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Weak Solution, Blow-up and Numerical Study of a Nonlinear Reaction Diffusion Problem with an Integral Condition

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Abstract: The purpose of this article is to investigate a semilinear nonlocal problem with a 2^{nd}-type integral condition applied in a specific category of nonlinear equations of parabolic type. The linear problem is analyzed using the Fadeo-Galarkin approach, and the primary objective of the study is to determine whether the weak solution is unique and existent. Significant results achieved for the linear problem are subjected to an iterative approach in order to extend this study to the semilinear problem. A special case of the semilinear problem and its finite-time blow-up solution are also examined in the work. Numerical examples are provided to confirm the precision and effectiveness of the suggested approaches, which employ a forward time-centered spatial scheme to solve the semilinear problem. The study is presented in a formal, technical way and contributes to the understanding of weak solutions for a particular category of nonlinear equations of parabolic type.

Keywords: Semilinear nonlocal problem, Weak solution, Finite blow-up solution, Reaction-diffusion problem, Existence and uniqueness, Fadeo-Galarkin method, Forward-time centred space scheme.

1 Introduction

To model a wide range of important phenomena in nature, a class of parabolic equations has been extensively studied [1, 2,3,4,5]. However, the nonlinear evolution equations involved in these models are often complex, and their theoretical analysis presents significant challenges [6,7,8]. In particular, many of these equations can be formulated with Nonlocal Boundary Conditions (NBCs), which has led to a growing interest in their study [9,10]. Numerous contemporary physics and technology problems are described by PDEs with nonlocal and integral conditions [11,12,13,14,15,16,17,18]. The following is an expression for the first kind of these conditions:

$$\int_{\Sigma} \Upsilon(\tau,s)\theta(\tau,s)d\tau = E(s),$$

In this context, the function Υ is specified over the region of interest $\Sigma \subset \mathbb{R}^n$. Whereas the second type of them, which are known as Dirichlet conditions or Neumann conditions modeled by integral forms, might be outlined by:

$$\theta(\tau,s)|_{\partial\Sigma} = \int \Upsilon(\tau,s)\theta(\tau,s)d\tau,$$

The Dirichlet and Neumann conditions, which belong to the second type of conditions, are represented using integral forms. These integrals, which are defined across the border $\partial\Sigma$ of the spatial domain $\Sigma \subset \mathbb{R}^n$, are employed when it is not

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feasible to measure the required quantity directly on the boundary, or when estimating its average or computing its total value is challenging. Here, s represents the time domain and Υ is a function that is known a priori. For further illustration, the reader may refer to the references [19,20,21,22,23,24,25].

The study of semilinear parabolic equations, particularly those including a novel integral condition of second type and a classical Dirichlet condition, served as the inspiration for the research described in this paper. This subject is of special relevance for our inquiry because this integral condition is more extensive than any prior conditions discovered. Stated otherwise, the following semilinear problem will be our focus:

$$\begin{cases} \frac{\partial \theta}{\partial s} - a\frac{\partial^2 \theta}{\partial \tau^2} + b\theta = f(\tau, s, \theta, \theta_\tau) & (1) \\ \theta(\tau,0) = \varphi(\tau) & (2) \\ \theta(0,s) = 0 & (3) \\ \frac{\partial \theta}{\partial \tau}(1,s) = \int_0^1 \Upsilon(\tau,s)\theta(\tau,s)d\tau. & (4) \end{cases} \qquad (P_1)$$

We define $a, b \in \mathbb{R}_+^*$, the functions f, φ, and Υ in $L^2(Q)$, where Q is a domain represented by $(\tau,s) \in \mathbb{R}^2$, $\tau \in \Sigma = (0,1)$, $s \in (0,T)$. Assuming that f is Lipschitzian, we require that for any $(\theta_1,\phi_1), (\theta_2,\phi_2) \in \left(L^2(Q)\right)^2$, a positive constant μ exists in which:

$$\|f(\tau,s,\theta_1,\phi_1) - f(\tau,s,\theta_2,\phi_2)\|_{L^2(Q)} \leq \mu\left(\|\theta_1-\theta_2\|_{L^2(Q)} + \|\phi_1-\phi_2\|_{L^2(Q)}\right). \qquad (1)$$

From this vantage point, for the linear problem, we use the Faedo-Galerkin approach to investigate the solvability of the weak solution [26]. An iterative method is then used to show that the weak solution to the semilinear problem exists and is unique. At the conclusion of this research, we numerically solve this semilinear problem using the forward time centered space scheme and provide multiple numerical examples to make sure the precision and efficacy of the recommended approach.

2 The linear problem

In the forthcoming portion, we will focus on the same rectangular area Q as in problem (P_1). We will begin by formulating the linear problem associated with this problem of the form:

$$\begin{cases} \frac{\partial \theta}{\partial s} - a\Delta\theta + b\theta = f(\tau,s) & \forall(\tau,s) \in Q \\ \theta(\tau,0) = \varphi(\tau) & \forall \tau \in (0,1) \\ \theta(0,s) = 0 & \forall s \in (0,T) \\ \theta(1,s) = \int_0^1 \Upsilon(\tau,s)\theta(\tau,s)d\tau & \forall s \in (0,T). \end{cases} \qquad (P_2)$$

The subsequent item is an expression for the parabolic equation associated with the previously mentioned problem:

$$\mathscr{L}\theta = \frac{\partial \theta}{\partial s} - \Delta\theta + b\theta = f(\tau,s), \qquad (2)$$

We apply the Dirichlet Boundary Condition (BC), Initial Condition (IC), and second-type integral condition to the parabolic equation, which is expressed in the following manner:

$$\ell\theta = \theta(\tau,0) = \varphi(\tau), \qquad \tau \in (0,1),$$

$$\theta(0,s) = 0, \qquad s \in (0,T),$$

and

$$\theta_\tau(1,s) = \int_0^1 \Upsilon(\tau,s)\theta(\tau,s)\,d\tau, \quad s \in (0,T).$$

Defining the space V as:

$$V = \left\{\theta \in H^1(\Sigma) : \phi(0) = 0\right\}.$$

Given the norm $\|\phi\|V = \|\phi\|H^1(\Sigma)$, this space actually turns into a Hilbert space. Therefore, using this perspective, given the following hypothesis, we can now investigate problem (P_2):

$$(H):\begin{cases} f \in L^2\left(0,T;\, L^2(\Sigma)\right) & (H.1) \\ \varphi \in H^1(\Sigma) & (H.2) \end{cases}.$$

Considering how the linear problem (P_2) is formulated, we can establish the next definition that will be extremely useful later on.

Definition 1. *The weak solution θ of problem (P_2) is a function that satisfying the following criteria:*

1. $\theta \in L^2\left(0,T;\, H^1(\Sigma)\right) \cap L^\infty\left(0,T;\, H^1(\Sigma)\right)$.
2. θ *admits a strong derivative* $\dfrac{\partial\theta}{\partial s} \in L^2\left(0,T;\, L^2(\Sigma)\right)$.
3. $\theta(0) = \varphi$.
4. θ *verify the following identity:*

$$(\theta_s,\phi) + a(\theta_\tau,\phi_\tau) + b(\theta,\phi) = (f,\phi) + \theta_\tau(1,s)\phi(1),$$

for all $s \in [0,T]$ and $\phi \in V$.

