Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110108

Novel Dynamic Inequalities of Ostrowski-Trapezoid-Grüss-type on q-Difference Operator

Ahmed A. El-Deeb[1,*] *and Barakah Almarri*[2]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt
[2] General Studies Department, Jubail Industrial College, 8244 Rd Number 6, Al Huwaylat, Al Jubail 35718, Saudi Arabia

Received: 30 May 2024, Revised: 11 Jun. 2024, Accepted: 12 Jun. 2024
Published online: 1 Jan. 2025

Abstract: We prove some new extensions of the Ostrowski inequality and its companion inequalities on q- difference operator by using two parameters for functions whose second q-derivatives are bounded. In addition to improving some results achieved by using weak conditions, the inequalities proved here include some results proven in the literature that are inferred as limited cases. Our results in the continuous and discrete instances lead to the derivation of several original integral and difference inequalities. So, furthermore to these generalizations, some continuous inequalities are obtained as special cases of main work. Our results are proved by using the integration by parts formula on q-difference operator.

Keywords: Montgomery identity, Ostrowski type inequality, q-derivatives, trapezoid type inequality, Grüss type inequality.

1 Introduction

For the sake to compute the absolute deviation of a differentiable function and its integral mean, Ostrowski [24] established the following sharp integral inequality:

Theorem 1. *Assume that the function $\Phi : [\theta,\zeta] \to \mathbb{R}$ is continuous on $[\theta,\zeta]$ and differentiable on (θ,ζ), then for all $\lambda \in [\theta,\zeta]$, we have*

$$\left|\Phi(\lambda) - \frac{1}{\zeta-\theta}\int_\theta^\zeta \Phi(\xi)d\xi\right| \le \sup_{\theta<\xi<\zeta} |\Phi'(\xi)|(\zeta-\theta)\left[\frac{\left(\lambda-\frac{\theta+\zeta}{2}\right)^2}{(\zeta-\theta)^2}+\frac{1}{4}\right]. \tag{1}$$

Clearly, inequality (1) estimates an upper bound for the absolute deviation between the value of Φ at a point λ in $[\theta,\zeta]$ and its integral mean over $[\theta,\zeta]$.

Grüss [16] proved the following inequality to estimate the absolute deviation of the integral mean of the product of two functions from the product of the integral means.

Theorem 2. *Let Φ and ϕ be continuous functions on $[\theta,\zeta]$ such that*

$$m_1 \le \Phi(\xi) \le M_1 \qquad and \qquad m_2 \le \phi(\xi) \le M_2, \qquad for\ all \quad \xi \in [\theta,\zeta].$$

Then the following inequality

$$\left|\frac{1}{\zeta-\theta}\int_\theta^\zeta \Phi(\xi)\phi(\xi)d\xi - \frac{1}{(\zeta-\theta)^2}\int_\theta^\zeta \Phi(\xi)d\xi\int_\theta^\zeta \phi(\xi)d\xi\right| \le \frac{1}{4}(M_1-m_1)(M_2-m_2) \tag{2}$$

holds.

* Corresponding author e-mail: ahmedeldeeb@azhar.edu.eg

Inequality (2) is known in the literature as the Grüss inequality.

One of the companion inequalities of the Ostrowski inequality is the following inequality which is known in the literature as the trapezoid inequality [22].

Theorem 3. *Assume that Φ is a twice differentiable function on $[\theta,\zeta]$, then*

$$\left|\frac{\Phi(\theta)+\Phi(\zeta)}{2}(\zeta-\theta)-\int_\theta^\zeta \Phi(\xi)d\xi\right| \le \sup_{\theta<\xi<\zeta}|\Phi''(\xi)|\frac{(\zeta-\theta)^3}{12}.$$

In [25], Pachpatte obtained the following trapezoid and Grüss type inequalities.

Theorem 4. *Assume that $\Phi:[\theta,\zeta]\to\mathbb{R}$ is continuous and differentiable on (θ,ζ), whose first derivative $\Phi':(\theta,\zeta)\to\mathbb{R}$ is bounded on (θ,ζ), then*

$$\left|\frac{\Phi^2(\zeta)-\Phi^2(\theta)}{2}-\frac{\Phi(\zeta)-\Phi(\theta)}{b-\theta}\int_\theta^\zeta \Phi(\xi)d\xi\right| \le \frac{M^2(\zeta-\theta)^2}{3},$$

where $M=\sup_{\theta<\xi<\zeta}\Phi'(\xi)$.

