal of H if and only if λ is an $(\in,\in \vee q_k)$-fuzzy interoir hyperideal of H.

*Proof.*The proof is similar to the proof of Theorem 3.2.

3.41 Corollary

(1) If a non-empty subset A of a semihypergroup H is an interior hyperideal of H then the characteristic function of A is a $(q,\in \vee q)$-fuzzy interior hyperideal of H.

(2) A non-empty subset A of a semihypergroup H is an interior hyperideal of H if and only if λ_A is a $(\in,\in \vee q)$-fuzzy interior hyperideal of H.

If we take $k = 0$ in Thoerem 3.40, then we get the following Corollary.

3.42 Corollary

Let A be a non-empty subset of H and λ be a fuzzy subset in H such that

$$\lambda(x) = \begin{cases} \geq 0.5 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) If A is an interior hyperideal of H then λ is a $(q,\in \vee q)$-fuzzy interoir hyperideal of H.

(2) A is an interior hyperideal of H if and only if λ is an $(\in,\in \vee q)$-fuzzy interoir hyperideal of H.

3.43 Theorem

A fuzzy subset λ of a semihypergroup H is *an $(\in,\in \vee q_k)$-fuzzy interior hyperideal of H if and only if it satisfies the following conditions,*

(1) $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$ for all $x,y \in H$.

(2) $\inf_{w \in x \circ a \circ y} \{\lambda(w)\} \geq \min\{\lambda(a), \frac{1-k}{2}\}$ for all $a,x,y \in H$.

*Proof.*The proof is similar to the proof of Theorem 3.5.

If we take $k = 0$ in Theorem 3.43, then we get the following corollary.

3.44 Corollary

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q)$-fuzzy interior hyperideal of H if and only if it satisfies the following conditions,

(1) $\inf_{z\in x\circ y}\{\lambda(z)\}\geq\min\{\lambda(x),\lambda(y),0.5\}$ *for all* $x,y\in H$.

(2) $\inf_{w\in x\circ a\circ y}\{\lambda(w)\}\geq\min\{\lambda(a),0.5\}$ *for all* $a,x,y\in H$.

3.45 Lemma

The intersection of any family of $(\in,\in\vee q_k)$-fuzzy interior hyperideals of a semihypergroup H is an $(\in,\in\vee q_k)$-fuzzy interior hyperideal of H.

Proof. Straightfoward.

3.46 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q_k)$-fuzzy interior hyperideal of H if and only if $U(\lambda;t)(\neq\emptyset)$ is an interior hyperideal of H for all $t\in(0,\frac{1-k}{2}]$.

Proof. The proof is similar to the proof of Theorem 3.7.

If we take $k=0$ in Theorem 3.46, then we get the following corollary.

3.47 Corollary

A fuzzy subset λ of a semihypergroup H is an $(\in,\in\vee q)$-fuzzy interior hyperideal of H if and only if $U(\lambda;t)(\neq\emptyset)$ is an interior hyperideal of H for all $t\in(0,0.5]$.

3.48 Lemma

Every $(\in,\in\vee q_k)$-fuzzy hyperideal of a semihypergroup H is an $(\in,\in\vee q_k)$-fuzzy interior hyperideal of H.

Proof. Let λ be an $(\in,\in\vee q_k)$-fuzzy hyperideal of H. Then

$$\inf_{\alpha\in x\circ y}\{\lambda(\alpha)\}\geq\min\{\lambda(x),\frac{1-k}{2}\}\geq\min\{\lambda(x),\lambda(y),\frac{1-k}{2}\}.$$

Thus λ is an $(\in,\in\vee q_k)$-fuzzy subsemihypergroup.

Let $x,a,y\in H$. Then for $w\in x\circ a\circ y=x\circ(a\circ y)$, so there exists $z\in a\circ y$, such that $w\in x\circ z$.

Thus

$$\inf_{w\in x\circ z}\{\lambda(w)\}\geq\min\{\lambda(z),\frac{1-k}{2}\}.$$

$$\Rightarrow\lambda(w)\geq\min\{\lambda(z),\frac{1-k}{2}\}.\qquad(*)$$

As $z\in a\circ y$, so

$$\inf_{\gamma\in a\circ y}\{\lambda(\gamma)\}\geq\min\{\lambda(a),\frac{1-k}{2}\}.$$

$$\Rightarrow\lambda(z)\geq\min\{\lambda(a),\frac{1-k}{2}\}.$$

$$\Rightarrow\min\{\lambda(z),\frac{1-k}{2}\}\geq\min\{\lambda(a),\frac{1-k}{2}\}.$$

Thus from (*) we have

$$\lambda(w)\geq\min\{\lambda(a),\frac{1-k}{2}\}.$$

$\Rightarrow\inf_{w\in x\circ a\circ y}\{\lambda(w)\}\geq\min\{\lambda(a),\frac{1-k}{2}\}$. Hence λ is an $(\in,\in\vee q_k)$-fuzzy interior hyperideal of H.

The following example shows that an $(\in,\in\vee q_k)$-fuzzy interior hyperideal of H need not be an $(\in,\in\vee q_k)$-fuzzy hyperideal of H. Also union of $(\in,\in\vee q_k)$-fuzzy interior hyperideals of H need not be an $(\in,\in\vee q_k)$-fuzzy interior hyperideal of H.

3.49 Example

Let $H=\{a,b,c,d\}$ be a semihypergroup with the following multiplication table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	$\{a,d\}$	a
c	a	a	a	a
d	a	a	a	a

Then the interior hyperideals of H are $\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,b,d\},\{a,c,d\}$ and H but $\{a,b\}$ is not a hyperideal of H. Define fuzzy subsets λ,μ of H by

$\lambda(a)=0.8=\lambda(b),\quad\lambda(c)=0=\lambda(d);$
$\mu(a)=0.8=\mu(c),\quad\mu(b)=0=\mu(d).$

Then we have

$$U(\lambda;t)=\begin{cases}\{a,b\} & \text{if } 0<t\leq0.8,\\ \emptyset & \text{if } 0.8<t\leq1.\end{cases}$$

$$U(\mu;t)=\begin{cases}\{a,c\} & \text{if } 0<t\leq0.8,\\ \emptyset & \text{if } 0.8<t\leq1.\end{cases}$$

Thus by Theorem 3.46, λ,μ are $(\in,\in\vee q_k)$-fuzzy interior hyperideals of H for every $k\in(0,1]$. But $U(\lambda\vee\mu;t)=\{a,b,c\}$ if $t\in(0,\frac{1-k}{2}]$ for $k=0.4$, which is not an interior hyperideal of H, so $\lambda\vee\mu$ is not an $(\in,\in\vee q_k)$-fuzzy interior hyperideal of H.

Also λ is not an $(\in,\in\vee q_k)$-fuzzy hyperideal of H because $\{a,b\}$ is not a hyperideal of H.

3.50 Definition

A fuzzy subset λ of a semihypergroup H is called an $(\in,\in \vee q_k)$-fuzzy quasi-hyperideal of H, if it satisfies,

$$\lambda(x) \geq \min\{(1\circ\lambda)(x),(\lambda\circ 1)(x),\frac{1-k}{2}\}.$$

3.51 Theorem

Let λ be an $(\in,\in \vee q_k)$-fuzzy quasi-hyperideal of H. Then the set $\lambda_0 = \{x \in H \mid \lambda(x) > 0\}$ is a quasi-hyperideal of H.

*Proof.*In order to show that λ_0 is a quasi-hyperideal of H, we have to show that $H\circ\lambda_0 \cap \lambda_0 \circ H \subseteq \lambda_0$. Let $a \in H\circ\lambda_0 \cap \lambda_0\circ H$. This means $a \in H\circ\lambda_0$ and $a \in \lambda_0\circ H$. So $a \in s\circ x$ and $a \in y\circ t$ for some $s,t \in H$ and $x,y \in \lambda_0$. Thus $\lambda(x) > 0$ and $\lambda(y) > 0$.