In this regard, we aim now to derive the variational formulation informally. For this purpose, we first multiply the equality:

$$\frac{\partial\theta}{\partial s} - a\frac{\partial^2\theta}{\partial\tau^2} + b\theta = f(\tau,s), \tag{3}$$

by an element $\phi \in V$, and then integrate the result over Σ to obtain:

$$\int_\Sigma \frac{\partial\theta}{\partial s}\cdot\phi d\tau - a\int_\Sigma \frac{\partial^2\theta}{\partial\tau^2}\cdot\phi d\tau + b\int_\Sigma \theta\cdot\phi d\tau = \int_\Sigma f\cdot\phi d\tau. \tag{4}$$

Consequently, with the help of using the BCs as well as the Green's formula, the above equality becomes:

$$(\theta_s,\phi) + a(\theta_\tau,\phi_\tau) + b(\theta,\phi) = (f,\phi) + \theta_\tau(1,s)\phi(1), \tag{5}$$

for which $(\cdot,\cdot)$ represents $L^2(\Sigma)$'s scalar product.

In what follows, in order to determine whether the problem's weak solution (P_2) exists and is unique, we will utilize the Faedo-Galerkin method. Actually, this method can be carried out by implementing three steps. We will state each step as subsection and applying it to the problem at hand.

2.1 Constructing the approximation solutions

Since the space V is thought to be separable, a series of functions $\eta_1,\eta_2,\ldots,\eta_m$ that meet the following criteria exists:

– $V_m = \langle\{\eta_1,\eta_2,\cdots,\eta_m\}\rangle$ is dense in V.
– $\eta_1,\eta_2,\cdots,\eta_m$ are linearly independent, $\forall m$.
– $\eta_i \in V$, $\forall i$.

We have, specifically:

$$\forall\varphi \in V \implies \exists(\alpha_{km})_m \in IN^*,\ \varphi_m = \sum_{k=1}^m \alpha_{km}\eta_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \tag{6}$$

The function θ in Faedo Galerkin's approximation is such that:

$$s \mapsto \theta_m(\tau,s) = \sum_{i=1}^m \vartheta_{im}(s)\,\eta_i(\tau),$$

for any integer $m \geq 1$. However, this function satisfies:

$$\begin{cases} \theta_m(s) \in V_m, \quad \forall s \in [0,T] \\ ((\theta_m(s))_s,\eta_k) + A(\theta_m(s),\eta_k) + b(\theta_m(s),\eta_k) = (f(s),\eta_k) \quad \forall k = \overline{1,m}, \end{cases} \tag{P_3}$$

where

$$((\theta_m(s))_s, \eta_k) = \left(\left(\sum_{i=1}^{m} \vartheta_{im}(s)\,\eta_i\right)_s, \eta_k\right) = \left(\sum_{i=1}^{m} \frac{\partial \vartheta_{im}}{\partial s}(s)\,\eta_i(\tau), \eta_k\right) = \sum_{i=1}^{m} (\eta_i, \eta_k)\,\frac{\partial \vartheta_{im}}{\partial s}(s), \tag{7}$$

and

$$\begin{aligned} A(\theta_m(s), \eta_k) &= A\left(\sum_{i=1}^{m} \vartheta_{im}(s)\,\eta_i, \eta_k\right) = a\sum_{i=1}^{m} \vartheta_{im}(s)\left[\int_{\Sigma} \frac{\partial \eta_i}{\partial \tau}\frac{\partial \eta_k}{\partial \tau}d\tau - \frac{\partial \eta_i}{\partial \tau}(1)\,\eta_k(1)\right] \\ &= a\sum_{i=1}^{m} \vartheta_{im}(s)\int_{\Sigma} \frac{\partial \eta_i(\tau)}{\partial \tau}\frac{\partial \eta_k(\tau)}{\partial \tau}d\tau - a\sum_{i=1}^{m} \vartheta_{im}(s)\,\frac{\partial \eta_i}{\partial \tau}(1)\,\eta_k(1) = \sum_{i=1}^{m} A(\eta_i, \eta_k)\,\vartheta_{im}(s). \end{aligned} \tag{8}$$

At the same time, we have:

$$\theta_m(0) = \sum_{i=1}^{m} \vartheta_{im}(0)\,\eta_i(\tau) = \varphi_m = \sum_{i=1}^{m} \alpha_{im}\eta_i(\tau).$$

The system of 1^{st}-order nonlinear ODEs that we therefore arrive at is provided by:

$$\begin{cases} \sum_{i=1}^{m} (\eta_i, \eta_k)\,\frac{\partial \vartheta_{im}}{\partial s}(s) + a\sum_{i=1}^{m}\left(\frac{\partial \eta_i}{\partial \tau}, \frac{\partial \eta_k}{\partial \tau}\right)\vartheta_{im}(s) + b\sum_{i=1}^{m} \vartheta_{im}(s)\,(\eta_i, \eta_k) \\ \qquad = (f(s), \eta_k) + a\sum_{i=1}^{m} \vartheta_{im}(s)\,\frac{\partial \eta_i}{\partial \tau}(l)\,\eta_k(l) \\ \qquad \vartheta_{im}(0) = \alpha_{im} \qquad \forall i = \overline{1,m}. \end{cases} \tag{P_4}$$

We now examine the matrix

$$B_m = ((\eta_i, \eta_j))_{\substack{1\le i\le m\\1\le j\le m}}, A_m = \left(\left(\frac{\partial \eta_i}{\partial \tau}, \frac{\partial \eta_j}{\partial \tau}\right)\right)_{\substack{1\le i\le m\\1\le j\le m}}$$

and the vector:

$$\vartheta_m = (\vartheta_{1m}(s), \cdots, \vartheta_{mm}(s)), f_m = ((f, \eta_1), \cdots, (f, \eta_m)),$$

and

$$C_m = \left(\frac{\partial \eta_i}{\partial \tau}(1)\cdot \eta_j(1)\right)_{\substack{1\le i\le m\\1\le j\le m}}.$$

Consequently, the following matrix form can be used to explain problem (P_4):

$$\begin{cases} B_m \frac{\partial \vartheta_m}{\partial s}(s) + aA_m\vartheta_m + bB_m\vartheta_m = f_m + aC_m\vartheta_m \\ \vartheta_m(0) = (\alpha_{im})_{1\le i\le m} \end{cases}.$$

Due to B_m is a diagonal matrix, then its entries are linearly independent. This, however, implies $\det B_m \neq 0$, which asserts that this matrix is invertible. Therefore, the function ϑ_m will be then a solution of the following problem:

$$\begin{cases} \frac{\partial \vartheta_m}{\partial s}(s) + \left(aB_m^{-1}A_m + bB_m^{-1}B_m - aB_m^{-1}C_m\right)\vartheta_m = B_m^{-1}f_m \\ \vartheta_m(0) = (\alpha_{im})_{1\le i\le m}. \end{cases} \tag{P_5}$$

Due to the standard solvability theorems employed for the system of ODEs reported in [27], we have the vector $B_m^{-1}f_m$ with continuous functions that majorize by integrable functions over $(0,T)$ and the matrix $\left(aB_m^{-1}A_m + bB_m^{-1}B_m - aB_m^{-1}C_m\right)$ with constant coefficients. We conclude from this perspective that there is a s_m that depends exclusively on $|\alpha_{im}|$, so that the nonhomogeneous problem (P_5) admits a single local solution $\vartheta_m(s) \in C[0, s_m]$, for which the interval $[0, s_m]$ contains the following: $\vartheta_m^{'}(s) \in L^2[0,T]$. However, since integrable functions on $(0,T)$ majorize the elements of the vector $B_m^{-1}f_m$, it is possible to expand the solution to $[0,T]$.