Theorem 5. *Assume that Φ, $\phi:[\theta,\zeta]\to\mathbb{R}$ are continuous and differentiable on (θ,ζ), whose first derivatives Φ', $\phi':(\theta,\zeta)\to\mathbb{R}$ are bounded on (θ,ζ), then*

$$\begin{aligned}&\left|\frac{1}{\zeta-\theta}\int_\theta^\zeta \Phi(\xi)\phi(\xi)d\xi-\left(\frac{1}{\zeta-\theta}\int_\theta^\zeta \Phi(\xi)d\xi\right)\left(\frac{1}{\zeta-\theta}\int_\theta^\zeta \phi(\xi)d\xi\right)\right|\\ &\le \frac{1}{2(\zeta-\theta)^2}\int_\theta^\zeta\Big[M|\phi(\xi)|+N|\Phi(\xi)|\Big]\left[\frac{(\zeta-\theta)^2}{4}+\left(\xi-\frac{\theta+\zeta}{2}\right)^2\right]d\xi, \qquad (3)\end{aligned}$$

where $M=\sup_{\theta<\xi<\zeta}\Phi'(\xi)$ *and* $N=\sup_{\theta<\xi<\zeta}\phi'(\xi)$.

Ostrowski's inequality has a significant importance in many fields, particularly in numerical analysis. One of its applications is the estimation of the error in the approximation of integrals.

Many generalizations and refinements of the Ostrowski inequality and its companion inequalities were done during the past several decades, we refer the reader to the articles [1,5,8,19,20,21,7,6,10,9,2], see also [14,28,11,12,13,12, 30], and the books [22,23] and the references cited therein.

The aim of the present paper is first to establish a new Ostrowski type inequality on q-difference operator for functions whose second q-derivatives are bounded. Then, we prove new generalized Trapezoid and Grüss type inequalities on q-difference operator. As special cases of our results, some continuous inequalities are obtained.

This paper is organized as follows: In Section 2, we briefly present the basic definitions and concepts related to the calculus on q-difference operator. In Section 3, we state and prove our main results.

2 Preliminaries

In this section, we introduce some q-notations, we can found some of the following concepts in [15] Let $0<q<1$ be fixed. The q-shifted factorial, is defined for $a\in\mathbb{C}$ and $n\in\mathbb{N}_0$ by

$$(a;q)_n=\begin{cases}\prod_{j=1}^{n}(1-aq^{j-1}), & \text{if } n\in\mathbb{N},\\ 1, & \text{if } n=0.\end{cases}$$

Since $0<q<1$, then the limit of $(a;q)_\infty$ as n tends to infinity exists and will be denoted by $(a;q)_\infty$. The multiple q-shifted factorial for complex numbers $a_1,\cdots,a_k$ is defined by

$$(a_1,a_2,\cdots,a_k;q)_n:=\prod_{j=1}^{k}(a_j;q)_n.$$

The q-binomial coefficients are [15]

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad k = 1,2,\cdots,n,$$

and satisfy the property

$$\binom{n+1}{k}_q = \binom{n}{k}_q q^k + \binom{n}{k-1}_q, \tag{4}$$

see also [27,26]. We say that $A \subseteq \mathbb{R}$ is a q-geometric set if for every $t \in A, qt \in A$. Let f be a real or complex valued function defined on a q-geometric set A. The following q-difference operator is defined by Jackson in [18]

$$D_q f(t) = \frac{f(t) - f(qt)}{t(1-q)}, \quad t \neq 0. \tag{5}$$

If $0 \in A$, the q-derivative at zero is defined in [29,3] by

$$D_q f(0) := \lim_{n \longrightarrow \infty} \frac{f(tq^n) - f(0)}{tq^n}, \quad t \in A. \tag{6}$$

Provided the limit exists and does not depend on t. As in clearly seen, the definition of $D_q f(t)$ on $A \subseteq \mathbb{R}$ does not require A to be q-geometric. However, if $0 \in A$, and we need to define $D_q f(0)$, A has to be q-geometric. In most reference, see e.g. [15], $D_q f(0)$ is defined to be $f'(0)$, which is more restrictive. Also defining $D_q f(t)$ does not need q to lie in $(0,1)$, we may take $q \in \mathbb{R}$. Nevertheless, in this paper we will always assume that $q \in (0,1)$. Exceptionally, we may consider $D_{q^{-1}} f(\cdot)$, which is defined by

$$D_{q^{-1}} f(t) = \begin{cases} D_q f(q^{-1}t), & t \neq 0, \\ D_q f(0), & t = 0. \end{cases} \tag{7}$$

provided that $D_q f(0)$ exists and A is suitably chosen. It is worthwhile to mention that although the q-difference operator (5) is attributed to Jackson, [18], it may go back to Heine, cf. [15]. The following rules concerning q-derivatives can be easily checked or found in related monographs like [4,15]. If f and g are defined on a q-geometric set A such that the q-derivatives of f and g exist for all $t \in A$, then

$$D_q(f \pm g)(t) = D_q f(t) \pm D_q g(t),$$
$$D_q(fg)(t) = D_q f(t) g(t) + f(qt) D_q g(t),$$

and if $f(t) \neq 0 \neq g(qt)$, then

$$D_q\left(\frac{f}{g}\right)(t) = \frac{D_q f(t) g(t) - f(t) D_q g(t)}{f(t) g(qt)}.$$