Since

$$\begin{aligned}(1\circ\lambda)(a) &= \bigvee_{a\in s\circ x}\{1(s)\wedge\lambda(x)\}\\ &\geq \{1(s)\wedge\lambda(x)\}\\ &= \{1\wedge\lambda(x)\}\\ &= \lambda(x).\end{aligned}$$

Similarly $(\lambda\circ 1)(a) \geq \lambda(y)$.
Thus

$$\begin{aligned}\lambda(a) &\geq \min\{(1\circ\lambda)(a),(\lambda\circ 1)(a),\frac{1-k}{2}\}\\ &\geq \min\{\lambda(x),\lambda(y),\frac{1-k}{2}\}\\ &> 0 \text{ because } \lambda(x) > 0 \text{ and } \lambda(y) > 0.\end{aligned}$$

Thus $a \in \lambda_0$. Hence λ_0 is a quasi-hyperideal of H.

3.52 Lemma

A non-empty subset Q of a semihypergroup H is a quasi-hyperideal of H if and only if the characteristic function λ_Q of Q is an $(\in,\in \vee q_k)$-fuzzy quasi-hyperideal of H.

*Proof.*Suppose Q is a quasi-hyperideal of H and λ_Q is the characteristic function of Q. Let $x \in H$. If $x \notin Q$ then $x \notin H\circ Q$ or $x \notin Q\circ H$. Thus $(1\circ\lambda_Q)(x) = 0$ or $(\lambda_Q\circ 1)(x) = 0$ and so $\min\{(1\circ\lambda_Q)(x),(\lambda_Q\circ 1)(x),\frac{1-k}{2}\} = 0 = \lambda_Q(x)$. If $x \in Q$ then $\lambda_Q(x) = 1 \geq \min\{(1\circ\lambda_Q)(x),(\lambda_Q\circ 1)(x),\frac{1-k}{2}\}$. Hence λ_Q is an $(\in,\in \vee q_k)$-fuzzy quasi-hyperideal of H.

Conversely, assume that λ_Q is an $(\in,\in \vee q_k)$-fuzzy quasi-hyperideal of H. Then Q is a quasi-hyperideal of H, by Theorem 3.51

3.53 Theorem

Every $(\in,\in \vee q_k)$-fuzzy left (right) hyperideal λ of H is an $(\in,\in \vee q_k)$-fuzzy quasi-hyperideal of H.

*Proof.*Let λ be an $(\in,\in \vee q_k)$-fuzzy left hyperideal λ of H and $x \in H$. Then

$$(1\circ\lambda)(x) = \bigvee_{x\in y\circ z}\{1(y)\wedge\lambda(z)\} = \bigvee_{x\in y\circ z}\lambda(z).$$

This implies that

$$\begin{aligned}(1\circ\lambda)(x)\wedge\frac{1-k}{2} &= \left(\bigvee_{x\in y\circ z}\lambda(z)\right)\wedge\frac{1-k}{2}\\ &= \bigvee_{x\in y\circ z}\left\{\lambda(z)\wedge\frac{1-k}{2}\right\}\\ &\leq \bigvee_{x\in y\circ z}\lambda(x)\end{aligned}$$

$$\left(\begin{array}{c}\text{Since } \lambda \text{ is an } (\in,\in \vee q_k)\text{-fuzzy}\\ \text{left hyperideal of } H, \text{ so}\\ \inf\limits_{x\in y\circ z}\lambda(x) \geq \lambda(z)\wedge\frac{1-k}{2}.\end{array}\right)$$

So $(1\circ\lambda)(x)\wedge\dfrac{1-k}{2} \leq \bigvee_{x\in y\circ z}\lambda(x) = \lambda(x)$.

Hence $\lambda(x) \geq (1\circ\lambda)(x)\wedge\frac{1-k}{2} \geq \min\{(1\circ\lambda)(x),(\lambda\circ 1)(x),\frac{1-k}{2}\}$. Thus λ is an $(\in,\in \vee q_k)$-fuzzy quasi-hyperideal of H.

3.54 Theorem

Every $(\in,\in \vee q_k)$-fuzzy quasi-hyperideal of a semihypergroup H is an $(\in,\in \vee q_k)$-fuzzy bi-hyperideal of H.

*Proof.*Suppose λ is an $(\in,\in \vee q_k)$-fuzzy quasi-hyperideal of semihypergroup H. Let $x,y \in H$. Now for every $\alpha \in x\circ y$, we have

$$\begin{aligned}\lambda(\alpha) &\geq \min\{(1\circ\lambda)(\alpha),(\lambda\circ 1)(\alpha),\frac{1-k}{2}\}\\ &= \left[\bigvee_{\alpha\in x\circ y}\{1(x)\wedge\lambda(y)\}\right]\wedge\\ &\quad \left[\bigvee_{\alpha\in x\circ y}\{\lambda(x)\wedge 1(y)\}\right]\wedge\frac{1-k}{2}\\ &\geq \{1(x)\wedge\lambda(y)\}\wedge\{\lambda(x)\wedge 1(y)\}\wedge\frac{1-k}{2}\\ &\geq \{1\wedge\lambda(y)\}\wedge\{\lambda(x)\wedge 1\}\wedge\frac{1-k}{2}\\ &= \lambda(y)\wedge\lambda(x)\wedge\frac{1-k}{2}.\end{aligned}$$

So $\inf\limits_{\alpha\in x\circ y}\{\lambda(\alpha)\} \geq \min\{\lambda(x),\lambda(y),\frac{1-k}{2}\}$ for all $x,y \in H$.

Also for all $x,y,z \in H$ and for every $w \in x \circ y \circ z$, there exists $a \in x \circ y$ such that $w \in a \circ z$. Also there exists $b \in y \circ z$ such that $w \in x \circ b$. Thus

$$\begin{aligned}\lambda(w) &\geq \min\{(1 \circ \lambda)(w), (\lambda \circ 1)(w), \frac{1-k}{2}\} \\ &= \left[\bigvee_{w \in a \circ z} \{1(a) \wedge \lambda(z)\}\right] \wedge \\ &\quad \left[\bigvee_{w \in x \circ b} \{\lambda(x) \wedge 1(b)\}\right] \wedge \frac{1-k}{2} \\ &\geq \{1(a) \wedge \lambda(z)\} \wedge \{\lambda(x) \wedge 1(b\} \wedge \frac{1-k}{2} \\ &\geq \{1 \wedge \lambda(z)\} \wedge \{\lambda(x) \wedge 1\} \wedge \frac{1-k}{2} \\ &= \lambda(z) \wedge \lambda(x) \wedge \frac{1-k}{2}.\end{aligned}$$

So $\inf\limits_{w \in x \circ y \circ z} \{\lambda(w)\} \geq \min\{\lambda(x), \lambda(z), \frac{1-k}{2}\}$ for all $x,y,z \in H$. Thus λ is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H.

The following example shows that the converse of the above Theorem is not true.

3.55 Example

Consider the semihypergroup $H = \{a,b,c,d\}$ with the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	$\{a,b\}$
d	a	a	$\{a,b\}$	$\{a,b,c\}$

One can easily check that $\{a,c\}$ is a bi-hyperideal of H but not a quasi-hyperideal of H.

Define a fuzzy subset λ of H by

$$\lambda(a) = 0.8 = \lambda(c), \qquad \lambda(b) = 0 = \lambda(d).$$

Then λ is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H for every $k \in [0,1)$ but not an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. Because $(1 \circ \lambda)(b) = 0.8 = (\lambda \circ 1)(b)$ but $\lambda(b) = 0 \ngeq \min\{(1 \circ \lambda)(b), (\lambda \circ 1)(b), \frac{1-k}{2}\}$.

4 Regular semihypergroups

In this section we characterize regular semihypergroups by the properties of their $(\in, \in \vee q_k)$-fuzzy hyperideals, $(\in, \in \vee q_k)$-fuzzy quasi- hyperideals and $(\in, \in \vee q_k)$-fuzzy bi-hyperideals.