2.2 Deriving an estimate a priori

In this subsection, a priori estimation of the problem's weak solution (P_2) is what we aim to obtain. For achieving the objective, we introduce the next theoretical result.

Lemma 1.*If*

$$b-\frac{\varepsilon}{2}-\frac{a\varepsilon}{l}-\frac{aK}{2\varepsilon}>0,\ \frac{1}{2}-\frac{1}{2\delta}-\frac{2Ta}{\varepsilon}>0,\ \frac{b}{2}-\|k\|_{\infty}^{2}>0 \text{ and } 1-\frac{aK}{2\varepsilon}>0,$$

for all $m\in\mathbb{N}^{*}$, then inequalities holding for the weak solution $\theta_m\in L^2(0,T\,;V_m)$ of (P_2), are given by:

$$\|\theta_m\|_{L^2(0,T;\,H^1(\Sigma))}\leq c_1$$
$$\left\|\frac{\partial\theta_m}{\partial s}\right\|_{L^2(0,T;\,L^2(\Sigma))}\leq c_2,$$

The constants c_1 and c_2 are both positive and independent of m.

*Proof.*The equation of (P_3) is multiplied by $\vartheta_{km}(s)$ in order to demonstrate this result, and the sum over k is then calculated to obtain:

$$\sum_{k=1}^{m}\left((\theta_m(s))_s,\eta_k\right)\cdot\vartheta_{km}(s)+a\sum_{k=1}^{m}\left(\frac{\partial\theta_m}{\partial\tau}(s),\frac{\partial\eta_k}{\partial\tau}\right)\cdot\vartheta_{km}(s)+b\sum_{k=1}^{m}\left(\theta_m(s),\eta_k\right)\cdot\vartheta_{km}(s)$$
$$=\sum_{k=1}^{m}\left(f(s),\eta_k\right)\cdot\vartheta_{km}(s)+a\sum_{i=1}^{m}\vartheta_{im}(s)\frac{\partial\eta_i}{\partial\tau}(1)\sum_{k=1}^{m}\vartheta_{km}(s)\eta_k(1).$$

Consequently, we obtain:

$$\left((\theta_m(s))_s,\theta_m(s)\right)+a\left(\frac{\partial\theta_m}{\partial\tau}(s),\frac{\partial\theta_m}{\partial\tau}(s)\right)+b\left(\theta_m(s),\theta_m(s)\right)$$
$$=\left(f(s),\theta_m(s)\right)+a\sum_{i=1}^{m}\vartheta_{im}(s)\frac{\partial\eta_i}{\partial\tau}(1)\sum_{k=1}^{m}\vartheta_{km}(s)\eta_k(1).$$

Thus, we get:

$$\frac{1}{2}\|\theta_m\|_{L^2(\Sigma)}^2-\frac{1}{2}\|\theta_m(0)\|_{L^2(\Sigma)}^2+a\int_0^s\left\|\frac{\partial\theta_m}{\partial\tau}\right\|_{L^2(\Sigma)}^2d\tau+b\int_0^s\|\theta_m\|_{L^2(\Sigma)}^2d\tau$$
$$\leq\frac{1}{2\varepsilon}\int_0^s\|f\|_{L^2(\Sigma)}^2d\tau+\frac{\varepsilon}{2}\int_0^s\|\theta_m\|_{L^2(\Sigma)}^2d\tau+a\int_0^s\left(\sum_{i=1}^{m}\vartheta_{im}(s)\frac{\partial\eta_i}{\partial\tau}(1)\sum_{k=1}^{m}\vartheta_{km}(s)\eta_k(1)\right)d\tau.$$

We must now estimate the third component of the side on the right of the aforementioned inequality with the aim of finishing our estimate of the weak solution of (P_2).

$$\int_0^s\left(\frac{\partial\theta_m}{\partial\tau}(1,\tau)\cdot\theta_m(1,\tau)\right)d\tau<\frac{\varepsilon}{2}\int_0^s\theta^2(1,\tau)\,d\tau+\frac{1}{2\varepsilon}\int_0^s\theta_\tau^2(1,\tau)\,d\tau,$$

for which $K=\max\int_Q k^2(\tau,s)\,d\tau ds$. As a consequence, integrating the result over Σ yields the following inequality:

$$\|\theta_m\|_{L^\infty(0,T;\,L^2(\Sigma))}^2+\left\|\frac{\partial\theta_m}{\partial\tau}\right\|_{L^2(0,T;\,L^2(\Sigma))}^2+\|\theta_m\|_{L^2(0,T;\,L^2(\Sigma))}^2\leq C_1\left(\|f\|_{L^2(0,T;\,L^2(\Sigma))}^2+\|\varphi_m\|_{L^2(\Sigma)}^2\right)=c_1,$$

or

$$C_1=\frac{1}{2\varepsilon\min\left\{\frac{1}{2},a(1-\varepsilon),\left(b-\frac{\varepsilon}{2}-a\varepsilon-\frac{aK}{2\varepsilon}\right)\right\}}.$$

The following equality may be established by using the same variational formulation (P_3), multiplying the new equation by $\vartheta'_{km}(s)$, and then computing the sum over k.

$$\int_Q \left(\frac{\partial\theta_m}{\partial s}\right)^2 d\tau ds + a\int_Q \frac{\partial\theta_m}{\partial\tau}\cdot\frac{\partial(\theta_m)_s}{\partial\tau}d\tau - a\int_0^\tau \frac{\partial\theta_m}{\partial\tau}\cdot\frac{\partial\theta_m}{\partial s}\Big|_{\tau=0}^{\tau=1} ds + b\int_Q \theta_m\cdot\frac{\partial\theta_m}{\partial s}d\tau = \int_Q f\cdot\frac{\partial\theta_m}{\partial s}d\tau.$$