Jackson in [17] introduced a q-integral which is denoted by

$$\int_a^b f(t) d_q t, \tag{8}$$

as a right inverse of the q-derivative. It is defined by

$$\int_a^b f(t) d_q t := \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \ \ a, b \in A, \tag{9}$$

where

$$\int_0^t f(t) d_q t := t(1-q) \sum_{n=0}^{\infty} q^n f(q^n t), \ \ t \in A, \tag{10}$$

provided that the series at the right-hand side of (10) converges at $t = a$ and b. The requirements for the derivation of a fundamental theorem of the q-calculus is less restrictive than classical calculus, cf. [3,29]. A function f defined on a q-geometric set A is said to be q-regular at zero if

$$\lim_{n \to \infty} f(tq^n) = f(0) \quad \text{for all} \quad t \in A. \tag{11}$$

It should be noted that continuity at zero implies q-regular at zero, but the converse is not necessarily true. A counter example could be found in [3]. Let f,g be a q-regular at zero functions defined on a q-geometric set A containing zero. Define

$$F(t) := \int_0^t f(s)d_qs, \quad t \in A.$$

Then F is q-regular at zero. Furthermore, $D_qF(t)$ exists for every $t \in A$ and

$$D_qF(t) = f(t), \qquad \forall t \in A. \tag{12}$$

Conversely, if a and b are two points in A, then

$$\int_a^b f(t)d_qt = F(b) - F(a), \tag{13}$$

also the rule of q-integration by parts is

$$\int_a^b f(t)D_qg(t)d_qt = f(b)g(b) - f(a)g(a) - \int_a^b D_q(f(t))g(qt)d_qt, \quad a,b \in A. \tag{14}$$

Let f,g be q-integrable on A, $k \in \mathbb{R}$ and $a,b,c \in A$. Then:
(i) $\int_a^a f(t)d_qt = 0$.
(ii) $\int_a^b kf(t)d_qt = k\int_a^b f(t)d_qt$.
(iii) $\int_a^b f(t)d_qt = -\int_b^a f(t)d_qt$.
(iv) $\int_a^b f(t)d_qt = \int_a^c f(t)d_qt + \int_c^b f(t)d_qt$.
(v) $\int_a^b (f(t)+g(t))d_qt = \int_a^b f(t)d_qt + \int_a^b g(t)d_qt$.

Next, we state and proof our main results.

3 Main Results

3.1 An Ostrowski-type inequality on q- difference operator

Theorem 6. *Let θ, ζ, λ, $\xi \in A$ and $\theta < \zeta$ where $0 \in A$ is q-geometric set. Further, assume that $\Phi : [\theta,\zeta] \to \mathbb{R}$ is a twice q-differentiable function. Then, for all $\lambda \in [\theta,\zeta]$ and $\eta,\gamma \in \mathbb{R}$, we have*

$$\begin{aligned}
&\Bigg|\Phi(\lambda) - \frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda \Phi(q\xi)d_q\xi + \frac{\gamma}{\zeta-\lambda}\int_\lambda^\zeta \Phi(q\xi)d_q\xi\right] \\
&-\frac{1}{\eta+\gamma}\left[\int_\theta^\zeta\int_\theta^\xi \frac{\eta}{\xi-\theta}\Upsilon(\lambda,\xi)D_q\Phi(qs)d_qsd_q\xi\right. \\
&\left.+\int_\theta^\zeta\int_\xi^\zeta \frac{\gamma}{\zeta-\lambda}\Upsilon(\lambda,\xi)D_q\Phi(qs)d_qsd_q\xi\right]\Bigg| \\
&\leqslant K\int_\theta^\zeta\int_\theta^\zeta |\Upsilon(\lambda,\xi)\Upsilon(\xi,s)|d_qsd_q\xi,
\end{aligned} \tag{15}$$

where

$$\Upsilon(\lambda,\xi) = \begin{cases} \dfrac{\eta}{\eta+\gamma}\left(\dfrac{\xi-\theta}{\lambda-\theta}\right), & \theta \le \xi < \lambda, \\ \dfrac{-\gamma}{\eta+\gamma}\left(\dfrac{\zeta-\xi}{\zeta-\lambda}\right), & \lambda \le \xi \le \zeta \end{cases}$$

and

$$K = \sup_{\theta<\xi<\zeta} |D_q^2\Phi(\xi)| < \infty.$$

Proof. Using integration by parts formula on q-difference operator (14), we have

$$\int_{\theta}^{\lambda}\frac{\eta}{\eta+\gamma}\Big(\frac{\xi-\theta}{\lambda-\theta}\Big)D_q\Phi(\xi)d_q\xi=\frac{\eta}{\eta+\gamma}\Phi(\lambda)-\frac{\eta}{(\eta+\gamma)(\lambda-\theta)}\int_{\theta}^{\lambda}\Phi(q\xi)d_q\xi, \tag{16}$$

and

$$\int_{\lambda}^{\zeta}\frac{-\gamma}{\eta+\gamma}\Big(\frac{\zeta-\xi}{\zeta-\lambda}\Big)D_q\Phi(\xi)d_q\xi=\frac{\gamma}{\eta+\gamma}\Phi(\lambda)-\frac{\gamma}{(\eta+\gamma)(\zeta-\lambda)}\int_{\lambda}^{\zeta}\Phi(q\xi)d_q\xi. \tag{17}$$