4.1 Definition

Let λ, μ be fuzzy subsets of H. We define the fuzzy subsets $\lambda_k, \lambda \wedge_k \mu, \lambda \vee_k \mu$ and $\lambda \circ_k \mu$ of H as follows;

$$\begin{aligned}\lambda_k(x) &= \lambda(x) \wedge \frac{1-k}{2} \\ (\lambda \wedge_k \mu)(x) &= (\lambda \wedge \mu)(x) \wedge \frac{1-k}{2} \\ (\lambda \vee_k \mu)(x) &= (\lambda \vee \mu)(x) \wedge \frac{1-k}{2} \\ (\lambda \circ_k \mu)(x) &= (\lambda \circ \mu)(x) \wedge \frac{1-k}{2} \qquad \text{for all } x \in H.\end{aligned}$$

4.2 Lemma

Let λ, μ be fuzzy subsets of H. Then the following hold.

(1) $(\lambda \wedge_k \mu) = (\lambda_k \wedge \mu_k)$

(2) $(\lambda \vee_k \mu) = (\lambda_k \vee \mu_k)$

(3) $(\lambda \circ_k \mu) = (\lambda_k \circ \mu_k)$.

Proof. Let $x \in H$.

(1)

$$\begin{aligned}(\lambda \wedge_k \mu)(x) &= (\lambda \wedge \mu)(x) \wedge \frac{1-k}{2} \\ &= \lambda(x) \wedge \mu(x) \wedge \frac{1-k}{2} \\ &= \left(\lambda(x) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(x) \wedge \frac{1-k}{2}\right) \\ &= \lambda_k(x) \wedge \mu_k(x) \\ &= (\lambda_k \wedge \mu_k)(x).\end{aligned}$$

(2)

$$\begin{aligned}(\lambda \vee_k \mu)(x) &= (\lambda \vee \mu)(x) \wedge \frac{1-k}{2} \\ &= (\lambda(x) \vee \mu(x)) \wedge \frac{1-k}{2} \\ &= \left(\lambda(x) \wedge \frac{1-k}{2}\right) \vee \left(\mu(x) \wedge \frac{1-k}{2}\right) \\ &= \lambda_k(x) \vee \mu_k(x) \\ &= (\lambda_k \vee \mu_k)(x).\end{aligned}$$

(3) If $x \notin y \circ z$ for all $y,z \in H$, then $(\lambda \circ \mu)(x) = 0$. Thus

$$(\lambda \circ_k \mu)(x) = (\lambda \circ \mu)(x) \wedge \frac{1-k}{2} = 0.$$

If $x \in y \circ z$ for some $y, z \in H$, then

$$\begin{aligned}(\lambda \circ_k \mu)(x) &= (\lambda \circ \mu)(x) \wedge \frac{1-k}{2} \\ &= \left[\bigvee_{x \in y \circ z} \{\lambda(y) \wedge \mu(z)\}\right] \wedge \frac{1-k}{2} \\ &= \bigvee_{x \in y \circ z} \left(\{\lambda(y) \wedge \mu(z)\} \wedge \frac{1-k}{2}\right) \\ &= \bigvee_{x \in y \circ z} \left\{\lambda(y) \wedge \mu(z) \wedge \frac{1-k}{2}\right\} \\ &= \bigvee_{x \in y \circ z} \left\{\left(\lambda(y) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(z) \wedge \frac{1-k}{2}\right)\right\} \\ &= \bigvee_{x \in y \circ z} \{\lambda_k(y) \wedge \mu_k(z)\} \\ &= (\lambda_k \circ \mu_k)(x).\end{aligned}$$

4.3 Lemma

Let A and B be nonempty subsets of a semihypergroup H. Then the following hold.
(1) $(\lambda_A \wedge_k \lambda_B) = (\lambda_{A\cap B})_k$
(2) $(\lambda_A \vee_k \lambda_B) = (\lambda_{A\cup B})_k$
(3) $(\lambda_A \circ_k \lambda_B) = (\lambda_{A\circ B})_k$.

Proof. Straightforward.

Next we show that if λ is an $(\in, \in \vee q_k)$-fuzzy left (right) hyperideal of H then λ_k is a fuzzy left (right) hyperideal of H.

4.4 Lemma

Let λ be an $(\in, \in \vee q_k)$-fuzzy left (right) hyperideal of H. Then λ_k is a fuzzy left (right) hyperideal of H.

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H. Then for all $x, y \in H$, we have $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(y) \wedge \frac{1-k}{2}$. This implies that

$$\inf_{z \in x \circ y} \{\lambda(z)\} \wedge \frac{1-k}{2} \geq \lambda(y) \wedge \frac{1-k}{2}.$$

So $\inf\limits_{z \in x \circ y} \{\lambda_k(z)\} \geq \lambda_k(y)$. Thus λ_k is a fuzzy left hyperideal of H.

4.5 Lemma

A nonempty subset L of H is a left (right) hyperideal of H if and and only if $(\lambda_L)_k$ is an $(\in, \in \vee q_k)$-fuzzy left (right) hyperideal of H.

Proof. The proof follows from Theorem 3.10.

4.6 Lemma

A non-empty subset Q of a semihypergroup H is a quasi-hyperideal of H if $(\lambda_Q)_k$ is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H.

Proof. The proof follows from Theorem 3.10.

4.7 Proposition

Let λ be an $(\in, \in \vee q_k)$-fuzzy left (right) hyperideal of H. Then λ_k is a fuzzy left (right) hyperideal of H.

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H. Then for all $x, y \in H$, we have $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(y) \wedge \frac{1-k}{2}$. This implies that

$$\inf_{z \in x \circ y} \{\lambda(z)\} \wedge \frac{1-k}{2} \geq \lambda(y) \wedge \frac{1-k}{2}.$$

So $\inf\limits_{z \in x \circ y} \{\lambda_k(z)\} \geq \lambda_k(y)$. Thus λ_k is a fuzzy left hyperideal of H.

4.8 Proposition

Let λ be a fuzzy subsemihypergroup H. Then $\lambda \circ_k \lambda \leq \lambda_k$.

Proof. Let λ be a fuzzy subsemihypergroup H. If $A_x = \{(y, z) \in H \times H : x \in y \circ z\} = \emptyset$, then

$$(\lambda \circ \lambda)(x) = 0 \leq \lambda(x).$$

If $A_x \neq \emptyset$, then $(\lambda \circ_k \lambda)(x) = (\lambda \circ \mu)(x) \wedge \frac{1-k}{2}$. As λ is a fuzzy subsemihypergroup of H, so for each $x \in y \circ z$,

$$\lambda(x) \geq \min\{\lambda(y), \lambda(z)\}, \text{ for all } y, z \in H.$$

This implies that

$$\lambda_k(x) = \lambda(x) \wedge \frac{1-k}{2} \geq \min\left\{\lambda(y), \lambda(z), \frac{1-k}{2}\right\}, \text{ for all } y, z \in H.$$

Hence $(\lambda \circ_k \lambda)(x) = (\lambda \circ \mu)(x) \wedge \frac{1-k}{2} = \bigvee\limits_{x \in y \circ z} \min\{\lambda(y), \lambda(z), \frac{1-k}{2}\} \leq \lambda_k(x)$. Thus, $\lambda \circ_k \lambda \leq \lambda_k$.

Next we characterize regular semihypergroups by the properties of $(\in, \in \vee q_k)$-fuzzy hyperideals, quasi-hyperideals, bi-hyperideals and generalized bi-hyperideals.

4.9 Lemma

Let λ be an $(\in, \in \vee q_k)$-fuzzy right hyperideal and μ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal of a semihypergroup H. Then $\lambda \circ_k \mu \leq \lambda \wedge_k \mu$.