Thus, we have:

$$\left\|\frac{\partial\theta_m}{\partial s}\right\|^2_{L^2(Q)} + \frac{a}{2}\left\|\frac{\partial\theta_m}{\partial\tau}(\tau)\right\|^2_{L^2(\Sigma)} + \frac{b}{2}\|\theta_m\|^2_{L^2(\Sigma)}$$
$$= \int_0^s \left(f(s), \frac{\partial\theta_m}{\partial s}\right) + a\int_0^\tau \left(\int_0^1 \Upsilon(\tau,s)\theta_m(\tau,s)d\tau\right)\cdot\frac{\partial\theta_m}{\partial s}(1,s)\,ds + \frac{a}{2}\|\varphi_m\|^2_{L^2(\Sigma)} + \frac{a}{2}\left\|\frac{\partial\varphi_m}{\partial\tau}\right\|^2_{L^2(\Sigma)}.$$

This, actually, implies:

$$\left(\frac{1}{2}-\frac{1}{2\delta}-\frac{2Ta}{\varepsilon}\right)\left\|\frac{\partial\theta_m}{\partial s}\right\|^2_{L^2(Q)} + \left(\frac{a}{2}-\frac{2a}{\varepsilon}\right)\left\|\frac{\partial\theta_m}{\partial\tau}(\tau)\right\|^2_{L^\infty(0,T;\,L^2(\Sigma))} + \left(\frac{b}{2}-\|k\|^2_\infty\right)\|\theta_m\|^2_{L^\infty(0,T;\,L^2(\Sigma))}$$
$$\leq \frac{a}{2}\|\varphi_m\|^2_{L^2(\Sigma)} + \left(\frac{a}{2}+\frac{a}{\varepsilon}\right)\|(\varphi_m)_\tau\|^2_{L^2(\Sigma)} + \frac{\delta}{2}\|f\|^2_{L^2(Q)}$$

Therefore, we finally have:

$$C_2 = \frac{\max\left\{\frac{a}{2}, \left(\frac{a}{2}+\frac{a}{\varepsilon}\right), \frac{\delta}{2}\right\}}{\min\left\{\left(\frac{1}{2}-\frac{1}{2\delta}-\frac{2Ta}{\varepsilon}\right), \left(\frac{a}{2}-\frac{2a}{\varepsilon}\right), \left(\frac{b}{2}-\|k\|^2_\infty\right)\right\}},$$

in which

$$c_2 = C_2\left(\|f\|^2_{L^2(Q)} + \|(\varphi_m)_\tau\|^2_{L^2(\Sigma)} + \|\varphi_m\|^2_{L^2(\Sigma)}\right).$$

This, immediately, gives:

$$\left\|\frac{\partial\theta_m}{\partial s}\right\|_{L^2(0,T;\,L^2(\Sigma))} \leq c_2. \tag{9}$$

Clearly, the solution to system (P_3) can be extended to $[0,T]$, as required by the first step. Consequently, as $m\to+\infty$, it can be inferred from (9) that:

$$\begin{cases} \theta_m \text{Uniformly Bounded (UB) in } L^\infty(0,T;\,L^2(\Sigma)) \\ \theta_m \text{ UB in } L^2(0,T;\,H^1(\Sigma)) \\ (\theta_m)_s \text{ UB in } L^2\left(0,T;\,L^2(\Sigma)\right) \end{cases}. \tag{10}$$

2.3 Testing the solution's convergence

The weak solution to problem (P_2) will be tested for convergence in this subsection. This actually would confirms the result of existence of such solution. To this aim, we present in what follow some novel results.

Theorem 1.*There exists a subsequence $(\theta_{m_k})_k \subseteq (\theta_m)_m$ of the function $u\in L^2(0,T;H^1(\Sigma))\cap L^\infty(0,T;L^2(\Sigma))$ in which $\frac{\partial\theta}{\partial s}\in L^2\left(0,T;\,L^2(\Sigma)\right)$ satisfying*

$$\begin{cases} \theta_{m_k} \rightharpoonup \theta & \text{in } L^2(0,T\,;H^1(\Sigma)) \\ \frac{\partial\theta_{m_k}}{\partial s} \rightharpoonup \frac{\partial\theta}{\partial s} & \text{in } L^2\left(0,T\,;L^2(\Sigma)\right) \end{cases},$$

when $m\longrightarrow+\infty$.

*Proof.*It should be noted that Lemma 1 implies the existence of two subsequences (θ_{m_k}) and $\left(\frac{\partial\theta_{m_k}}{\partial s}\right)$ of (θ_m) and $(\theta_m)_s$ respectively, for which

$$\theta_{m_k} \rightharpoonup \theta \qquad \text{in } L^2(0,T;\,H^1(\Sigma))\;. \tag{11}$$

$$\frac{\partial \theta_{m_k}}{\partial s} \rightharpoonup \eta \qquad \text{in } L^2\left(0,T;\, L^2\left(\Sigma\right)\right) \quad . \tag{12}$$

Rellich's Theorem, which ensures that each weakly convergent sequence in $H^1(Q)$ has a subsequence that converges strongly in $L^2(Q)$, can be used to establish the compactness of the injection of $H^1(Q)$ into $L^2(Q)$. Then, we obtain:

$$\theta_{m_k} \longrightarrow \theta \qquad \text{in } L^2(Q) \quad . \tag{13}$$

However, Lemma 1 allows us to deduce that there is a subsequence of $(\theta_{m_k})_k$, still represented by θ_{m_k}, that converges nearly everywhere to θ in a way that:

$$\theta_{m_k} \longrightarrow \theta \qquad \text{a.e } Q \quad . \tag{14}$$

Therefore, it is still necessary to demonstrate that $\eta = \frac{\partial \theta}{\partial s}$ is true. Stated otherwise, it is enough to demonstrate:

$$\theta(s) = \varphi + \int_0^s \eta(\tau)d\tau. \tag{15}$$

The proof of (15) will actually be completed by simply proving that $\theta_{m_k} \rightharpoonup \varphi + \chi$ in $L^2(0,T\,;L^2(\Sigma))$, since $\theta_{m_k} \rightharpoonup \theta$ in $L^2(0,T\,;L^2(\Sigma))$, i.e.,

$$\lim \left(\theta_{m_k} - \varphi - \chi, \phi\right)_{L^2(0,T\,;L^2(\Sigma))} = 0,\ \forall \phi \in L^2(0,T\,;L^2(\Sigma)),$$

where

$$\chi(s) = \int_0^s \eta(\tau)d\tau.$$

As a matter of fact, with the help of using the equality:

$$\theta_{m_k} - \varphi_{m_k} = \int_0^s \frac{\partial \theta_{m_k}}{\partial \tau} d\tau, \text{ for all } s \in [0,T],$$

the following assertion hold:

$$\begin{aligned}
&\left(\theta_{m_k} - \varphi - \int_0^s \eta(\tau)d\tau, \phi\right)_{L^2(0,T\,;L^2(\Sigma))} \\
&= \left(\theta_{m_k} - \varphi_{m_k} - \int_0^s \eta(\tau)d\tau, \phi\right)_{L^2(0,T\,;L^2(\Sigma))} + \left(\varphi_{m_k} - \varphi, \phi\right)_{L^2(0,T\,;L^2(\Sigma))} \\
&= \int_0^s \left(\frac{\partial \theta_{m_k}}{\partial \tau} - \eta(\tau), \phi\right)_{L^2(0,T\,;L^2(\Sigma))} d\tau + \left(\varphi_{m_k} - \varphi, \phi\right)_{L^2(0,T\,;L^2(\Sigma))}, \text{ for all } s \in [0,T],
\end{aligned}$$

where $\theta_{m_k} \in L^2\left(0,T;V_{m_k}\right)$ and $\left(\theta_{m_k}\right)_s \in L^2\left(0,T;V_{m_k}\right)$. However, we also get:

$$\lim_{k\longrightarrow\infty} \int_0^s \left(\frac{\partial \theta_{m_k}}{\partial \tau} - \eta(\tau), \phi\right)_{L^2(0,T\,;L^2(\Sigma))} d\tau = 0, \text{ for } s \in [0,T]. \tag{16}$$