Adding (16) and (17), we get

$$\int_{\theta}^{\zeta}\Upsilon(\lambda,\xi)D_q\Phi(\xi)d_q\xi=\Phi(\lambda)-\frac{1}{\eta+\gamma}\Big[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda}\Phi(q\xi)d_q\xi+\frac{\gamma}{\zeta-\lambda}\int_{\lambda}^{\zeta}\Phi(q\xi)d_q\xi\Big]. \tag{18}$$

Similarly, we have

$$\begin{aligned}&\int_{\theta}^{\zeta}\Upsilon(\xi,s)D_q^2\Phi(s)d_qs\\&=D_q\Phi(\xi)-\frac{1}{\eta+\gamma}\Big[\frac{\eta}{\xi-\theta}\int_{\theta}^{\xi}D_q\Phi(qs)d_qs+\frac{\gamma}{\zeta-\xi}\int_{\xi}^{\zeta}D_q\Phi(qs)d_qs\Big].\end{aligned} \tag{19}$$

Substituting (19) into (18) leads to

$$\begin{aligned}&\int_{\theta}^{\zeta}\int_{\theta}^{\zeta}\Upsilon(\lambda,\xi)\Upsilon(\xi,s)D_q^2\Phi(s)d_qsd_q\xi\\&\quad+\frac{1}{\eta+\gamma}\Big[\int_{\theta}^{\zeta}\int_{\theta}^{\xi}\frac{\eta}{\xi-\theta}\Upsilon(\lambda,\xi)D_q\Phi(qs)d_qsd_q\xi\\&\quad+\int_{\theta}^{\zeta}\int_{\xi}^{\zeta}\frac{\gamma}{\zeta-\xi}\Upsilon(\lambda,\xi)D_q\Phi(qs)d_qsd_q\xi\Big]\\&=\Phi(\lambda)-\frac{1}{\eta+\gamma}\Big[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda}\Phi(q\xi)d_q\xi+\frac{\gamma}{\zeta-\lambda}\int_{\lambda}^{\zeta}\Phi(q\xi)d_q\xi\Big].\end{aligned} \tag{20}$$

Inequality (15) follows directly from (20) and the properties of modulus. This completes the proof.

Corollary 1. *If we take $q\longrightarrow 1^-$ in Theorem 6, then, inequality (15) becomes*

$$\begin{aligned}&\Big|\Phi(\lambda)-\frac{1}{\eta+\gamma}\Big[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda}\Phi(\xi)d\xi+\frac{\gamma}{\zeta-\lambda}\int_{\lambda}^{\zeta}\Phi(\xi)d\xi\Big]\\&-\frac{1}{\eta+\gamma}\Big[\int_{\theta}^{\zeta}\int_{\theta}^{\xi}\frac{\eta}{\xi-\theta}\Upsilon(\lambda,\xi)\Phi'(s)dsd\xi\\&+\int_{\theta}^{\zeta}\int_{\xi}^{\zeta}\frac{\gamma}{\zeta-\lambda}\Upsilon(\lambda,\xi)\Phi'(s)dsd\xi\Big]\Big|\\&\leqslant K\int_{\theta}^{\zeta}\int_{\theta}^{\zeta}\big|\Upsilon(\lambda,\xi)\Upsilon(\xi,s)\big|dsd\xi,\end{aligned}$$

where

$$\Upsilon(\lambda,\xi)=\begin{cases}\frac{\eta}{\eta+\gamma}\Big(\frac{\xi-\theta}{\lambda-\theta}\Big), & \theta\le\xi<\lambda,\\ \frac{-\gamma}{\eta+\gamma}\Big(\frac{\zeta-\xi}{\zeta-\lambda}\Big), & \lambda\le\xi\le\zeta\end{cases}$$

and

$$K=\sup_{\theta<\xi<\zeta}|\Phi''(\xi)|<\infty.$$

3.2 A Trapezoid-type inequality on q- difference operator

Theorem 7. *Under the same assumptions as in Theorem 6, we have*

$$\Bigg|\Phi^2(\zeta)-\Phi^2(\theta)-\frac{1}{\eta+\gamma}\int_\theta^\zeta\Bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda\left[\Phi(q\xi)+\Phi(q^2\xi)\right]d_q\xi$$
$$+\frac{\gamma}{\zeta-\lambda}\int_\lambda^\zeta\left[\Phi(q\xi)+\Phi(q^2\xi)\right]d_q\xi\Bigg]d_q\lambda\Bigg|$$
$$\leq M\big(M+P\big)\int_\theta^\zeta\int_\theta^\zeta\left|\Upsilon(\lambda,\xi)\right|d_q\xi d_q\lambda, \tag{21}$$

where

$$\Upsilon(\lambda,\xi)=\begin{cases}\dfrac{\eta}{\eta+\gamma}\Big(\dfrac{\xi-\theta}{\lambda-\theta}\Big), & \theta\leq\xi<\lambda,\\ \dfrac{-\gamma}{\eta+\gamma}\Big(\dfrac{\zeta-\xi}{\zeta-\lambda}\Big), & \lambda\leq\xi\leq\zeta\end{cases}$$

and

$$M=\sup_{\theta<\xi<\zeta}\left|D_q\Phi(\xi)\right| \qquad \textit{and} \qquad P=\sup_{\theta<\xi<\zeta}\left|D_q\Phi(q\xi)\right|.$$