*Proof.*Let λ be an $(\in, \in \vee q_k)$-fuzzy right hyperideal and μ be an

$(\in, \in \vee q_k)$-fuzzy left hyperideal of H. Let $y, z \in H$. Then for every $a \in y \circ z$ we have

$$\begin{aligned}(\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in y \circ z} \{\lambda(y) \wedge \mu(z)\}\right) \wedge \frac{1-k}{2} \\ &= \bigvee_{a \in y \circ z} \left(\{\lambda(y) \wedge \mu(z)\} \wedge \frac{1-k}{2}\right) \\ &= \bigvee_{a \in y \circ z} \left\{ \begin{array}{c} (\lambda(y) \wedge \frac{1-k}{2}) \wedge \\ (\mu(z) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} \end{array} \right\} \\ &\leq \bigvee_{a \in y \circ z} \left\{\lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2}\right\} \\ &= \lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2} \\ &= (\lambda \wedge_k \mu)(a).\end{aligned}$$

So $(\lambda \circ_k \mu) \leq (\lambda \wedge_k \mu)$.

4.10 Theorem

The following assertions are equivalent for a semihypergroup H.

(1) H is regular.

(2) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy left hyperideal μ of H.

Proof.(1) $\Rightarrow$ (2) Since H is regular, so there exists $x \in H$ such that $a \in a \circ x \circ a = (a \circ x) \circ a$. Thus there exists some $\beta \in a \circ x$ such that $a \in \beta \circ a$. So

$$\begin{aligned}(\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in c \circ d} \{\lambda(c) \wedge \mu(d)\}\right) \wedge \frac{1-k}{2} \\ &\geq \{\lambda(\beta) \wedge \mu(a)\} \wedge \frac{1-k}{2} \\ & \left(\begin{array}{c} \text{Since } \beta \in a \circ x \text{ and } \lambda \text{ is an} \\ (\in, \in \vee q_k)\text{-fuzzy right} \\ \text{hyperideal of } H, \text{ so} \\ \inf_{\gamma \in \theta \circ \delta} \{\lambda(z)\} \geq \lambda(\theta) \wedge \frac{1-k}{2} \\ \text{therefore } \lambda(\beta) \geq \lambda(\theta) \wedge \frac{1-k}{2}. \end{array} \right)\end{aligned}$$

$$\begin{aligned}\text{Therefore } (\lambda \circ_k \mu)(a) &\geq \left\{\lambda(a) \wedge \frac{1-k}{2} \wedge \mu(a)\right\} \wedge \frac{1-k}{2} \\ &= \{\lambda(a) \wedge \mu(a)\} \wedge \frac{1-k}{2} \\ &= (\lambda \wedge_k \mu)(a).\end{aligned}$$

So $\lambda \circ_k \mu \geq \lambda \wedge_k \mu$. But by Lemma 4.9, $\lambda \circ_k \mu \leq \lambda \wedge_k \mu$. Hence $\lambda \wedge_k \mu = \lambda \circ_k \mu$.

(2) $\Rightarrow$ (1) Let R and L be right and left hyperideals of H. Then by Lemma 4.5, $(\lambda_R)_k$ and $(\lambda_L)_k$ are $(\in, \in \vee q_k)$-fuzzy right and $(\in, \in \vee q_k)$-fuzzy left hyperideals of H, respectively. Thus by hypothesis

$$\begin{aligned}(\lambda_{R \circ L})_k &= (\lambda_R \circ_k \lambda_L) \\ &= (\lambda_R \wedge_k \lambda_L) \text{ by (2)} \\ &= (\lambda_{R \cap L})_k.\end{aligned}$$

This implies $R \cap L = R \circ L$. Hence it follows from Proposition 2.2 that H is regular.

4.11 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) H is regular.

(2) $(\lambda \wedge_k \mu \wedge_k \nu) \leq (\lambda \circ_k \mu \circ_k \nu)$ for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ, for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal μ and for every $(\in, \in \vee q_k)$-fuzzy left hyperideal ν of H.

(3) $(\lambda \wedge_k \mu \wedge_k \nu) \leq (\lambda \circ_k \mu \circ_k \nu)$ for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ, for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal μ and for every $(\in, \in \vee q_k)$-fuzzy left hyperideal ν of H.

(4) $(\lambda \wedge_k \mu \wedge_k \nu) \leq (\lambda \circ_k \mu \circ_k \nu)$ for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ, for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal μ and for every $(\in, \in \vee q_k)$-fuzzy left hyperideal ν of H.

Proof.(1) $\Rightarrow$ (2) Let λ, μ and ν be any $(\in, \in \vee q_k)$-fuzzy right hyperideal, $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal and $(\in, \in \vee q_k)$-fuzzy left hyperideal of H, respectively. Let $a \in H$. Since H is regular, so there exists $x \in H$ such that $a \in a \circ x \circ a = (a \circ x) \circ a$. Thus there exists $\beta \in a \circ x$ such that $a \in \beta \circ a$. So

$$\begin{aligned}(\lambda \circ_k \mu \circ_k \nu)(a) &= (\lambda \circ \mu \circ \nu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in \beta \circ a} \{\lambda(\beta) \wedge (\mu \circ \nu)(a)\}\right) \wedge \frac{1-k}{2} \\ &\geq \{\lambda(\beta) \wedge (\mu \circ \nu)(a)\} \wedge \frac{1-k}{2} \\ & \left(\text{since } \inf_{\beta \in a \circ x} \{\lambda(\beta)\} \geq \lambda(a) \wedge \frac{1-k}{2}.\right)\end{aligned}$$

$$\text{so, } (\lambda \circ_k \mu \circ_k \nu)(a) \geq \left(\lambda(a) \wedge \frac{1-k}{2}\right) \wedge (\mu \circ \nu)(a) \wedge \frac{1-k}{2} \quad \text{(i)}$$

Since $a \in a \circ x \circ a = a \circ (x \circ a)$, so there exists $\gamma \in x \circ a$, such that $a \in a \circ \gamma$. Thus

$$\begin{aligned}(\mu \circ \nu)(a) &= \bigvee_{a \in a \circ \gamma} \{\mu(a) \wedge \nu(\gamma)\} \\ &\geq \{\mu(a) \wedge \nu(\gamma)\} \\ &\geq \left(\mu(a) \wedge \nu(a) \wedge \frac{1-k}{2}\right).\end{aligned}$$

Thus substitiuting value of $(\mu \circ \nu)(a)$ in (i), we have

$$(\lambda \circ_k \mu \circ_k \nu)(a) \geq \left(\lambda(a) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(a) \wedge \nu(a) \wedge \frac{1-k}{2}\right)$$
$$= (\lambda(a) \wedge \mu(a) \wedge \nu(a)) \wedge \frac{1-k}{2}$$
$$= (\lambda \wedge_k \mu \wedge_k \nu)(a).$$

$(2) \Rightarrow (3) \Rightarrow (4)$ Straightforward.

