Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110111

A Solution of Fuzzy Time Fractional Diffusion Equation by Modified Natural Daftardar-Jafari Method

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Received: 2 May 2024, Revised: 22 Jun. 2024, Accepted: 13 Jul. 2024
Published online: 1 Jan. 2025

Abstract: In this article, The Natural Daftardar-Jafari method is reformulated and applied for the first time to solve the fuzzy time-fractional diffusion equation, with a fractional order of $0 < \alpha \leq 1$. The initial and boundary conditions are expressed using Triangular fuzzy numbers with convex normalization to represent the degree of fuzziness, and the Caputo formula is implemented to define the time-fractional derivative. A practical example with numerical values is provided to illustrate the effectiveness of the method and the results obtained are found to agree with those of other established methods used to solve the same problem. By incorporating symmetry through fuzzy set theory and fuzzy numbers, this article achieves accurate modelling and analysis of the fuzzy time-fractional diffusion equation (FTFDE), enhancing our understanding of complex system dynamics.

Keywords: Fuzzy time-fractional diffusion equation, natural Daftardar-Jafari method, fuzzy number, Caputo formula

1 Introduction

Over the past few years, there has been a growing interest in partial differential equations with fractional order due to their relation in various fields such as engineering, science, and medicine [1-5]. In light of their importance, the researchers have been working on developing effective approaches for solving these equations. However, despite significant advancements in this domain, there is no consensus on a standardized methodology for solving FPDEs. The substitution of a fractional derivative for the first-order time derivative in diffusion equations result in the time-fractional diffusion equation (TFDE), which is useful in modelling various phenomena in physical environments such as the movement of bacteria cells in biofilms, the motion of charge carriers in amorphous semiconductors, and diffusion in critical percolation networks [6-8]. The equation'solutions have been analysed using various techniques such as fractional calculus, Green's functions, and numerical simulations.

TFDE have been studied by many authors [9-14]. Several authors have aimed to develop analytical solutions for TFDE. Wyss [9] developed analytical solutions for TFDEs. The Fox functions were employed to find the corresponding Green functions. By employing the Fox functions, Wyss obtained analytical solutions for TFDEs and explored the properties of the derived Green functions. After that, Schneider and Wyss [10] investigated analytical solutions for TFDEs using the Fox functions. They derived the associated Green functions and examined their properties. They obtained analytical solutions for TFDEs by utilizing the Fox functions and studied the resulting Green functions. Gorenflo et al. [11] applied the same approach with the Laplace transform to TFDE is expressed in terms of the Wright function. Additionally, Liu, et al. [12] used Fourier-Laplace transforms to obtain a solution for the fractional advection-dispersion equation with time-fractional order equations in whole-space and half-space. Liu et.al [13] utilized the finite difference method in both space and time directions to solve the TFDE and established some stability conditions for this approach.

The conventional analysis of diffusion phenomena assumes that the parameters and variables involved are clear and precise, but in actuality, they may be uncertain and imprecise due to errors in measurement and experimentation. To handle this issue, mathematicians discovered there is a need to use the fuzzy fractional diffusion equation instead of the fractional diffusion equation. The FTFDE has been studied by many authors [14-18]. Ghazanfari and Ebrahimi [14] utilized the differential transformation method (DTM) to tackle the FTFDE. The solution was approximated as a series

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with terms that can be computed. The DTM proved to be a simple and highly effective approach to obtaining analytical solutions for FTFDE. Salah et al. [15] applied a method called the homotopy analysis transform method to obtain a solution for the fuzzy fractional heat and wave equations. In another study, a new computational technique was presented by Chakraverty and Tampaswini [16] to address uncertainties in the initial conditions of the FTFDE. Their approach involved converting the fuzzy diffusion equation into an interval-based fuzzy differential equation using a single parametric form of fuzzy numbers. The resulting equation was then transformed into a crisp form using the double parametric form of fuzzy numbers and solved using the Adomian decomposition method to establish the uncertain solution bounds. Kumar and Cupta (2021) [17] presented an approximation solution for FTFDE using the vibrational iteration method. The fractional derivative for time was taken in the Caputo definition. The solution is shown as a series solution with simple predictable terms. Zureigat et al. [18] reformulated and implemented the fourth-order Crank-Nicholson approach to handle the FTFDE. Caputo' definition is utilized for defining the fractional time derivative.