Proof. From (18) we have

$$\Phi(\lambda)=\int_\theta^\zeta\Upsilon(\lambda,\xi)D_q\Phi(\xi)d_q\xi+\frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda\Phi(q\xi)d_q\xi+\frac{\gamma}{\zeta-\lambda}\int_\lambda^\zeta\Phi(q\xi)d_q\xi\right] \tag{22}$$

and similarly

$$\Phi(q\lambda)\ =\int_\theta^\zeta\Upsilon(\lambda,\xi)D_q\Phi(q\xi)d_q\xi+\frac{1}{\eta+\gamma}\Bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda\Phi(q^2\xi)d_q\xi$$
$$+\frac{\gamma}{\zeta-\lambda}\int_\lambda^\zeta\Phi(q^2\xi)d_q\xi\Bigg]. \tag{23}$$

Now, adding (22) and (23) produces

$$\Phi(\lambda)+\Phi(q\lambda)=\int_\theta^\zeta\Upsilon(\lambda,\xi)\left[D_q\Phi(\xi)+D_q\Phi(q\xi)\right]d_q\xi$$
$$+\frac{1}{\eta+\gamma}\Bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda\left[\Phi(q\xi)+\Phi(q^2\xi)\right]d_q\xi$$
$$+\frac{\gamma}{\zeta-\lambda}\int_\lambda^\zeta\left[\Phi(q\xi)+\Phi(q^2\xi)\right]d_q\xi\Bigg].$$

Multiplying the last identity by $D_q\Phi(\lambda)$, using (14) and integrating the resulting identity with respect to λ from θ to ζ yields

$$\Phi^2(\zeta)-\Phi^2(\theta)=\int_\theta^\zeta\int_\theta^\zeta D_q\Phi(\lambda)\Upsilon(\lambda,\xi)\left[D_q\Phi(\xi)+D_q\Phi(q\xi)\right]d_q\xi d_q\lambda$$
$$+\frac{1}{\eta+\gamma}\int_\theta^\zeta D_q\Phi(\lambda)\Bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda\left[\Phi(q\xi)+\Phi(q^2\xi)\right]d_q\xi$$
$$+\frac{\gamma}{\zeta-\lambda}\int_\lambda^\zeta\left[\Phi(q\xi)+\Phi(q^2\xi)\right]d_q\xi\Bigg]d_q\lambda.$$

Equivalently

$$\Phi^2(\zeta)-\Phi^2(\theta)-\frac{1}{\eta+\gamma}\int_\theta^\zeta D_q\Phi(\lambda)\Bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda\left[\Phi(q\xi)+\Phi(q^2\xi)\right]d_q\xi$$
$$+\frac{\gamma}{b-\lambda}\int_\lambda^\zeta\left[\Phi(q\xi)+\Phi(q^2\xi)\right]d_q\xi\Bigg]d_q\lambda$$
$$=\int_\theta^\zeta\int_\theta^\zeta D_q\Phi(\lambda)\Upsilon(\lambda,\xi)\left[D_q\Phi(\xi)+D_q\Phi(q\xi)\right]d_q\xi d_q\lambda.$$

Taking the absolute value on both sides, we get

$$\Bigg|\Phi^2(\zeta)-\Phi^2(\theta)-\frac{1}{\eta+\gamma}\int_\theta^\zeta D_q\Phi(\lambda)\Bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda\big[\Phi(q\xi)+\Phi(q^2\xi)\big]d_q\xi$$
$$+\frac{\gamma}{\zeta-\lambda}\int_\lambda^\zeta\big[\Phi(q\xi)+\Phi(q^2\xi)\big]d_q\xi\Bigg]d_q\lambda\Bigg|$$
$$=\Bigg|\int_\theta^\zeta\int_\theta^\zeta D_q\Phi(\lambda)\Upsilon(\lambda,\xi)\big[D_q\Phi(\xi)+D_q\Phi(q\xi)\big]d_q\xi d_q\lambda\Bigg|$$
$$\leq\int_\theta^\zeta\int_\theta^\zeta\big|D_q\Phi(\lambda)\big|\big|\Upsilon(\lambda,\xi)\big|\Big[\big|D_q\Phi(\xi)\big|+\big|D_q\Phi(q\xi)\big|\Big]d_q\xi d_q\lambda$$
$$\leq M\big(M+P\big)\int_\theta^\zeta\int_\theta^\zeta\big|\Upsilon(\lambda,\xi)\big|d_q\xi d_q\lambda.$$

This shows the validity of (21).