$(4) \Rightarrow (1)$ Let λ and ν be any $(\in, \in \vee q_k)$-fuzzy right and $(\in, \in \vee q_k)$-fuzzy left hyperideals of H, respectively. Since "1" is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H, so by hypothesis, we have

$$(\lambda \wedge_k \nu)(a) = (\lambda \wedge \nu)(a) \wedge \frac{1-k}{2}$$
$$= (\lambda \wedge 1 \wedge \nu)(a) \wedge \frac{1-k}{2}$$
$$= (\lambda \wedge_k 1 \wedge_k \nu)(a)$$
$$\leq (\lambda \circ_k 1 \circ_k \nu)(a)$$
$$= (\lambda \circ 1 \circ \nu)(a) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a \in b \circ c} \{(\lambda \circ 1)(b) \wedge \nu(c)\}\right) \wedge \frac{1-k}{2}$$
$$\left(\text{As } (\lambda \circ 1)(b) = \bigvee_{b \in p \circ q} \{\lambda(p) \wedge 1(q)\}\right)$$

Thus,

$$(\lambda \wedge_k \nu)(a) = \left(\bigvee_{a \in b \circ c}\left\{\left(\bigvee_{b \in p \circ q}\{\lambda(p) \wedge 1(q)\}\right) \wedge \nu(c)\right\}\right) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a \in b \circ c}\left\{\left(\bigvee_{b \in p \circ q}\{\lambda(p) \wedge 1\}\right) \wedge \nu(c)\right\}\right) \wedge \frac{1-k}{2}.$$
$$= \left(\bigvee_{a \in b \circ c}\left\{\left(\bigvee_{b \in p \circ q}\lambda(p)\right) \wedge \nu(c)\right\}\right) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a \in b \circ c}\left\{\left(\bigvee_{b \in p \circ q}\lambda(p)\right) \wedge \nu(c)\right\} \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a \in b \circ c}\left\{\left(\bigvee_{b \in p \circ q}\left\{\lambda(p) \wedge \frac{1-k}{2}\right\}\right) \wedge \nu(c)\right\}\right) \wedge \frac{1-k}{2}$$
$$\left(\begin{array}{c}\text{Since } \lambda \text{ is an } (\in, \in \vee q_k)\text{-fuzzy right} \\ \text{hyperidealof } H, \text{ so} \\ \inf_{b \in p \circ q}\{\lambda(b)\} \geq \lambda(p) \wedge \frac{1-k}{2} \text{ that is} \\ \lambda(b) \geq \lambda(p) \wedge \frac{1-k}{2} \text{ for every } b \in p \circ q.\end{array}\right)$$
$$\text{Thus, } (\lambda \wedge_k \nu)(a) \leq \left(\bigvee_{a \in b \circ c}\{\lambda(b) \wedge \nu(c)\}\right) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a \in b \circ c}\lambda(b) \wedge \nu(c)\right) \wedge \frac{1-k}{2}$$
$$= (\lambda \circ_k \nu)(a).$$

Thus it follows that $\lambda \wedge_k \nu \leq \lambda \circ_k \nu$ for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ of H and for every $(\in, \in \vee q_k)$-fuzzy left hyperideal ν of H. But by Lemma 4.9, $\lambda \wedge_k \nu \geq \lambda \circ_k \nu$. So $\lambda \wedge_k \nu = \lambda \circ_k \nu$. Hence by Theorem 4.10, H is regular.

4.12 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) H is regular.

(2) $\lambda_k = (\lambda \circ_k 1 \circ_k \lambda)$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal λ of H.

(3) $\lambda_k = (\lambda \circ_k 1 \circ_k \lambda)$ for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal λ of H.

(4) $\lambda_k = (\lambda \circ_k 1 \circ_k \lambda)$ for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ of H.

Proof. $(1) \Rightarrow (2)$ Let λ be an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H and $a \in H$. Since H is regular, so there exists $x \in H$ such that $a \in a \circ x \circ a = (a \circ x) \circ a$. Thus there exists $\beta \in a \circ x$ such that $a \in \beta \circ a$. So

$$\begin{aligned}(\lambda \circ_k 1 \circ_k \lambda)(a) &= (\lambda \circ 1 \circ \lambda)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in \beta \circ a} \{(\lambda \circ 1)(\beta) \wedge \lambda(a)\}\right) \wedge \frac{1-k}{2} \\ &\geq (\lambda \circ 1)(\beta) \wedge \lambda(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{\beta \in a \circ x} \{\lambda(a) \wedge 1(x)\}\right) \wedge \lambda(a) \wedge \frac{1-k}{2} \\ &\geq \{\lambda(a) \wedge 1(x)\} \wedge \lambda(a) \wedge \frac{1-k}{2} \\ &= \{\lambda(a) \wedge 1\} \wedge \lambda(a) \wedge \frac{1-k}{2} \\ &= \lambda_k(a).\end{aligned}$$

Thus $(\lambda \circ_k 1 \circ_k \lambda) \geq \lambda_k$.

Since λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H. So we have

$$\begin{aligned}(\lambda \circ_k 1 \circ_k \lambda)(a) &= (\lambda \circ 1 \circ \lambda)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in x \circ y} \{(\lambda \circ 1)(x) \wedge \lambda(y)\}\right) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in x \circ y} \left\{\left(\bigvee_{x \in p \circ q} \{\lambda(p) \wedge 1(q)\}\right) \wedge \lambda(y)\right\}\right) \\ &\quad \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in x \circ y} \left\{\left(\bigvee_{x \in p \circ q} \{\lambda(p) \wedge 1\}\right) \wedge \lambda(y)\right\}\right) \\ &\quad \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in x \circ y} \left\{\bigvee_{x \in p \circ q} \{\lambda(p) \wedge \lambda(y)\}\right\}\right) \wedge \frac{1-k}{2} \\ &= \bigvee_{a \in x \circ y} \left\{\bigvee_{x \in p \circ q} \{\lambda(p) \wedge \lambda(y)\} \wedge \frac{1-k}{2}\right\}\end{aligned}$$

$$\left(\begin{array}{c}\text{Since } a \in p \circ q \circ y \text{ and } \lambda \text{ is an } (\in, \in \vee q_k)\text{-fuzzy} \\ \text{generalized bi-hyperideal of } H \text{ so} \\ \inf_{a \in p \circ q \circ y} \{\lambda(a)\} \geq \{\lambda(p) \wedge \lambda(y)\} \wedge \frac{1-k}{2}.\end{array}\right)$$

Thus

$$\begin{aligned}(\lambda \circ_k 1 \circ_k \lambda)(a) &\leq \lambda(a) \wedge \frac{1-k}{2} \\ &= \lambda_k(a).\end{aligned}$$

This implies $(\lambda \circ_k 1 \circ_k \lambda) \leq \lambda_k$. Thus $\lambda_k = (\lambda \circ_k 1 \circ_k \lambda)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) Let A be any quasi-hyperideal of H. Then λ_A is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. Hence, by hypothesis,

$$(\lambda_A)_k = (\lambda_A \circ_k 1 \circ_k \lambda_A) = (\lambda_A \circ_k \lambda_H \circ_k \lambda_A) = (\lambda_{A \circ H \circ A})_k.$$

This implies $A = A \circ H \circ A$. Hence it follows from Proposition 2.2, that H is regular.

4.13 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) H is regular.

(2) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy hyperideal μ of H.

(3) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy interior hyperideal μ of H.

(4) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy hyperideal μ of H.

(5) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy interior hyperideal μ of H.

(6) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy hyperideal μ of H.

(7) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy interior hyperideal μ of H.

Proof.(1) $\Rightarrow$ (7) Let λ and μ be any $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal and $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H, respectively. Then

$$\begin{aligned}(\lambda \circ_k \mu \circ_k \lambda)(a) &= (\lambda \circ \mu \circ \lambda)(a) \wedge \frac{1-k}{2} \\ &\leq (\lambda \circ 1 \circ \lambda)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in x \circ y} \{(\lambda \circ 1)(x) \wedge \lambda(y)\}\right) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in x \circ y} \left\{\left(\bigvee_{x \in p \circ q} \{\lambda(p) \wedge 1(q)\}\right) \wedge \lambda(y)\right\}\right) \\ &\quad \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in x \circ y} \left\{\left(\bigvee_{x \in p \circ q} \{\lambda(p) \wedge 1\}\right) \wedge \lambda(y)\right\}\right) \\ &\quad \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in x \circ y} \left\{\bigvee_{x \in p \circ q} \{\lambda(p) \wedge \lambda(y)\}\right\}\right) \wedge \frac{1-k}{2} \\ &= \bigvee_{a \in x \circ y} \left\{\bigvee_{x \in p \circ q} \{\lambda(p) \wedge \lambda(y)\} \wedge \frac{1-k}{2}\right\}.\end{aligned}$$