The Daftardar-Jafari method presents numerous advantages in the resolution of both ordinary and partial differential equations. This is particularly evident when addressing fractional partial differential equations, as the simplicity of the NDJM facilitates the straightforward derivation of solutions for partial differential equations involving fractional orders, surpassing the comparative complexity of alternative methods. After conducting a review of existing literature, the previously published articles used the NDJM to solve the time-fractional diffusion equation. However, in the present study, we have innovatively developed and reformulated the NDJM for the first time, specifically to tackle the fuzzy time-fractional diffusion equation within the framework of Hukuhara differentiability. In particular, the main aim of this paper is to carry out a study to solve the FTFDE using transform iterative methods. Specifically, we will discuss, develop and apply the Natural Daftardar-Jafari method for solving the FTFDE with a fractional order of $0 < \alpha \leq 1$.

2 Time Fractional diffusion Equation in Fuzzy Environment

In this section, we introduce the overall structure of the TFDE within a fuzzy setting, employing fundamental fuzzy properties as outlined in references [19–21]. Consider the one dimensional of FTFDE with the initial and boundary conditions [22]

$$\frac{\partial^{\alpha}\widetilde{u}(x,t,\alpha)}{\partial^{\alpha}t} = \widetilde{C}^2\frac{\partial^2\widetilde{u}(x,t)}{\partial x^2} + \widetilde{q}(x,t) \quad , 0<\alpha\leq 1, \ (x,t)\varepsilon\ \Omega=[0,l]\times\ [0,T]$$

$$\widetilde{u}(x,0)=\widetilde{f}_1(x), \frac{\partial\widetilde{u}}{\partial t}(x,0)=\widetilde{f}_2(x), \widetilde{u}(0,t)=\widetilde{v},\ \ \widetilde{u}(l,t)=\widetilde{y}, \tag{1}$$

Where:
$\widetilde{u}(x,t,\alpha)$: is a fuzzy function of variables t and x and α is arbitrary fractional order.
$\widetilde{q}(x,t)$: The nonhomogeneous term is a fuzzy function of variable t and x.
$\frac{\partial^{\alpha}\widetilde{u}(x,t,\alpha)}{\partial^{\alpha}t}$: is the fuzzy time fractional derivative of order α.
$\frac{\partial^2\widetilde{U}(x,t)}{\partial x^2}$: is a second order fuzzy partial Hukuhara derivatives.

Furthermore, in Eq. (1) the fuzzy initial conditions are $\widetilde{u}(0,x)$, $\frac{\partial u}{\partial t}(x,0)$ and $\widetilde{f}_1(x)$and $\widetilde{f}_2(x)$ are fuzzy functions of x. The boundary conditions in the fuzzy form are $\widetilde{u}(0,t)$ and $\widetilde{u}(l,0)$ and are equal to the fuzzy convex numbers $\widetilde{v}$ and $\widetilde{y}$, respectively. Finally, in Eq. (1) the fuzzy functions are defined as follows [23]:

$$\begin{cases}\widetilde{q}(x,t)=\widetilde{\omega}_1 s_1(x)\\ \widetilde{f}_1(x)=\widetilde{\omega}_2 s_2(x)\\ \widetilde{f}_2(x)=\widetilde{\omega}_3 s_3(x)\end{cases} \tag{2}$$

where $s_1(x)$, $s_2(x)$ and $s_3(x)$ are the crisp functions of the crisp variable x with $\widetilde{\omega}_1$,$\widetilde{\omega}_2$, and $\widetilde{\omega}_3$ being the fuzzy convex numbers.

The FTFDE in this section is represented using the r-level cut approach under Hukuhara derivatives. We may write Eq. (1) for all $r\in[0,1]$ is as follows [24]

$$[\widetilde{u}(x,t)]_r = \underline{u}(x,t;r), \overline{u}(x,t;r) \tag{3}$$

$$\left[\frac{\partial^{\alpha}\widetilde{u}(x,t,\alpha)}{\partial^{\alpha}t}\right]_r = \frac{\partial^{\alpha}\underline{u}(x,t,\alpha;r)}{\partial^{\alpha}t}, \frac{\partial^{\alpha}\overline{u}(x,t,\alpha;r)}{\partial^{\alpha}t} \tag{4}$$

$$\left[\frac{\partial^2\widetilde{u}(x,t)}{\partial x^2}\right]_r = \frac{\partial^2\underline{u}(x,t;r)}{\partial x^2}, \frac{\partial^2\overline{u}(x,t;r)}{\partial x^2} \tag{5}$$