Corollary 2. *If we take $q\longrightarrow 1^-$ in Theorem 7, then, inequality (21) becomes*

$$\Bigg|\frac{\Phi^2(\zeta)-\Phi^2(\theta)}{2}-\frac{1}{\eta+\gamma}\int_\theta^\zeta\Phi'(\lambda)\Bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda\Phi(\xi)d\xi+\frac{\gamma}{\zeta-\lambda}\int_\lambda^\zeta\Phi(\xi)d\xi\Bigg]d\lambda\Bigg|$$
$$\leq M^2\int_\theta^\zeta\int_\theta^\zeta\big|\Upsilon(\lambda,\xi)\big|d\xi d\lambda,$$

where

$$\Upsilon(\lambda,\xi)=\begin{cases}\dfrac{\eta}{\eta+\gamma}\Big(\dfrac{\xi-\theta}{\lambda-\theta}\Big), & \theta\leq\xi<\lambda,\\ \dfrac{-\gamma}{\eta+\gamma}\Big(\dfrac{\zeta-\xi}{\zeta-\lambda}\Big), & \lambda\leq\xi\leq\zeta\end{cases}$$

and

$$M=\sup_{\theta<\xi<\zeta}\big|\Phi'(\xi)\big|.$$

3.3 A Grüss-type inequality on q- difference operator

Theorem 8. *Let θ, ζ, λ, $\xi\in A$ and $\theta<\zeta$ where $0\in A$ is q-geometric set. Moreover, assume that Φ, $\phi:[\theta,\zeta]\to\mathbb{R}$ are q-differentiable functions. Then, for all $\lambda\in[\theta,\zeta]$ and $\eta,\gamma\in\mathbb{R}$, we have*

$$\Bigg|2\int_\theta^\zeta\Phi(\lambda)\phi(\lambda)d_q\lambda-\frac{1}{\eta+\gamma}\Bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\zeta\int_\theta^\lambda\Big(\Phi(q\xi)\phi(\lambda)+\phi(q\xi)\Phi(\lambda)\Big)d_q\xi d_q\lambda$$
$$+\frac{\gamma}{\zeta-\lambda}\int_\theta^\zeta\int_\lambda^\zeta\Big(\Phi(q\xi)\phi(\lambda)+\phi(q\xi)\Phi(\lambda)\Big)d_q\xi d_q\lambda\Bigg]\Bigg|$$
$$\leq\int_\theta^\zeta\int_\theta^\zeta\big|\Upsilon(\lambda,\xi)\big|\Big[M\big|\phi(\lambda)\big|+N\big|\Phi(\lambda)\big|\Big]d_q\xi d_q\lambda, \tag{24}$$

where

$$\Upsilon(\lambda,\xi)=\begin{cases}\dfrac{\eta}{\eta+\gamma}\Big(\dfrac{\xi-\theta}{\lambda-\theta}\Big), & \theta\leq\xi<\lambda,\\ \dfrac{-\gamma}{\eta+\gamma}\Big(\dfrac{\zeta-\xi}{\zeta-\lambda}\Big), & \lambda\leq\xi\leq\zeta\end{cases}$$

and

$$M=\sup_{\theta<\xi<\zeta}|D_q\Phi(\xi)|<\infty\qquad\text{and}\qquad N=\sup_{\theta<\xi<\zeta}|D_q\phi(\xi)|<\infty.$$

Proof. From (18) we have

$$\Phi(\lambda)=\int_\theta^\zeta\Upsilon(\lambda,\xi)D_q\Phi(\xi)d_q\xi+\frac{1}{\eta+\gamma}\Bigg[\frac{\eta}{\lambda-\theta}\int_\theta^\lambda\Phi(q\xi)d_q\xi+\frac{\gamma}{\zeta-\lambda}\int_\lambda^\zeta\Phi(q\xi)d_q\xi\Bigg] \tag{25}$$

and similarly

$$\phi(\lambda)=\int_{\theta}^{\zeta}\Upsilon(\lambda,\xi)D_q\phi(\xi)d_q\xi+\frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\lambda}\phi(q\xi)d_q\xi+\frac{\gamma}{\zeta-\lambda}\int_{\lambda}^{\zeta}\phi(q\xi)d_q\xi\right]. \tag{26}$$

Multiplying (25) by $\phi(\lambda)$ and (26) by $\Phi(\lambda)$, adding them and integrating the resulting identity with respect to λ from θ to ζ yield

$$\begin{aligned}2\int_{\theta}^{\zeta}\Phi(\lambda)\phi(\lambda)d_q\lambda &= \int_{\theta}^{\zeta}\int_{\theta}^{\zeta}\Upsilon(\lambda,\xi)\left[D_q\Phi(\xi)\phi(\lambda)+D_q\phi(\xi)\Phi(\lambda)\right]d_q\xi d_q\lambda\\ &+\frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\zeta}\int_{\theta}^{\lambda}\Big(\Phi(q\xi)\phi(\lambda)+\phi(q\xi)\Phi(\lambda)\Big)d_q\xi d_q\lambda\right.\\ &\left.+\frac{\gamma}{\zeta-\lambda}\int_{\theta}^{\zeta}\int_{\lambda}^{\zeta}\Big(\Phi(q\xi)\phi(\lambda)+\phi(q\xi)\Phi(\lambda)\Big)d_q\xi d_q\lambda\right].\end{aligned}$$