Since $a \in p \circ q \circ y$ and λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H so

$$\inf_{a \in p \circ q \circ y} \{\lambda(a)\} \geq \{\lambda(p) \wedge \lambda(y)\} \wedge \frac{1-k}{2}.$$

Thus

$$(\lambda \circ_k \mu \circ_k \lambda)(a) \leq \bigvee_{a\in p\circ q\circ y} \left\{\lambda(a) \wedge \frac{1-k}{2}\right\}$$
$$\leq \lambda(a) \wedge \frac{1-k}{2}$$
$$= \lambda_k(a).$$

Thus $(\lambda \circ_k \mu \circ_k \lambda)(a) \leq \lambda_k(a)$. Also

$$(\lambda \circ_k \mu \circ_k \lambda)(a) \leq (1 \circ_k \mu \circ_k 1)(a)$$
$$= (1 \circ \mu \circ 1)(a) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a\in x\circ y} \{(1\circ\mu)(x) \wedge 1(y)\}\right) \wedge \frac{1-k}{2}$$
$$\left(\begin{array}{c}\text{Since } a \in x\circ y \text{ but for } x \in p\circ q,\\ \text{we have}\\ (1\circ\mu)(x) = \bigvee_{x\in p\circ q}\{1(p)\wedge\mu(q)\}\end{array}\right)$$
$$= \left(\bigvee_{a\in x\circ y}\left\{\left(\bigvee_{x\in p\circ q}\{1(p)\wedge\mu(q)\}\right)\wedge 1(y)\right\}\right)$$
$$\wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a\in x\circ y}\left\{\left(\bigvee_{x\in p\circ q}\{1\wedge\mu(q)\}\right)\wedge 1\right\}\right) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a\in x\circ y}\left\{\bigvee_{x\in p\circ q}\mu(q)\right\}\right) \wedge \frac{1-k}{2}$$
$$= \bigvee_{a\in x\circ y\subseteq p\circ q\circ y}\left\{\mu(q)\wedge\frac{1-k}{2}\right\}$$

Since μ is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H, so there exist $r, s, \theta \in H$ such that $\inf\limits_{z\in r\circ\theta\circ s}\{\mu(z)\} \geq \mu(\theta) \wedge \frac{1-k}{2}$. Thus $\mu(a) \geq \mu(q) \wedge \frac{1-k}{2}$. But $a \in x\circ y \subseteq (p\circ q)\circ y$, because $x \in p\circ q$, therefore,

$$(\lambda \circ_k \mu \circ_k \lambda)(a) \leq \bigvee_{a\in p\circ q\circ y}\left\{\mu(a)\wedge\frac{1-k}{2}\right\}$$
$$= \mu(a) \wedge \frac{1-k}{2} = \mu_k(a).$$

Hence $(\lambda \circ_k \mu \circ_k \lambda) \leq (\lambda_k \wedge \mu_k) = (\lambda \wedge_k \mu)$.

Now let $a \in H$. Since H is regular, so there exists $x \in H$ such that $a \in a\circ x\circ a = a\circ(x\circ a\circ x\circ a)$. Thus there exists $\gamma \in x\circ a\circ x$, and $\beta \in \gamma\circ a$ such that $a \in a\circ\beta$. So

$$(\lambda \circ_k \mu \circ_k \lambda)(a) = (\lambda\circ\mu\circ\lambda)(a) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a\in a\circ\beta}\{\lambda(a)\wedge(\mu\circ\lambda)(\beta)\}\right) \wedge \frac{1-k}{2}$$
$$\geq \{\lambda(a)\wedge(\mu\circ\lambda)(\beta)\} \wedge \frac{1-k}{2}$$
$$= \lambda(a) \wedge \left(\bigvee_{\beta\in\gamma\circ a}\{\mu(\gamma)\wedge\lambda(a)\}\right) \wedge \frac{1-k}{2}$$
$$\left(\begin{array}{c}\text{Since every } (\in,\in\vee q_k)\text{-fuzzy}\\ \text{hyperideal of } H \text{ is an}\\ (\in,\in\vee q_k)\text{-fuzzy hyperideal of } H\\ \text{so } \inf\limits_{\gamma\in x\circ a\circ x}\{\mu(\gamma)\} \geq \mu(a)\wedge\frac{1-k}{2}.\end{array}\right)$$
$$\geq \lambda(a) \wedge \left(\mu(a)\wedge\frac{1-k}{2}\wedge\lambda(a)\right) \wedge \frac{1-k}{2}$$
$$= \lambda(a)\wedge\mu(a)\wedge\frac{1-k}{2}$$
$$= (\lambda \wedge_k \mu)(a).$$

Hence $(\lambda \circ_k \mu \circ_k \lambda) \leq (\lambda_k \wedge \mu_k) = (\lambda \wedge_k \mu)$.

Now let $a \in H$. Since H is regular, so there exists $x \in H$ such that $a \in a\circ x\circ a = a\circ(x\circ a\circ x\circ a)$. Thus for each $\gamma \in x\circ a\circ x$, $\beta \in \gamma\circ a$ and thus $a \in a\circ\beta$. So

$$(\lambda \circ_k \mu \circ_k \lambda)(a) = (\lambda\circ\mu\circ\lambda)(a) \wedge \frac{1-k}{2}$$
$$= \left(\bigvee_{a\in a\circ\beta}\{\lambda(a)\wedge(\mu\circ\lambda)(\beta)\}\right) \wedge \frac{1-k}{2}$$
$$\geq \{\lambda(a)\wedge(\mu\circ\lambda)(\beta)\} \wedge \frac{1-k}{2}$$
$$= \lambda(a) \wedge \left(\bigvee_{\beta\in\gamma\circ a}\{\mu(\gamma)\wedge\lambda(a)\}\right) \wedge \frac{1-k}{2}$$
$$\geq \lambda(a) \wedge \left(\mu(a)\wedge\frac{1-k}{2}\wedge\lambda(a)\right) \wedge \frac{1-k}{2}$$
$$= \lambda(a)\wedge\mu(a)\wedge\frac{1-k}{2}$$
$$= (\lambda \wedge_k \mu)(a).$$

So $(\lambda \circ_k \mu \circ_k \lambda) \geq (\lambda \wedge_k \mu)$. Hence $(\lambda \circ_k \mu \circ_k \lambda) = (\lambda \wedge_k \mu)$.

$(7) \Rightarrow (5) \Rightarrow (3) \Rightarrow (2)$ and $(7) \Rightarrow (6) \Rightarrow (4) \Rightarrow (2)$ are obvious.

$(2) \Rightarrow (1)$ Let λ be any $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. Then, since "1" is an $(\in, \in \vee q_k)$-fuzzy two-sided hyperideal of H, we have

$$\lambda_k(a) = \lambda(a) \wedge \frac{1-k}{2}$$
$$= (\lambda \wedge 1)(a) \wedge \frac{1-k}{2}$$
$$= (\lambda \wedge_k 1)(a)$$
$$= (\lambda \circ_k 1 \circ_k \lambda)(a).$$

Thus it follows from Theorem 4.12 that H is regular.