$$\left[\widetilde{c^2}\right]_r = \underline{c^2}, \overline{c^2} \tag{6}$$

$$[\widetilde{q}(x,t)]_r = \underline{q}(x,t;r), \overline{q}(x,t;r) \tag{7}$$

$$[\widetilde{u}(x,0)]_r = \underline{u}(x,0;r), \overline{u}(x,0;r) \tag{8}$$

$$\left[\frac{\partial \widetilde{u}}{\partial t}(x,0)\right]_r = \frac{\partial \underline{u}}{\partial t}(x,0;r), \frac{\partial \overline{u}}{\partial t}(x,0;r) \tag{9}$$

$$[\widetilde{u}(0,t)]_r = \underline{u}(0,t;r), \overline{u}(0,t;r) and\, [\widetilde{u}(l,t)]_r = \underline{u}(l,t;r), \overline{u}(l,t;r) \tag{10}$$

$$\left[\widetilde{f}(x)\right]_r = \underline{f}(x;r), \overline{f}(x;r) \tag{11}$$

$$\begin{cases} [\widetilde{v}]_r = \underline{v}(r), \overline{v}(r) \\ [\widetilde{y}]_r = \underline{y}(r), \overline{y}(r) \end{cases} \tag{12}$$

where

$$\begin{cases} [\widetilde{q}(x,t)]_r = [\underline{\omega}(r)_1, \overline{\omega}_1(r)]\, s_1(x) \\ \left[\widetilde{f}_1(x)\right]_r = [\underline{\omega}(r)_2, \overline{\omega}_2(r)]\, s_2(x) \\ \left[\widetilde{f}_2(x)\right]_r = [\underline{\omega}(r)_3, \overline{\omega}_3(r)]\, s_3(x) \end{cases} \tag{13}$$

The fuzzy extension principle is employed to establish the membership function [25]

$$\begin{cases} \underline{u}(x,t;r) = min\{\widetilde{u}(\widetilde{\mu}(r),t))|\widetilde{\mu}(r) \in \widetilde{u}(x,t;r)\} \\ \overline{u}(x,t;r) = max\{\widetilde{u}(\widetilde{\mu}(r),t)|\widetilde{\mu}(r) \in \widetilde{u}(x,t;r)\} \end{cases} \tag{14}$$

According to [19], by fuzzification of Eq. (1) and defuzzification of Eq.(4 - 14), we can rewrite the Eq. (1) in the following new formula.

$$\begin{cases} \frac{\partial^\alpha \underline{u}(x,t,\alpha)}{\partial^{\alpha} t} = \underline{c}^2(x)\frac{\partial^2 \underline{u}(x,t;r)}{\partial x^2} + [\underline{\omega}(r)_1]\, s_1(x) \\ \underline{u}(x,0;r) = \underline{\omega}(r)_2 s_2(x) \\ \frac{\partial \underline{u}}{\partial t}(x,0;r) = \underline{\omega}(r)_3 s_3(x) \\ \underline{u}(0,t;r) = \underline{v}(r), \underline{u}(l,t;r) = \underline{y}(r) \end{cases} \tag{15}$$

$$\begin{cases} \frac{\partial^\alpha \overline{u}(x,t,\alpha)}{\partial^{\alpha} t} = \overline{c}^2 \frac{\partial^2 \overline{u}(x,t;r)}{\partial x^2} + [\overline{\omega}_1(r)]\, s_1(x) \\ \overline{u}(x,0;r) = \overline{\omega}(r)_2 s_2(x) \\ \frac{\partial \overline{u}}{\partial t}(x,0;r) = \overline{\omega}(r)_3 s_3(x) \\ \overline{u}(0,t;r) = \overline{v}(r), \overline{u}(l,t;r) = \overline{y}(r) \end{cases} \tag{16}$$

Eq.(15) and Eq.(16) presented the lower and upper bounds respectively of the general form of FTFDE.

3 The Fuzzy Natural Daftardar-Jafari Method for the Solution of FTFDE

In this section, the Natural Daftardar-Jafari Method is developed and applied for the first time to obtain a numerical solution of FTFDE.