In addition, we might yield:

$$\lim_{k\longrightarrow\infty} (\varphi_m - \varphi, \phi)_{L^2(0,T\,;L^2(\Sigma))} = 0. \tag{17}$$

Hence, we get:

$$\lim_{k\longrightarrow\infty} \left(\theta_{m_k} - \varphi - \chi, \phi\right)_{L^2(0,T\,;L^2(\Sigma))} = 0, \quad \forall \phi \in L^2\left(0,T;L^2(\Sigma)\right).$$

Theorem 2. *The weak solution to the problem (P_2) according to Definition 1 is the function θ in Theorem 1.*

*Proof.*We have demonstrated in Theorem 1 that the limit function θ satisfies the first two requirements of Definition 1. As a result, we shall now illustrate the final two requirements, 3 and 4. In order to address requirement 3, we should be aware that the Theorem 1 allows us to have:

$$\theta_{m_k}(0) \rightharpoonup \theta(0) \qquad \text{in } L^2(\Sigma) \quad .$$

Additionally, we have:

$$\theta_{m_k}(0) \longrightarrow \varphi \qquad \text{in } L^2(\Sigma) \quad .$$

This gives:

$$\theta_{m_k}(0) \rightharpoonup \varphi \qquad \text{in } L^2(\Sigma) \quad .$$

This limit's uniqueness allows us to obtain $\theta(0) = \varphi$, confirming condition 3 in Definition 1. Thus, it remains to demonstrate the fourth one, i.e., for all $s \in [0,T]$ and $\phi \in V$, we get

$$(\theta_s, \phi) + a(\theta, \phi) + b(\theta, \phi) = (f, \phi).$$

To this aim, we integrate (P_3) over $(0,T)$ to obtain:

$$\int_0^s ((\theta_m(s))_s, \eta_k)\, d\tau + \int_0^s a(\theta_m(s), \eta_k)\, d\tau + b\int_0^s (\theta_m(s), \eta_k)\, d\tau = \int_0^s (f(s), \eta_k)\, d\tau, \tag{18}$$

$\forall k = \overline{1,m}$. After passing to the limit in (18) and using (9) to observe that V_m is dense in V, we obtain the following:

$$\int_0^S (\theta_s, \eta_k)\, d\tau + \int_0^S a(\theta, \eta_k)\, d\tau + b\int_0^S (\theta, \eta_k)\, d\tau = \int_0^S (f, \eta_k)\, d\tau, \quad \forall s \in [0,T].$$

Since condition 4 is satisfied, the proof of this result is therefore finished.

Corollary 1.*The estimate of the problem (9) directly demonstrates the uniqueness of the weak solution of (P_2).*

3 The semilinear problem

The subject of this part is the proof that the weak solution to the given semilinear problem (P_1) exists and is unique. In order to do this, we presume:

$$\theta = y + w,$$

where problem (P_1) has a solution here, θ, and the following problem has a solution here, w:

$$\begin{cases} \dfrac{\partial w}{\partial s} - a\Delta\eta(\tau,s) + b\eta(\tau,s) = 0 & \forall(\tau,s) \in Q \\ w(\tau,0) = \varphi(\tau) & \forall\tau \in (0,1) \\ \eta(0,s) = 0 & \forall s \in (0,T) \\ \dfrac{\partial w}{\partial \tau}(1,s) = \int_0^1 \Upsilon(\tau,s)\eta(\tau,s)d\tau. & \forall s \in (0,T) \end{cases}.$$

At the same time, we note:

$$y = \theta - w,$$

which satisfies

$$\mathscr{L}y = \frac{\partial y}{\partial s} - a\Delta y(\tau,s) + by(\tau,s) = 0 = G(\tau,s,y,y_\tau),$$

$$\begin{aligned} y(x,0) &= 0, \quad \forall\tau \in (0,1), \\ y(0,s) &= 0 \quad \forall s \in (0,s), \\ \frac{\partial y}{\partial \tau}(1,s)d\tau &= 0 \quad \forall s \in (0,s), \end{aligned}$$

where

$$G(\tau,s,y,y_\tau) = f(\tau,s,y+w,(y+w)_\tau).$$

The function G is Lipchitzian, similarly to the function f, meaning that a positive constant k exists for which:

$$\| G(\tau,s,\theta_1,\phi_1) - G(\tau,s,\theta_2,\phi_2)_{L^2(Q)} \|_{L^2(Q)} \le k\left(\|\theta_1-\theta_2\|_{L^2\left(0,T,H^1(0,1)\right)} + \|\phi_1-\phi_2\|_{L^2\left(0,T,H^1(0,1)\right)}\right).$$

We now put forth the idea of the solution under study. In other words, we assume that $v=\phi(\tau,s)$ represent any function of V, by which:

$$V=\left\{\phi\in C^1(Q),\ \phi(1,s)=\phi(0,s)=0,\ s\in[0,T]\right\}.$$

Multiplying the following equation:

$$\frac{\partial y}{\partial s} - a\frac{\partial^2 y}{\partial \tau^2} + by = f(\tau,s,y,y_\tau)$$

by v, applying the integration by parts to the result with the conditions imposed on y and v after integrating the result over Q_τ yields the following assertion:

$$\int_{Q_\tau} \frac{\partial y}{\partial s}(\tau,s)\cdot\phi(\tau,s)d\tau ds + a\int_{Q_\tau}\frac{\partial y}{\partial \tau}(\tau,s)\cdot\frac{\partial v}{\partial \tau}(\tau,s)d\tau ds + b\int_{Q_\tau} y(\tau,s)\cdot\phi(\tau,s)d\tau ds \tag{19}$$
$$= \int_{Q_\tau} G(\tau,s,y,y_\tau)\cdot\phi(\tau,s)d\tau ds.$$

Consequently, we have from (19) the following assertion:

$$A(y,\phi) = \int_{Q_\tau} G(\tau,s,y,y_\tau)\cdot\phi(\tau,s)d\tau ds, \tag{20}$$

or

$$A(y,\phi) = \int_{Q_\tau} \frac{\partial y}{\partial s}(\tau,s)\cdot\phi(\tau,s)d\tau ds + a\int_{Q_\tau}\frac{\partial y}{\partial \tau}(\tau,s)\cdot\frac{\partial v}{\partial \tau}(\tau,s)d\tau ds + b\int_{Q_\tau} y(\tau,s)\cdot\phi(\tau,s)d\tau ds.$$