By using modulus properties, we obtain

$$\begin{aligned}&\left|2\int_{\theta}^{\zeta}\Phi(\lambda)\phi(\lambda)d_q\lambda-\frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\zeta}\int_{\theta}^{\lambda}\Big(\Phi(q\xi)\phi(\lambda)+\phi(q\xi)\Phi(\lambda)\Big)d_q\xi d_q\lambda\right.\right.\\ &\left.\left.+\frac{\gamma}{\zeta-\lambda}\int_{\theta}^{\zeta}\int_{\lambda}^{\zeta}\Big(\Phi(q\xi)\phi(\lambda)+\phi(q\xi)\Phi(\lambda)\Big)d_q\xi d_q\lambda\right]\right|\\ &=\left|\int_{\theta}^{\zeta}\int_{\theta}^{\zeta}\Upsilon(\lambda,\xi)\left[D_q\Phi(\xi)\phi(\lambda)+D_q\phi(\xi)\Phi(\lambda)\right]d_q\xi d_q\lambda\right|\\ &\le\int_{\theta}^{\zeta}\int_{\theta}^{\zeta}|\Upsilon(\lambda,\xi)|\Big[|D_q\Phi(\xi)||\phi(\lambda)|+|D_q\phi(\xi)||\Phi(\lambda)|\Big]d_q\xi d_q\lambda\\ &\le\int_{\theta}^{\zeta}\int_{\theta}^{\zeta}|\Upsilon(\lambda,\xi)|\Big[M|\phi(\lambda)|+N|\Phi(\lambda)|\Big]d_q\xi d_q\lambda.\end{aligned}$$

This concludes the proof.

Corollary 3. *If we take $q\longrightarrow 1^-$ in Theorem 8, then, inequality (24) becomes*

$$\begin{aligned}&\left|2\int_{\theta}^{\zeta}\Phi(\lambda)\phi(\lambda)d\lambda-\frac{1}{\eta+\gamma}\left[\frac{\eta}{\lambda-\theta}\int_{\theta}^{\zeta}\int_{\theta}^{\lambda}\Big(\Phi(\xi)\phi(\lambda)+\phi(\xi)\Phi(\lambda)\Big)d\xi d\lambda\right.\right.\\ &\left.\left.+\frac{\gamma}{\zeta-\lambda}\int_{\theta}^{\zeta}\int_{\lambda}^{\zeta}\Big(\Phi(\xi)\phi(\lambda)+\phi(\xi)\Phi(\lambda)\Big)d\xi d\lambda\right]\right|\\ &\le\int_{\theta}^{\zeta}\int_{\theta}^{\zeta}|\Upsilon(\lambda,\xi)|\Big[M|\phi(\lambda)|+N|\Phi(\lambda)|\Big]d\xi d\lambda,\end{aligned}$$

where

$$\Upsilon(\lambda,\xi)=\begin{cases}\dfrac{\eta}{\eta+\gamma}\Big(\dfrac{\xi-\theta}{\lambda-\theta}\Big), & \theta\le\xi<\lambda,\\ \dfrac{-\gamma}{\eta+\gamma}\Big(\dfrac{\zeta-\xi}{\zeta-\lambda}\Big), & \lambda\le\xi\le\zeta\end{cases}$$

and

$$M=\sup_{\theta<\xi<\zeta}|\Phi'(\xi)|<\infty \qquad \text{and} \qquad N=\sup_{\theta<\xi<\zeta}|\phi'(\xi)|<\infty.$$

4 Conclusion

In this manuscript we discussed some new investigations of the Ostrowski inequality and its companion inequalities on q-difference operator by using two parameters. These inequalities have certain conditions that have not been studied before. For example, in Theorem 6, we are dealing with a function Φ whose second q-derivative is bounded, while all the existing literature deals with functions whose first derivatives are bounded. Besides that, in order to obtain some new inequalities as special cases, we also extended our inequalities to continuous calculus.

References

[1] F. Ahmed, P. Cerone, S. S. Dragomir and N. A. Mir, On some bounds of Ostrowski and Čebyšev type, *J. Math. Ineq.* **4**, 53–65 (2010).

[2] M. H. Ali, H. M. El-Owaidy, H. M. Ahmed, A. A. El-Deeb and I. Samir, Optical solitons and complexitons for generalized Schrödinger–Hirota model by the modified extended direct algebraic method, *Opt. Quant. Electr.* **55**, (2023).

[3] M. H. Annaby and Z. S. Mansour, *q-Fractional calculus and equations*, Springer, Vol. **2056**, 2012.

[4] M. H. Annaby and Z. S. Mansour,q-Taylor and interpolation series for Jackson q-difference operators, *J. Math. Anal. Appl.* **344**, 472–483 (2008).