At the first, the FTFDE in Eq.(15) and Eq.(16) is Let we represented as the following general form:

$$\begin{cases} \mathscr{D}_t^\alpha \underline{u}(x,t;\alpha,r) + R(\underline{u}(x,t;\alpha,r)) + F(\underline{u}(x,t;\alpha,r)) = \underline{q}(x,t;\alpha,r) \\ \mathscr{D}_t^\alpha \overline{u}(x,t;\alpha,r) + R(\overline{u}(x,t;\alpha,r)) + F(\overline{u}(x,t;\alpha,r)) = \overline{q}(x,t;\alpha,r) \end{cases} \tag{17}$$

With the fuzzy initial condition

$$\begin{cases}\underline{u}^{(i)}(x,0,r)=\frac{\partial^i \underline{u}(x,0,r)}{\partial t^i}\\ \overline{u}^{(i)}(x,0,r)=\frac{\partial^i \overline{u}(x,0,r)}{\partial t^i}\end{cases},\ i=0,1,2,\dots,p-1$$

where, $\widetilde{D}_t^{\alpha}=\frac{\partial^\alpha}{\partial t^\alpha}$ is the fuzzy Caputo time fractional derivative, $R(\widetilde{u}(x,t))$ is the fuzzy linear partial differential operator, $F(\widetilde{u}(x,t))$ represent the nonlinear partial terms and $\widetilde{q}(x,t)$ is a fuzzy source term.

Now, in the first step, we take natural transform on both sides of Eq.(17)

$$\begin{cases}\mathscr{N}[\mathscr{D}_t^{\alpha}\underline{u}(x,t;\alpha,r)]+\mathscr{N}[R(\underline{u}(x,t;\alpha,r))]+\mathscr{N}[\ F(\underline{u}(x,t;\alpha,r))]=\mathscr{N}[\underline{q}(x,t;\alpha,r)]\\ \mathscr{N}[\mathscr{D}_t^{\alpha}\overline{u}(x,t;\alpha,r)]+\mathscr{N}[R(\overline{u}(x,t;\alpha,r))]+\mathscr{N}[\ F(\overline{u}(x,t;\alpha,r))]=\mathscr{N}[\overline{q}(x,t;\alpha,r)]\end{cases}\tag{18}$$

As define in [25]. The natural transform of fuzzy Caputo derivative $\mathscr{N}[\mathscr{D}_t^{\alpha}\widetilde{u}(x,t)]$ is defined as follows:

$$\mathscr{N}[\mathscr{D}_t^{\alpha}\widetilde{u}(x,t)]=\left(\frac{s}{m}\right)^{\alpha}\widetilde{\psi}(s,m)-\sum_{i=0}^{p-1}\frac{1}{s}\left(\frac{s}{m}\right)^{\alpha-i}y^{(i)}(0),\ \alpha\in(p-1;p]\tag{19}$$

Simplifying the Eq.(18) and applying the initial conditions to get:

$$\begin{cases}\underline{\psi}(x,s,u;\alpha,r)=\left(\frac{m}{s}\right)^{\alpha}\sum_{i=0}^{p-1}\frac{1}{s}\left(\frac{s}{m}\right)^{\alpha-i}y^{(i)}(0)+\left(\frac{m}{s}\right)^{\alpha}\mathscr{N}\left[\underline{q}(x,t;\alpha,r)\right]-\\ \qquad\left(\frac{m}{s}\right)^{\alpha}\left[\mathscr{N}[R(\underline{u}(x,t;\alpha,r))]+\mathscr{N}[F(\underline{u}(x,t;\alpha,r))]\right]\\ (\overline{\psi}(x,s,u;\alpha,r)=\left(\frac{m}{s}\right)^{\alpha}\sum_{i=0}^{p-1}\frac{1}{s}\left(\frac{s}{m}\right)^{\alpha-i}y^{(i)}(0)+\left(\frac{m}{s}\right)^{\alpha}\mathscr{N}\left[\overline{q}(x,t;\alpha,r)\right]-\\ \qquad\left(\frac{m}{s}\right)^{\alpha}\left[\mathscr{N}[R(\overline{u}(x,t;\alpha,r))]+\mathscr{N}[F(\overline{u}(x,t;\alpha,r))]\right]\end{cases}\tag{20}$$

In the second step, we take the inverse natural transform on both sides of Eq.(20) to obtain

$$\begin{cases}\underline{u}(x,t;\alpha,r)=\mathscr{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\sum_{i=0}^{p-1}\frac{1}{s}\left(\frac{s}{m}\right)^{\alpha-i}y^{(i)}(0)+\left(\frac{m}{s}\right)^{\alpha}\mathscr{N}[\underline{q}(x,t;r)]\right]-\\ \qquad\mathscr{N}^{-1}\left[\left(\frac{m}{s}\right)^{\mu}\left[\mathscr{N}[R(\underline{u}(x,t;r))]+\mathscr{N}[F(\underline{u}(x,t;r))]\right]\right]\\ \overline{u}(x,t;\alpha,r)=\mathscr{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\sum_{i=0}^{p-1}\frac{1}{s}\left(\frac{s}{m}\right)^{\alpha-i}y^{(i)}(0)+\left(\frac{m}{s}\right)^{\alpha}\mathscr{N}[\overline{q}(x,t;r)]\right]-\\ \qquad\mathscr{N}^{-1}\left[\left(\frac{m}{s}\right)^{\mu}\left[\mathscr{N}[R(\overline{u}(x,t;r))]+\mathscr{N}[F(\overline{u}(x,t;r))]\right]\right]\end{cases}\tag{21}$$