With $y^{(0)}=0$ as the beginning point, we now want to construct a recurrent series $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ for $n=1,2,3,\cdots$. The following is a definition for this sequence: Considering the element $y^{(n-1)}$, the following problem must be resolved:

$$\begin{cases} \frac{\partial y^{(n)}}{\partial s} - a\Delta y^{(n)} + by^{(n)} = G\left(\tau,s,y^{(n-1)},y_\tau^{(n-1)}\right) \\ y^{(n)}(\tau,0)=0 \\ y^{(n)}(0,s)=0 \\ y_\tau^{(n)}(1,s)=0 \end{cases}. \tag{P_6}$$

Given the linear problem study, problem (P_6) admits a unique solution $y^{(n)}(\tau,s)$ if we fix n at each time. This solution can then be written using the Fadeo-Galarkin approach. Now, we suppose:

$$z^{(n)}(\tau,s) = y^{(n+1)}(\tau,s) - y^{(n)}(\tau,s).$$

This implies a new problem of the form:

$$\begin{cases} \frac{\partial z^{(n)}}{\partial s} - a\Delta z^{(n)} + bz^{(n)} = p^{(n-1)}(\tau,s) \\ z^{(n)}(\tau.0)=0 \\ z^{(n)}(0,s)=0 \\ z^{(n)}(1,s)d\tau=0 \end{cases}, \tag{P_7}$$

or

$$p^{(n-1)}(\tau,s) = G\left(\tau,s,y^{(n)},y_\tau^{(n)}\right) - G\left(\tau,s,y^{(n-1)},y_\tau^{(n-1)}\right).$$

The following equality is obtained by multiplying the equations (P_7) by $z^{(n)}$ and then integrating the result over Q_ρ:

$$\frac{1}{2}\int_0^1 (z^{(n)}(\tau,\rho))^2 d\tau + a\int_{Q_\rho}\left(\frac{\partial z^{(n)}}{\partial \tau}(\tau,s)\right)^2 d\tau ds + b\int_{Q_\rho}\left(z^{(n)}(\tau,s)\right)^2 d\tau ds = \int_{Q_\rho} p^{(n-1)}(\tau,s)\cdot z^{(n)}(\tau,s)d\tau ds.$$

The following is the result of applying the Cauchy Schwarz inequality to the second portion of the equation above:

$$\frac{1}{2}\int_0^1 (z^{(n)}(\tau,\rho))^2 d\tau + a\int_{Q_\rho}\left(\frac{\partial z^{(n)}}{\partial \tau}(\tau,s)\right)^2 d\tau ds + b\int_{Q_\rho}\left(z^{(n)}(\tau,s)\right)^2 d\tau ds$$

$$\leqslant \frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|_{L^2(0,T,H^1(0,l))} + 2\left(\frac{\varepsilon}{2}-b\right)\int_{Q^\rho}\left(z^{(n)}(\tau,s)\right)^2 d\tau ds.$$

In addition, applying Grenwell's Lemma yields:

$$\int_0^1 (z^{(n)}(\tau,\rho))^2 d\tau + 2a\int_{Q_\rho}\left(\frac{\partial z^{(n)}}{\partial \tau}(\tau,s)\right)^2 d\tau ds \leqslant \frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|_{L^2(0,T,H^1(0,l))}\exp((\varepsilon-2b)T).$$

The following is obtained by integrating the aforementioned inequality over s:

$$\int_{Q_S} (z^{(n)}(\tau,\rho))^2 d\tau ds + 2Ta\int_{Q_S}\left(\frac{\partial z^{(n)}}{\partial \tau}(\tau,s)\right)^2 d\tau ds \leqslant \frac{Tk^2}{\varepsilon}\left\|z^{(n-1)}\right\|_{L^2(0,T,H^1(0,l))}\exp((\varepsilon-2b)T)$$

$$\int_{Q_S} (z^{(n)}(\tau,\rho))^2 d\tau ds + \int_{Q_S}\left(\frac{\partial z^{(n)}}{\partial \tau}(\tau,s)\right)^2 d\tau ds \leqslant \frac{Tk^2\exp((\varepsilon-2b)T)}{\varepsilon\min(1,2Ta)}\left\|z^{(n-1)}\right\|_{L^2(0,T,H^1(0,1))}.$$

Now, letting $c = \frac{Tk^2\exp((\varepsilon-2b)T)}{\varepsilon\min(1,2Ta)}$ yields the following inequality:

$$\| z^{(n)} \| \leqslant c\left\|z^{(n-1)}\right\|_{L^2(0,T,H^1(0,1))},$$

such that

$$\sum_{i=1}^{n-1} z^{(i)} = y^{(n)}.$$

The series $\sum_{n=1}^{\infty} z^{(n)}$ converges if $|c|<1$, as per the convergence condition. This, immediately, implies:

$$\left|\frac{Tk^2\exp((\varepsilon-2b)T)}{\varepsilon\min(1,2Ta)}\right| < 1 \Rightarrow$$

$$k\sqrt{\frac{T\exp((\varepsilon-2b)T)}{\varepsilon\min(1,2Ta)}} < 1,$$

i.e.,

$$k < \sqrt{\frac{\varepsilon\min(1,2Ta)}{2T\exp((\varepsilon-2b)T)}}.$$

Hence, in $L^2(0,T,H^1(0,l))$, $(y^{(n)})_n$ converges to an element y. Now, we attempt to demonstrate that the problem's solution of (P_6) is

$$\lim_{n\longrightarrow\infty} y^{(n)}(\tau,s) = y(\tau,s).$$

This, really, can be achieved by showing that y meets the following assertion:

$$A(y,\phi) = \int_{Q_\rho} G(\tau,s,y,y_\tau)\cdot\phi(\tau,s)d\tau ds.$$

In order to achieve this, we formulate the weak formulation of the problem (P_1) as follows:

$$A\left(y^{(n)},\phi\right) = \int_{Q_\rho}\frac{\partial y^{(n)}}{\partial s}(\tau,s)\cdot\phi(\tau,s)d\tau ds + a\int_{Q_\rho}\frac{\partial y^{(n)}}{\partial \tau}(\tau,s)\cdot\frac{\partial v}{\partial \tau}(\tau,s)d\tau ds + b\int_{Q_\rho} y^{(n)}(\tau,s)\cdot\phi(\tau,s)d\tau ds.$$