4.14 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) *H is regular.*

(2) *$(\lambda \wedge_k \mu) \leq (\lambda \circ_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy left hyperideal μ of H.*

(3) *$(\lambda \wedge_k \mu) \leq (\lambda \circ_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy left hyperideal μ of H.*

(4) *$(\lambda \wedge_k \mu) \leq (\lambda \circ_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy left hyperideal μ of H.*

Proof.(1) $\Rightarrow$ (4) Let λ and μ be an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal and any $(\in, \in \vee q_k)$-fuzzy left hyperideal of H, respectively. Let $a \in H$. Since H is regular, so there exists $x \in H$ such that $a \in a \circ x \circ a = a \circ (x \circ a)$. Then there exists $\beta \in x \circ a$ such that $a \in a \circ \beta$. Thus we have

$$\begin{aligned}(\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in a \circ \beta} \{\lambda(a) \wedge \mu(\beta)\} \right) \wedge \frac{1-k}{2} \\ &\geq \lambda(a) \wedge \mu(\beta) \wedge \frac{1-k}{2} \\ &\geq \lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \\ &= \lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2} \\ &= (\lambda \wedge_k \mu)(a).\end{aligned}$$

So $(\lambda \circ_k \mu) \geq (\lambda \wedge_k \mu)$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (1) Let λ and μ be an $(\in, \in \vee q_k)$-fuzzy right hyperideal and any $(\in, \in \vee q_k)$-fuzzy left hyperideal of H, respectively. Since every $(\in, \in \vee q_k)$-fuzzy right hyperideal of H is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. So $(\lambda \circ_k \mu) \geq (\lambda \wedge_k \mu)$. By Lemma 4.9, $\lambda \circ_k \mu \leq \lambda \wedge_k \mu$. Thus $(\lambda \circ_k \mu) = (\lambda \wedge_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy right hyperideal and for every $(\in, \in \vee q_k)$-fuzzy left hyperideal of H. Hence by Theorem 4.10 H is regular.

5 Intra-regular semihypergroups

Recall that a semihypergroup H is intra-regular if for each $a \in H$, there exist $x, y \in H$ such that $a \in x \circ a \circ a \circ y$. In general neither intra-regular semihypergroups are regular nor regular semihypergroups are intra-regular semihypergroups. However, in commutative semihypergroups both the concepts coincide.

5.1 Theorem

A semihypergroup H is intra-regular if and only if $L \cap R \subseteq L \circ R$ for every left hyperideal L and for every right hyperideal R of H.

*Proof.*Let H be an intra-regular semihypergroup and L, R are left and right hyperideals of H respectively. Let $a \in L \cap R$ then $a \in L$ and $a \in R$. Since H is intra-regular so there exist $x, y \in H$ such that $a \in x \circ a \circ a \circ y = (x \circ a) \circ (a \circ y) \subseteq L \circ R$. Thus $L \cap R \subseteq L \circ R$.

Conversely, assume that $a \in L \cap R \subseteq L \circ R$. This implies that $a \in L \circ R = \bigcup_{l \circ r} \{l \circ r : l \in L, r \in R\}$. Since L is a left hyperideal so for some $x \in H$ such that $l = x \circ a$. Also R is a right hyperideal so for some $y \in H$ such that $r = a \circ y$. Thus

$$a \in l \circ r = (x \circ a) \circ (a \circ y) = x \circ a \circ a \circ y.$$

Hence H is intra-regular.

5.2 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) *H is intra-regular.*

(2) *$(\lambda \wedge_k \mu) \leq (\lambda \circ_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy left hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy right hyperideal μ of H.*

Proof.(1) $\Rightarrow$ (2) Let λ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal and μ be an $(\in, \in \vee q_k)$-fuzzy right hyperideal of H. For $a \in H$, there exist $x, y \in H$ such that $a \in x \circ a \circ a \circ y = (x \circ a) \circ (a \circ y)$. Thus there exists $\beta \in x \circ a$ and $\gamma \in a \circ y$, such that $a \in \beta \circ \gamma$.

Thus

$$\begin{aligned}(\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in c \circ d} \{\lambda(c) \wedge \mu(d)\} \right) \wedge \frac{1-k}{2} \\ &\geq \{\lambda(\beta) \wedge \mu(\gamma)\} \wedge \frac{1-k}{2} \\ & \left(\begin{array}{c} \text{Since } \beta \in x \circ a \text{ and } \lambda \text{ is an} \\ (\in, \in \vee q_k)\text{-fuzzy left hyperideal of } H, \\ \text{so } \inf\limits_{z \in x \circ a} \{\lambda(z)\} \geq \lambda(a) \wedge \frac{1-k}{2}. \\ \text{Thus } \lambda(\beta) \geq \lambda(a) \wedge \frac{1-k}{2}. \text{ Also since } \mu \text{ is} \\ \text{an } (\in, \in \vee q_k)\text{-fuzzy right hyperideal of } H, \\ \text{so } \inf\limits_{z \in a \circ x} \{\mu(z)\} \geq \mu(a) \wedge \frac{1-k}{2}. \\ \text{Thus } \mu(\gamma) \geq \mu(a) \wedge \frac{1-k}{2} \end{array} \right) \\ &\geq \left\{ \left(\lambda(a) \wedge \frac{1-k}{2} \right) \wedge \left(\mu(a) \wedge \frac{1-k}{2} \right) \right\} \wedge \frac{1-k}{2} \\ &= \lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2} \\ &= (\lambda \wedge_k \mu)(a).\end{aligned}$$

$(\lambda \circ_k \mu) \geq (\lambda \wedge_k \mu)$.

(2) $\Rightarrow$ (1) Let R and L be right and left hyperideals of H. Then by Lemma 4.5, $(\lambda_R)_k$ and $(\lambda_L)_k$ are $(\in, \in \vee q_k)$-fuzzy right and $(\in, \in \vee q_k)$-fuzzy left hyperideals of H, respectively. Thus by hypothesis we have

$$\begin{aligned}(\lambda_{L \circ R})_k &= (\lambda_L \circ_k \lambda_R) \\ &\geq (\lambda_L \wedge_k \lambda_R) \\ &= (\lambda_{L \cap R})_k.\end{aligned}$$

Thus $L \cap R \subseteq L \circ R$. Hence it follows from Theorem 5.1, that H is intra-regular.

5.3 Theorem

The following conditions are equivalent for a semihypergroup H.

(1) *H is both regular and intra-regular.*

(2) *$\lambda \circ_k \lambda = \lambda_k$ for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ of H.*

(3) *$\lambda \circ_k \lambda = \lambda_k$ for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal λ of H.*

(4) *$\lambda \circ_k \mu \geq \lambda \wedge_k \mu$ for all $(\in, \in \vee q_k)$-fuzzy quasi-hyperideals λ, μ of H.*

(5) *$\lambda \circ_k \mu \geq \lambda \wedge_k \mu$ for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal μ of H.*

(6) *$\lambda \circ_k \mu \geq \lambda \wedge_k \mu$ for all $(\in, \in \vee q_k)$-fuzzy bi-hyperideals λ, μ of H.*

Proof.(1) $\Rightarrow$ (6) Let λ, μ be $(\in, \in \vee q_k)$-fuzzy bi-hyperideals of H and $a \in H$. Then there exist $x, y, z \in H$ such that $a \in a \circ x \circ a$ and $a \in y \circ a \circ a \circ z$. So

$$\begin{aligned}a &\in a \circ x \circ a \\ &\subseteq a \circ x \circ a \circ x \circ a \\ &= (a \circ x) \circ a \circ (x \circ a) \\ &\subseteq a \circ x \circ (y \circ a \circ a \circ z) \circ x \circ a \\ &= (a \circ x \circ y \circ a) \circ (a \circ z \circ x \circ a).\end{aligned}$$

Thus there exist $p \in x \circ y, q \in z \circ x, b \in a \circ p \circ a$ and $c \in a \circ q \circ a$ such that $a \in b \circ c$.

Therefore

$$\begin{aligned}(\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in d \circ e} \{\lambda(d) \wedge \mu(e)\}\right) \wedge \frac{1-k}{2} \\ &\geq \{\lambda(b) \wedge \mu(c)\} \wedge \frac{1-k}{2} \\ &\quad \left(\begin{array}{c}\text{Since } b \in a \circ p \circ a \text{ and } \lambda \text{ is an } (\in, \in \vee q_k)\text{-fuzzy} \\ \text{bi-hyperideal of } H, \text{ we have} \\ \inf_{\alpha \in a \circ p \circ a} \{\lambda(\alpha)\} \geq \min\{\lambda(a), \lambda(a), \frac{1-k}{2}\}. \\ \text{Thus } \lambda(b) \geq \min\{\lambda(a), \frac{1-k}{2}\}. \\ \text{Similarly } \mu(c) \geq \min\{\mu(a), \frac{1-k}{2}\}.\end{array}\right) \\ &\geq \left[\left(\lambda(a) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(a) \wedge \frac{1-k}{2}\right)\right] \wedge \frac{1-k}{2} \\ &= [\lambda(a) \wedge \mu(a)] \wedge \frac{1-k}{2} = (\lambda \wedge_k \mu)(a).\end{aligned}$$

Thus $\lambda \circ_k \mu \geq \lambda \wedge_k \mu$ for all $(\in, \in \vee q_k)$-fuzzy bi-hyperideals λ, μ of H.