The Eq.(21) can be rewritten as follows:

$$\begin{aligned}\underline{u}(x,t;\alpha,r)&=\underline{Q}(x,t;r)-\mathscr{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\mathscr{N}[R(\underline{u}(x,t;\alpha,r))]+\mathscr{N}[F(\underline{u}(x,t;\alpha,r))]\right]\right]\\ \overline{u}(x,t;\alpha,r)&=\overline{Q}(x,t;r)-\mathscr{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\mathscr{N}[R(\overline{u}(x,t;\alpha,r))]+\mathscr{N}[F(\overline{u}(x,t;\alpha,r))]\right]\right]\end{aligned}\tag{22}$$

where $\underline{Q}(x,t;r)$, $\overline{Q}(x,t;r)$ are the term due to the lower and upper initial conditions respectively.

Now in the final step, an iterative method known as the fuzzy Daftardar-Jafari method (FDJM) [26] is applied, the solution of Eq.(17) is written as an infinite series as follows:

$$\begin{cases}\underline{u}(x,t;r)=\sum_{n=0}^{\infty}\underline{u_n}(x,t;r)\\ \overline{u}(x,t;r)=\sum_{n=0}^{\infty}\overline{u_n}(x,t;r)\end{cases},\tag{23}$$

substituting Eq.(23) into Eq.(22) gives to obtain

$$\begin{cases}\sum_{n=0}^{\infty}\underline{u_n}(x,t;r)=\underline{Q}(x,t;r)-\mathscr{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\mathscr{N}\left[R\left(\sum_{n=0}^{\infty}\underline{u_n}\right)\right]+\mathscr{N}\left[F\left(\sum_{n=0}^{\infty}\underline{u_n}\right)\right]\right]\right]\\ \sum_{n=0}^{\infty}\overline{u_n}(x,t;r)=\overline{Q}(x,t;r)-\mathscr{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\mathscr{N}\left[R\left(\sum_{n=0}^{\infty}\overline{u_n}\right)\right]+\mathscr{N}\left[F\left(\sum_{n=0}^{\infty}\overline{u_n}\right)\right]\right]\right]\end{cases}\tag{24}$$

the non-linear term is decomposed as [26],

$$\begin{cases} F\left(\sum_{n=0}^{\infty} \underline{u}(x,t;r)\right) = F\left(\underline{u_0}(x,t;r)\right) + \sum_{n=1}^{\infty}\left[F\left(\sum_{k=0}^{n} \underline{u_k}\right) - F\left(\sum_{k=0}^{n-1} \underline{u_k}\right)\right] \\ F\left(\sum_{n=0}^{\infty} \overline{u}(x,t;r)\right) = F\left(\overline{u_0}(x,t;r)\right) + \sum_{n=1}^{\infty}\left[F\left(\sum_{k=0}^{n} \overline{u_k}\right) - F\left(\sum_{k=0}^{n-1} \overline{u_k}\right)\right] \end{cases} \tag{25}$$

Substituting Eq.(25) into Eq.(24) to get:

$$\begin{cases} \sum_{n=0}^{\infty} \underline{u_n}(x,t;r) = \underline{Q}(x,t;r) - \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[R\sum_{n=0}^{\infty} \underline{u_n}(x,t;r)\right]\right] \\ \quad -\mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[F\left(\underline{u_0}(x,t)\right) + \sum_{n=1}^{\infty}\left[F\left(\sum_{k=0}^{n} \underline{u_k}\right) - F\left(\sum_{k=0}^{n-1} \underline{u_k}\right)\right]\right]\right] \\ \sum_{n=0}^{\infty} \overline{u_n}(x,t;r) = \overline{Q}(x,t;r) - \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[R\sum_{n=0}^{\infty} \overline{u_n}(x,t;r)\right]\right] \\ \quad -\mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[F\left(\overline{u_0}(x,t)\right) + \sum_{n=1}^{\infty}\left[F\left(\sum_{k=0}^{n} \overline{u_k}\right) - F\left(\sum_{k=0}^{n-1} \overline{u_k}\right)\right]\right]\right] \end{cases}$$