Given that A is linear, we get:

$$A\left(y^{(n)},\phi\right)=A\left(y^{(n)}-y,\phi\right)+A\left(y,\phi\right)$$

$$=\int\limits_{Q_\rho}\frac{\partial(y^{(n)}-y)}{\partial s}(\tau,s)\cdot\phi(\tau,s)d\tau ds+a\int\limits_{Q_\rho}\frac{\partial(y^{(n)}-y)}{\partial\tau}(\tau,s)\cdot\frac{\partial v}{\partial\tau}(\tau,s)d\tau ds+b\int\limits_{Q_\rho}(y^{(n)}-y)(\tau,s)\cdot\phi(\tau,s)d\tau ds$$

$$+\int\limits_{Q_\rho}\frac{\partial y}{\partial s}(\tau,s)\cdot\phi(\tau,s)d\tau ds+a\int\limits_{Q_\rho}\frac{\partial y}{\partial\tau}(\tau,s)\cdot\frac{\partial v}{\partial\tau}(\tau,s)d\tau ds+b\int\limits_{Q_\rho}y(\tau,s)\cdot\phi(\tau,s)d\tau ds.$$

This, consequently, implies:

$$A\left(y^{(n)}-y,\phi\right)=\int\limits_{Q_\rho}\frac{\partial(y^{(n)}-y)}{\partial s}(\tau,s)\cdot\phi(\tau,s)d\tau ds+a\int\limits_{Q_\rho}\frac{\partial(y^{(n)}-y)}{\partial\tau}(\tau,s)\cdot\frac{\partial v}{\partial\tau}(\tau,s)d\tau ds$$
$$+b\int\limits_{Q_\rho}(y^{(n)}-y)(\tau,s)\cdot\phi(\tau,s)d\tau ds.$$

The Cauchy Schwartz inequality is applied, and the result is:

$$A\left(y^{(n)}-y,\phi\right)\leqslant C(\left\|(y^{(n)}-y)_s\right\|_{L^2(0,T,H^1(0,l))}+\left\|(y^{(n)}-y)_\tau\right\|_{L^2(0,T,H^1(0,l))}$$
$$+\left\|y^{(n)}-y\right\|_{L^2(0,T,H^1(0,l))})\left(\|\,v\,\|_{L^2(Q_\rho)}+\|\,\phi_\tau\,\|_{L^2(Q_\rho)}\right),$$

or

$$C=\frac{\max(1,a,b)}{\min(1,a,b)}.$$

Now, due to $y^{(n)}\longrightarrow y$ in $L^2\left(0,T,H^1\left(0,l\right)\right)\approxeq H^1\left(Q\right)$, we have:

$$y^{(n)}\longrightarrow y\quad\text{in } L^2\left(Q\right),$$
$$y_s^{(n)}\longrightarrow y_s\quad\text{in } L^2\left(Q\right),$$
$$y_\tau^{(n)}\longrightarrow y_\tau\quad\text{in } L^2\left(Q\right).$$

Thus, as $n\longrightarrow+\infty$, we obtain:

$$\lim_{n\longrightarrow+\infty}A\left(y^{(n)}-y,\phi\right)=0.$$

4 Finite-time blow-up solution

The semilinear problem's blow-up solution (P_1) will be examined in this section. This might be applied by assuming $f(\tau,s,\theta,\theta_\tau)=\theta^p$. We examine the following Sturm-Liouville problem for this purpose:

$$\begin{cases}-\Delta\psi=\lambda^2\psi\\ \psi(0)=0\\ \psi_\tau(l)=0.\end{cases}$$

By considering the solution corresponding to the first eigenvalue, we can get:

$$\psi(\tau)=B\sin\left(\frac{\pi}{2}\tau\right).$$

Now, by letting:

$$\Pi(s)=\int_0^1\psi(\tau)\theta(\tau,s)d\tau,$$

as well as multiplying the equation:

$$\frac{\partial\theta}{\partial s}-a\frac{\partial^2\theta}{\partial\tau^2}+b\theta=\theta^p$$

by ψ and subsequently integrating the outcome across the domain $\Sigma = (0,1)$, we arrive at:

$$\int_0^1 \psi(\tau)\frac{\partial\theta}{\partial s}d\tau - a\int_0^1 \psi(\tau)\frac{\partial^2\theta}{\partial\tau^2}d\tau - b\int_0^1 \psi(\tau)u d\tau = \int_0^1 \psi(\tau)\theta^p d\tau.$$

This, consequently, implies:

$$\begin{aligned}\Pi'(s) + \left(a\lambda^2 + b\right)\Pi(s) &= a\left[\int_0^1 \Upsilon(\tau,s)\theta(\tau,s)d\tau\right]\psi(1) + \int_0^1 \psi(\tau)\theta^p d\tau \\ &\geq a\min_{x,y\in Q}(\Upsilon(\tau,s))\int_0^1 \psi(\tau)\theta(\tau,s)d\tau + \int_0^1 \psi(\tau)\theta^p d\tau. \end{aligned} \tag{21}$$

Applying Jensen's inequality:

$$\int_0^l \psi(\tau)\theta^p d\tau \geq \left(\frac{\pi}{2}\right)^{p-1}(\Pi(s))^p$$

yields the following assertion:

$$\Pi'(s) + \left(b + a\lambda^2 - a\min_{x,y\in Q}(\Upsilon(\tau,s))\right)\Pi(s) \geq \left(\frac{\pi}{2}\right)^{p-1}(\Pi(s))^p.$$

Now, we will consider the following equation:

$$\Pi'(s) + \underbrace{\left(b + a\lambda^2 - a\min_{x,y\in Q}(\Upsilon(\tau,s))\right)}_{C}\Pi(s) - \left(\frac{\pi}{2}\right)^{p-1}(\Pi(s))^p = 0. \tag{22}$$

In reality, the following form of the equation's solution is obtained by setting $K = (p-1)C$ and using the change of variable $v = \Pi^{1-p}$ to solve it:

$$\Pi(s) = \left(\left((\Pi(0))^{1-p} - \frac{1}{K}(1-p)\left(\frac{\pi}{2}\right)^{p-1}\right)e^{-Kt} + \frac{1}{K}(1-p)\left(\frac{\pi}{2}\right)^{p-1}\right)^{\frac{1}{1-p}},$$

or

$$\Pi(s) = \left(\frac{1}{\left((\Pi(0))^{1-p} - \frac{1}{K}(1-p)\left(\frac{\pi}{2}\right)^{p-1}\right)e^{-Kt} + \frac{1}{K}(1-p)\left(\frac{\pi}{2}\right)^{p-1}}\right)^{\frac{1}{p-1}}.$$

Now, as $\frac{1}{p-1} > 0$, then we have:

$$\Pi \to \infty \text{ if } \left((\Pi(0))^{1-p} - \frac{1}{K}(1-p)\left(\frac{\pi}{2}\right)^{p-1}\right)e^{-Kt} + \frac{1}{K}(1-p)\left(\frac{\pi}{2}\right)^{p-1} \to 0.$$

Consequently, we have:

$$T = \frac{1}{K}\ln\left(\frac{\frac{1}{K}(p-1)\left(\frac{\pi}{2}\right)^{p-1}}{\left((\Pi(0))^{1-p} - \frac{1}{K}(1-p)\left(\frac{\pi}{2}\right)^{p-1}\right)}\right).$$