[5] P. Cerone, S. S. Dragomir and J. Roumeliotis, An inequality of Ostrowski-Grüss type for twice differentiable mappings and applications in numerical integration, *Kyungpook Math. J.* **9**, 333–341 (1999).

[6] S. S. Dragomir, A generalization of Ostrowski integral inequality for mappings whose derivatives belong to $L_1\ [a,b]$ and applications in numerical integration, *J. Comput. Anal. Appl.* **3**, 343–360 (2001).

[7] S. S. Dragomir, A generalization of the Ostrowski integral inequality for mappings whose derivatives belong to $L_p[a,b]$ and applications in numerical integration, *J. Math. Anal. Appl.* **255**, 605–626 (2001).

[8] S. S. Dragomir, P. Cerone and J. Roumeliotis, A new generalization of Ostrowski's integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means, *Appl. Math. Lett.* **13**, 19–25 (2000).

[9] A. A. El-Deeb, H. A. El-Sennary and Z. A. Khan, Some Steffensen-type dynamic inequalities on time scales, *Adv. Differ. Equ.* **2019**, 1–14 (2019).

[10] A. A. El-Deeb, H. A. El-Sennary and Z. A. Khan,Some reverse inequalities of Hardy type on time scales, *Adv. Differ. Equ.* **2020**, 1–18 (2020).

[11] A. A. El-Deeb and J. Awrejcewicz, Novel fractional dynamic Hardy–Hilbert-type inequalities on time scales with applications, *Mathematics* **9**, (2021).

[12] A. A. El-Deeb and D. Baleanu, Some new dynamic Gronwall–Bellman–Pachpatte type inequalities with delay on time scales and certain applications, *J. Ineq. Appl.* **2022** (1),(2022).

[13] A. A. El-Deeb, S. D. Makharesh, S. S. Askar and D. Baleanu, Bennett-Leindler nabla type inequalities via conformable fractional derivatives on time scales, *AIMS Math.* **7**(8), 14099-1411 (2022).

[14] A. A. El-Deeb, S. D. Makharesh and D. Baleanu, Dynamic Hilbert-type inequalities with Fenchel-Legendre transform, *Symmetry* **12**, (2020).

[15] G. Gasper, M. Rahman and G. George, *Basic hypergeometric series*, Cambridge University Press, **96**, 2004.

[16] G. Grüss, *Über das Maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx$, *Math. Z.* **39**, 215–226 (1935).

[17] D. O. Jackson, T. Fukuda, O. Dunn and E. Majore, On q-definite integrals, *Quart. J. Pure Appl. Math.*, Citeseer, (1910).

[18] F. H. Jackson, XI.—on q-functions and a certain difference operator, *Earth Environ. Sci. Trans. Royal Soc. Edinburgh*, **46**, 253–281 (1909).

[19] W. Liu, Several error inequalities for a quadrature formula with a parameter and applications, *Comput. Math. Appl.* **56**, 1766–1772, 2008.

[20] W.-J. Liu, Y. Huang and X.-X. Pan, New weighted Ostrowski-Grüss-Čebyšev type inequalities, *Bull. Korean Math. Soc.* **45**, 477–483 (2008).

[21] W.-J. Liu, Q.-L. Xue and S.-F. Wang, Several new perturbed Ostrowski-like type inequalities, *JIPAM. J. Inequal. Pure Appl. Math.*, **8**, (2007).

[22] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Mathematics and its Applications (East European Series, Kluwer Academic Publishers Group, Dordrecht, **53**, 1991.

[23] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and new inequalities in analysis*, Mathematics and its Applications (East European Series, Kluwer Academic Publishers Group, Dordrecht, **61**, 1993.

[24] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helvetici*, **10**, 226–227 (1937).

[25] B. G. Pachpatte, On trapezoid and Grüss-like integral inequalities, *Tamkang J. Math.* **34**, 365–369 (2003).

[26] G. Pólya and G. L Alexanderson, Gaussian binomial coefficients, *Elemente Math.* **26**, 102–109 (1971).

[27] G. Pólya and G. Szegö, Problems and theorems in analysis: series, integral calculus, theory of functions, Springer, 1972.

[28] C. S. K. Raju, M.D. Kumar, N. A. Ahammad, A. A. El-Deeb, B. Almarri and N. Ali Shah, Non-linear dynamic movements of CNT/graphene/aluminum oxide and copper/silver/cobalt ferrite solid particles in a magnetized and suction-based internally heated surface: Sensitivity and response surface optimization, *Mathematics* **10**, (2022).

[29] M. H. Abu Risha, M. H. Annaby, Z. S. Mansour and M. E. H. Ismail, Linear q-difference equations, *Zeitschrift Anal. Anwendungen*, **26**, 481–194 (2007).

[30] M. M Tharwat, M. M Ahmed, A. Nosheen, K. A. Khan, I. Shahzadi, D. Baleanu and A. A El-Deeb, Dynamic inequalities of Gruss, Ostrowski and Trapezoid type via diamond-α integrals and Montgomery identity, *AIMS Math.* **9**, 12778–12799 (2024).