The following iteration is then deduced:

$$\begin{aligned} \underline{u_0}(x,t;\alpha,r) &= \underline{Q}(x,t;\alpha,r), \\ \underline{u_1}(x,t;\alpha,r) &= -\mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[R\left(\underline{u_0}(x,t;r)\right)\right] - \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[F\left(\underline{u_0}(x,t;r)\right)\right]\right]\right. \\ \underline{u_2}(x,t;\alpha,r) &= -\mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[R\left(\underline{u_1}(x,t;r)\right)\right] - \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[F\left(\underline{u_1}+\alpha_0\right) - F\left(\underline{u_0}\right)\right]\right]\right., \\ \underline{u_n}(x,t;\alpha,r) &= -\mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[R\left(\underline{u_{n-1}}\right)\right]\right] \\ &\quad -\mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[F\left(\underline{u_0}+\ldots+\underline{u_{n-1}}\right) - F\left(\underline{u_0}+\ldots+\underline{u_{n-2}}\right)\right]\right], n = 3,4,\ldots \end{aligned} \tag{26}$$

$$\begin{aligned} \overline{u_0}(x,t;\alpha,r) &= \overline{Q}(x,t;r), \\ \overline{u_1}(x,t;\alpha,r) &= -\mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[R\left(\overline{u_0}(x,t;r)\right)\right] - \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[F\left(\overline{u_0}(x,t;r)\right)\right]\right]\right. \\ \overline{u_2}(x,t;\alpha,r) &= -\mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[R\left(\overline{u_1}(x,t;r)\right)\right] - \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[F\left(\overline{u_1}+\alpha_0\right) - F\left(\overline{u_0}\right)\right]\right]\right., \\ \overline{u_n}(x,t;\alpha,r) &= -\mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[R\left(\overline{u_{n-1}}\right)\right]\right] \\ &\quad -\mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha} \mathcal{N}\left[F\left(\overline{u_0}+\ldots+\overline{u_{n-1}}\right) - F\left(\overline{u_0}+\ldots+\overline{u_{n-2}}\right)\right]\right], n = 3,4,\ldots \end{aligned} \tag{27}$$

Where the Eq.(26) and Eq.(27) represent the fuzzy lower and fuzzy upper approximation solution of Eq.(15) and Eq.(16) respectively.

4 Numerical Example

In this section, we implement the modified Natural Daftardar-Jafari Method which is introduced and discussed in section 3, to solve FTFDE. The Wolfram Mathematica 11.2 was used to perform our proposed approach's numerical experiment.

Consider the one dimensional of FTFDE with the initial and boundary conditions [15]

$$\frac{\partial^{\alpha}\widetilde{u}(x,t,\alpha;r)}{\partial^{\alpha}t} = \frac{1}{2}\,x^2\frac{\partial^2\widetilde{u}(x,t;r)}{\partial x^2} \quad , 0<\alpha\leq 1,\ \ (x,t)\varepsilon\ \Omega = [0,1]\times\ [0,1] \tag{28}$$

subject to the fuzzy boundary conditions $\widetilde{u}(0,t) = 0$, $\widetilde{u}(l,t) = 0$, and fuzzy initial condition

$$\widetilde{u}(x,0) = \widetilde{k}(r)x \quad , \quad \frac{\partial\widetilde{u}}{\partial t}(x,0) = \widetilde{k}(r)x^2,$$

where $\widetilde{k}(r) = [0.75+0.25\,r\,,\ \ 1.25-0.25\,r]$ for all $r\in[0,1]$

Now let $\widetilde{R}\left(u(x,t)\right) = \frac{1}{2}\,x^2\frac{\partial^2\widetilde{u}(x,t;r)}{\partial x^2}$, $\widetilde{F}\left(u(x,t;r)\right) = 0$ *and* $\widetilde{q}(x,t;r) = 0$

we take the natural transform on both sides of (28) and use the initial conditions, this obtain:

$$\begin{aligned} \underline{\psi}(x,s,u;\alpha,r) &= (0.75+0.25\,r)\ \left(\frac{x}{s}+\frac{m\,x^2}{s^2}\right) + \left(\frac{m}{s}\right)^{\alpha}\ \left[\mathcal{N}\left[\ \frac{1}{2}\,x^2\frac{\partial^2\widetilde{u}(x,t;r)}{\partial x^2}\right]\right. \\ \overline{\psi}(x,s,u;\alpha,r) &= (1.25-0.25\,r)\ \left(\frac{x}{s}+\frac{m\,x^2}{s^2}\right) + \left(\frac{m}{s}\right)^{\alpha}\ \left[\mathcal{N}\left[\ \frac{1}{2}\,x^2\frac{\partial^2\widetilde{u}(x,t;r)}{\partial x^2}\right]\right. \end{aligned} \tag{29}$$