(6) $\Rightarrow$ (5) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (2) Take $\lambda = \mu$ in (4), we get $\lambda \circ_k \lambda \geq \lambda_k$. Since every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup, so $\lambda \circ_k \lambda \leq \lambda_k$. Hence $\lambda \circ_k \lambda = \lambda_k$.

(6) $\Rightarrow$ (3) Take $\lambda = \mu$ in (6), we get $\lambda \circ_k \lambda \geq \lambda_k$. Since every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup, so $\lambda \circ_k \lambda \leq \lambda_k$. Hence $\lambda \circ_k \lambda = \lambda_k$.

(3) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) Let Q be a quasi-hyperideal of H. Then by Lemma 3.52, λ_Q is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. Hence, by hypothesis, $\lambda_Q \circ_k \lambda_Q = (\lambda_Q)_k$. Thus $(\lambda_{Q \circ Q})_k = \lambda_Q \circ_k \lambda_Q = (\lambda_Q)_k$ implies $Q \circ Q = Q$. So by Theorem **??**, S is both regular and intra-regular.

5.4 Theorem

The following conditions are equivalent for a semihypergroup H.

(1) *H is both regular and intra-regular.*

(2) *$(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy left hyperideal μ of H.*

(3) *$(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal μ of H.*

(4) *$(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal μ of H.*

(5) *$(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal μ of H.*

(6) *$(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ for every $(\in, \in \vee q_k)$-fuzzy left hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal μ of H.*

(7) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every* $(\in, \in \vee q_k)$-*fuzzy left hyperideal* λ *and for every* $(\in, \in \vee q_k)$-*fuzzy bi-hyperideal* μ *of* H.

(8) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every* $(\in, \in \vee q_k)$-*fuzzy left hyperideal* λ *and for every* $(\in, \in \vee q_k)$-*fuzzy generalized bi-hyperideal* μ *of* H.

(9) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every* $(\in, \in \vee q_k)$-*fuzzy quasi-hyperideals* λ *and* μ *of* H.

(10) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every* $(\in, \in \vee q_k)$-*fuzzy quasi-hyperideal* λ *and for every* $(\in, \in \vee q_k)$-*fuzzy bi-hyperideal* μ *of* H.

(11) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every* $(\in, \in \vee q_k)$-*fuzzy quasi-hyperideal* λ *and for every* $(\in, \in \vee q_k)$-*fuzzy generalized bi-hyperideal* μ *of* H.

(12) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every* $(\in, \in \vee q_k)$-*fuzzy bi-hyperideals* λ *and* μ *of* H.

(13) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every* $(\in, \in \vee q_k)$-*fuzzy bi-hyperideal* λ *and for every* $(\in, \in \vee q_k)$-*fuzzy generalized bi-hyperideal* μ *of* H.

(14) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every* $(\in, \in \vee q_k)$-*fuzzy generalized bi-hyperideals* λ *and* μ *of* H.

Proof.(1) $\Rightarrow$ (14) Let λ, μ be $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideals of H and $a \in H$. Then there exist $x, y, z \in H$ such that $a \in a \circ x \circ a$ and $a \in y \circ a \circ a \circ z$. So

$$\begin{aligned} a &\in a \circ x \circ a \\ &\subseteq a \circ x \circ a \circ x \circ a \\ &= (a \circ x) \circ a \circ (x \circ a) \\ &\subseteq a \circ x \circ (y \circ a \circ a \circ z) \circ x \circ a \\ &= (a \circ x \circ y \circ a) \circ (a \circ z \circ x \circ a). \end{aligned}$$

Thus there exist $p \in x \circ y, q \in z \circ x, b \in a \circ p \circ a$ and $c \in a \circ q \circ a$ such that $a \in b \circ c$.

Therefore

$$\begin{aligned} (\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in d \circ e} \{\lambda(d) \wedge \mu(e)\} \right) \wedge \frac{1-k}{2} \\ &\geq \{\lambda(b) \wedge \mu(c)\} \wedge \frac{1-k}{2} \end{aligned}$$

(Since $b \in a \circ p \circ a$ and $c \in a \circ q \circ a$ and λ and μ are $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideals of H, so $\inf_{w \in x \circ y \circ z} \{\lambda(w)\} \geq \min\{\lambda(x), \lambda(z), \frac{1-k}{2}\}$ and $\inf_{w \in x \circ y \circ z} \{\mu(w)\} \geq \min\{\mu(x), \mu(z), \frac{1-k}{2}\}$ Thus $\lambda(b) \geq \lambda(a) \wedge \frac{1-k}{2}$ and $\mu(c) \geq \mu(a) \wedge \frac{1-k}{2}$.)

$$\begin{aligned} \text{Therefore } (\lambda \circ_k \mu)(a) &\geq \left[\begin{array}{c} (\lambda(a) \wedge \frac{1-k}{2}) \wedge \\ (\mu(a) \wedge \frac{1-k}{2}) \end{array} \right] \wedge \frac{1-k}{2} \\ &= [\lambda(a) \wedge \mu(a)] \wedge \frac{1-k}{2} = (\lambda \wedge_k \mu)(a). \end{aligned}$$

Similarly we can prove that $(\mu \circ_k \lambda) \geq (\lambda \wedge_k \mu)$. Hence $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$.

(14) $\Rightarrow$ (13) $\Rightarrow$ (12) $\Rightarrow$ (10) $\Rightarrow$ (9) $\Rightarrow$ (3) $\Rightarrow$ (2),
(14) $\Rightarrow$ (11) $\Rightarrow$ (10),
(14) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (6) $\Rightarrow$ (2) and
(14) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1) Let λ be an $(\in, \in \vee q_k)$-fuzzy right hyperideal and μ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H.

For $a \in H$, we have

$$\begin{aligned} (\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in y \circ z} \{\lambda(y) \wedge \mu(z)\} \right) \wedge \frac{1-k}{2} \\ &= \bigvee_{a \in y \circ z} \left\{ \lambda(y) \wedge \mu(z) \wedge \frac{1-k}{2} \right\} \\ &= \bigvee_{a \in y \circ z} \left\{ \begin{array}{c} (\lambda(y) \wedge \frac{1-k}{2}) \wedge \\ (\mu(z) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} \end{array} \right\} \end{aligned}$$

$$\left(\begin{array}{c} \text{Because } \inf_{a \in y \circ z} \{\lambda(a)\} \geq \lambda(y) \wedge \frac{1-k}{2} \text{ and} \\ \inf_{a \in y \circ z} \{\mu(a)\} \geq \mu(z) \wedge \frac{1-k}{2} \end{array} \right).$$

Thus

$$\begin{aligned} (\lambda \circ_k \mu)(a) &\leq \bigvee_{a \in y \circ z} \left\{ \lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2} \right\} \\ &= \lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2} \\ &= (\lambda \wedge_k \mu)(a). \end{aligned}$$

So $(\lambda \circ_k \mu) \leq (\lambda \wedge_k \mu)$. By hypothesis $(\lambda \circ_k \mu) \geq (\lambda \wedge_k \mu)$. Thus $(\lambda \circ_k \mu) = (\lambda \wedge_k \mu)$. Hence by Theorem 4.10, H is regular. Also by hypothesis $(\lambda \circ_k \mu) \geq (\lambda \wedge_k \mu)$, so by Theorem 5.2, H is intra-regular.

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