Now, take the inverse natural transform of (15) to get

$$\underline{u}(x,t;\alpha,r) = (0.75+0.25\,r)\ \left(x+t\,x^2\right) + \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\ \mathcal{N}\left[\ \frac{1}{2}\,x^2\,\frac{\partial^2\widetilde{u}(x,t;r)}{\partial x^2}\ \right]\right]\right]$$
$$\overline{u}(x,t;\alpha,r) = (1.25-0.25\,r)\ \left(x+t\,x^2\right) + \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\ \mathcal{N}\left[\ \frac{1}{2}\,x^2\,\frac{\partial^2\widetilde{u}(x,t;r)}{\partial x^2}\ \right]\right]\right] \tag{30}$$

By implementing the FNDJM to get the following terms:

$$\begin{aligned}
\underline{u_0}(x,t;\alpha,r) &= (0.75+0.25\,r)\ \left(x+t\,x^2\right),\\
\underline{u_1}(x,t;\alpha,r) &= \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\ \mathcal{N}\left[\ \frac{1}{2}\,x^2\,\frac{\partial^2\underline{u_0}(x,t)}{\partial x^2}\ \right]\right]\right] = \frac{(r-1)\ x^2t^{\alpha+1}}{\Gamma(\alpha+2)}\\
\underline{u_2}(x,t;\alpha,r) &= \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\ \mathcal{N}\left[\ \frac{1}{2}\,x^2\,\frac{\partial^2\underline{u_1}(x,t)}{\partial x^2}\ \right]\right]\right] = \frac{(r-1)\ x^2t^{2\alpha+1}}{\Gamma(2\alpha+2)}\\
\underline{u_n}(x,t;\alpha,r) &= \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\ \mathcal{N}\left[\ \frac{1}{2}\,x^2\,\frac{\partial^2\underline{u_{n-1}}(x,t)}{\partial x^2}\ \right]\right]\right] = \frac{(r-1)\ x^2t^{n\alpha+1}}{\Gamma(n\alpha+2)}
\end{aligned} \tag{31}$$

$$\begin{aligned}
\overline{u_0}(x,t;\alpha,r) &= (1.25-0.25\,r)\ \left(x+t\,x^2\right),\\
\overline{u_1}(x,t;\alpha,r) &= \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\ \mathcal{N}\left[\ \frac{1}{2}\,x^2\,\frac{\partial^2\overline{u_0}(x,t;r)}{\partial x^2}\ \right]\right]\right] = \frac{(1-r)\ x^2t^{\alpha+1}}{\Gamma(\alpha+2)}\\
\overline{u_2}(x,t;\alpha,r) &= \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\ \mathcal{N}\left[\ \frac{1}{2}\,x^2\,\frac{\partial^2\overline{u_1}(x,t;r)}{\partial x^2}\ \right]\right]\right] = \frac{(1-r)\ x^2t^{2\alpha+1}}{\Gamma(2\alpha+2)}\\
\overline{u_n}(x,t;\alpha,r) &= \mathcal{N}^{-1}\left[\left(\frac{m}{s}\right)^{\alpha}\left[\ \mathcal{N}\left[\ \frac{1}{2}\,x^2\,\frac{\partial^2\overline{u_{n-1}}(x,t;r)}{\partial x^2}\ \right]\right]\right] = \frac{(1-r)\ x^2t^{n\alpha+1}}{\Gamma(n\alpha+2)}
\end{aligned} \tag{32}$$

By simplifying the Eq.(31-32) to get the general solution of FTFDE:

$$\begin{cases}\underline{u}(x,t;r) = \left[(0.75+0.25\,r)\left(x+\,x^2\right)\right]\ \sum_{n=0}^{\infty}\frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)}\\ \overline{u}(x,t;r) = \left[(1.25-0.25\,r)\ \left(x+\,x^2\right)\right]\ \sum_{n=0}^{\infty}\frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)}\end{cases} \tag{33}$$

Table 1: Fuzzy solutions of Eq.(28) by the FNDJM for all $t \in [0,1]$ when $\alpha = 0.5, x = 0.5$, $r = 0$

t	Fuzzy lower solution by FNDJM	Fuzzy upper solution by FNDJM
0.1	0.07309044812425813	0.12181741354043021
0.2	0.16559479112858172	0.2759913185476362
0.3	0.27543402033759407	0.45905670056265674
0.4	0.40297134095806153	0.6716189015967692
0.5	0.5492257857377686	0.9153763095629477
0.6	0.7155980368518778	1.1926633947531298
0.7	0.9037872786173959	1.506312131028993
0.8	1.1157653392788622	1.8596088987981036
0.9	1.35377456687104	2.2562909447850665
1	1.6203380139375587	2.700563356562598

Table 2: Fuzzy lower and upper solutions of Eq.(28) by FNDJM for all $r \in [0,1]$ at $x,t = 0.5$ *and* $\alpha = 0.5$

r	Fuzzy lower solution by FNDJM	Fuzzy upper solution by FNDJM
0	0.5492257857377686	0.9153763095629477
0.2	0.5858408381202866	0.8787612571804297
0.4	0.6224558905028044	0.8421462047979118
0.6	0.6590709428853224	0.8055311524153941
0.8	0.6956859952678401	0.7689161000328761
1	0.7323010476503582	0.7323010476503582

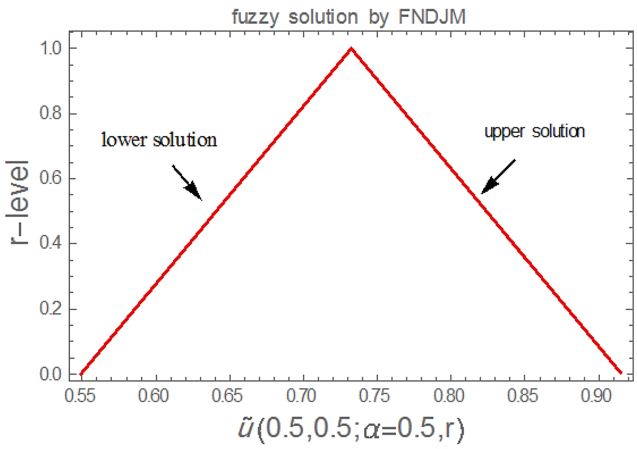

Figure 1: Fuzzy solution of Eq.(28) by FNDJM at $t = 0.5,\ x = 0.5$ *and* $\alpha = 0.5$ for all $r \in [0,1]$

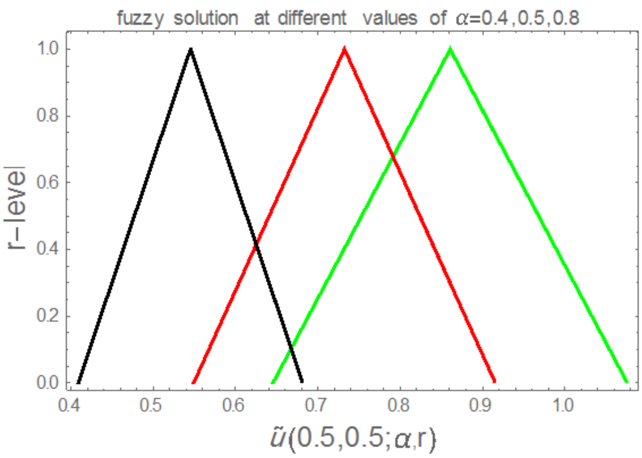

Figure 2: Fuzzy solution of Eq. (28) by FNDJM at $t = 0.5,\ x = 0.5$ *and* $\alpha = 0.4,\ 0.5\ ,0.8$ for all $r \in [0,1]$

Table 1, Table 2, and Figure 1 show that the fuzzy solution of the FTFDE by the FNDJM method satisfies the properties of fuzzy numbers by attaining the triangular fuzzy number shape. Furthermore, as seen in Figure 2, the fuzzy solution of FTFDE by FNDJM at different values of α indicates that the FNDJM method is feasible, accurate and satisfied the fuzzy properties and theories.

5 Conclusion

In conclusion, this article developed and applied the Natural Daftardar-Jafari method for the first time to solve the fuzzy time-fractional diffusion equation, extending its utility to problems with a fractional order of $0 < \alpha \geq 1$. The initial and boundary conditions are expressed using Triangular fuzzy numbers with convex normalization to represent the degree of fuzziness. The Caputo formula is implemented to define the time-fractional derivative. Finally, a practical numerical example demonstrates the approach's effectiveness, yielding results consistent with established methods. This research contributes to accurate modeling and analysis of the fuzzy time-fractional diffusion equation, advancing our comprehension of complex system dynamics.

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