5 Construction of approximate solutions

We want to use the forward time centred space (FTCS) method [28,29] for the purpose of obtaining an exact solution of (P_1). We start by taking the two positive integers N and M in order to accomplish this purpose. Additionally, with $k = T/N$, and $h = l/M$, respectively, the intervals $[0,l]$ and $[0,T]$ are separated into M and N subintervals of the same length. In parallel, at the i^{th} grid-point and n^{th} time step, we approximate the solution θ as θ_i^n. For $i = 0,1,2,\ldots,M$, $s_n = nk$, and $n = 0,1,2,\ldots,N$, the Grid point $(\tau_i,\ s_n)$ is obtained in this instance using $\tau_i = ih$. Additionally, the notations θ_i^n, f_i^n, φ_i, and k_i^n are used to implement the finite difference approximations of $\theta(\tau_i,s_n)$, $f(\tau_i,s_n)$, $\varphi(\tau_i)$, and $\Upsilon(\tau_i,s_n)$, respectively. Specifically, the temporal derivative in this case can be roughly estimated using the forward difference quotient. The spatial derivative of the first and second order can then be estimated using the centered second-order approximation. This implies the following assertion:

$$\frac{\theta_i^{n+1}-\theta_i^n}{k}-a(\frac{\theta_{i+1}^n-2\theta_i^n+\theta_{i-1}^n}{h^2})+b\theta_i^n=f(\tau_i,s_n,\theta_i^n,\frac{\theta_{i+1}^n-\theta_{i-1}^n}{2h}).$$

As a matter of fact, the above approach can be written as:

$$\theta_i^{n+1}=\frac{ak}{h^2}(\theta_{i+1}^n-2\theta_i^n+\theta_{i-1}^n)-bk\theta_i^n+\theta_i^n+kf(\tau_i,s_n,\theta_i^n,\frac{\theta_{i+1}^n-\theta_{i-1}^n}{2h})$$
$$\theta_i^{n+1}=r(\theta_{i+1}^n-2\theta_i^n+\theta_{i-1}^n)-bk\theta_i^n+\theta_i^n+kf(\tau_i,s_n,\theta_i^n,\frac{\theta_{i+1}^n-\theta_{i-1}^n}{2h}),$$

where $r=\frac{ak}{h^2}$. Consequently,for $n=0,1,\cdots,N$ and $i=1,2,\cdots,M-1$, we have:

$$\theta_i^{n+1}=r(\theta_{i-1}^n+\theta_{i+1}^n)+(1-bk-2r)\theta_i^n+kf(\tau_i,s_n,\theta_i^n,\frac{\theta_{i+1}^n-\theta_{i-1}^n}{2h}). \tag{23}$$

One can observe that this approach is explicit, and hence we do not need to solve the gained nonlinear algebraic equations. So, letting $i=1$ makes θ_0^n needed to be known. Actually, θ_0^n can be determined by using the given left BC (3). That is, $\theta_0^n=0$ for $n=0,1,\cdots,N$. However, by using the proper BC (4), the first derivative can be approximated using the center-second-order finite difference approximation, and then the trapezoidal rule of integration to approximate the integral in question. Since this numerical rule has the same second-order precision in the space, we have actually picked it. It also refers to the strategies employed to address the internal aspect of the problem at hand. In brief, we can obtain the following equalities:

$$\frac{\theta_{M+1}^n-\theta_{M-1}^n}{2h}=\int_0^1 k(\tau,s^n)\,\theta(\tau,s^n)\,d\tau=\frac{h}{2}(k_0^n\theta_0^n+2\sum_{i=1}^{M-1}k_i^n\theta_i^n+k_M^n\theta_M^n).$$

This, immediately, implies:

$$\theta_{M+1}^n=h^2(2\sum_{i=1}^{M-1}k_i^n\theta_i^n+k_M^n\theta_M^n)+\theta_{M-1}^n. \tag{24}$$

Eliminating of the fictitious value θ_{M+1}^n from the above equation gives the following numerical formula:

$$\theta_M^{n+1}=r(2\theta_{M-1}^n+h^2(2\sum_{i=1}^{M-1}k_i^n\theta_i^n)+(1-bk-2r+rh^2k_M^n)\theta_M^n+kf(\tau_i,s_n,\theta_i^n,\frac{h}{2}(2\sum_{i=1}^{M-1}k_i^n\theta_i^n+k_M^n\theta_M^n)). \tag{25}$$

6 Numerical experiments

To evaluate the algorithm outlined in the previous section, we will illustrate two numerical examples that deal with problem (P_1) with its known analytical solution.

*Example 1.*Consider problem (P_1) with $a=1$ and $b=2$. In order for the function $\theta(\tau,s)=xe^{x+t}$ to be the precise problem's solution of (P_1), the functions f and k are selected. In summary, the following problem will be examined:

$$\frac{\partial\theta}{\partial s}-\frac{\partial^2\theta}{\partial\tau^2}+2\theta=f(\tau,s,\theta,\theta_\tau), \tag{26}$$

Subject to the IC:

$$\theta(\tau,0)=\tau e^{\tau},\qquad 0<\tau<1, \tag{27}$$

the BC:

$$\theta(0,s)=0,\qquad 0<s\le T, \tag{28}$$

and the NBC:

$$\frac{\partial\theta}{\partial\tau}(1,s)=\int_0^1 2e^1\theta(\tau,s)\,d\tau, 0<s\le T, \tag{29}$$

where $f(\tau,s,\theta,\theta_\tau)=-3e^{\tau+s}+\theta+\theta_\tau$. We may see the obtained numerical findings in Table 1 by taking $h=1/80$, $r=0.4$, $\tau=0.25;0.5;0.75$ and $s=1$ when using the finite difference formula. In the same regard, Table 2 displays the numerical solutions' maximum errors according to the experimental order of convergence. The following defines the maximum errors in this instance:

$$Er = \|\theta - \theta_{hk}\|_\infty = ma\tau_{0\le k\le N}\{ma\tau_{0\le i\le M}|\theta(\tau_i, s_k) - \theta_i^k|\},$$

and the following formula is used to determine the experiment order of convergence for the suggested scheme:

$$order = \frac{ln(Er(h_{i-1})/Er(h_i))}{ln(h_{i-1}/h_i)}.$$

Table 1: Comparison between the FTCS and exact solutions of Example 1 wit $t = 1$, $h = \frac{1}{80}$ and $r = 0.4$.

τ_i	Exact solution	FTCS's solution
0.25	0.872753454838	0.872585739365
0.5	2.241218325540	2.240844535169
0.75	4.316629062927	4.315952007004

Table 2: The experiment sequence of convergence and maximum errors for Example 1.

h	Maximum errors	Convergence order
h=1/20	2.20×10^{-2}	
h=1/40	$5.48\cdot10^{-3}$	2.004
h=1/80	$1.37\cdot10^{-3}$	2.001

Based on Table 1 and Table 2, it is evident that the approximate numerical results from Example 1 accord well with the absolute ones. Furthermore, the suggested plan has second-order accuracy in the space. However, a graphical comparison of the precise and FTCS solutions for Example 1 is shown in Figure 1.

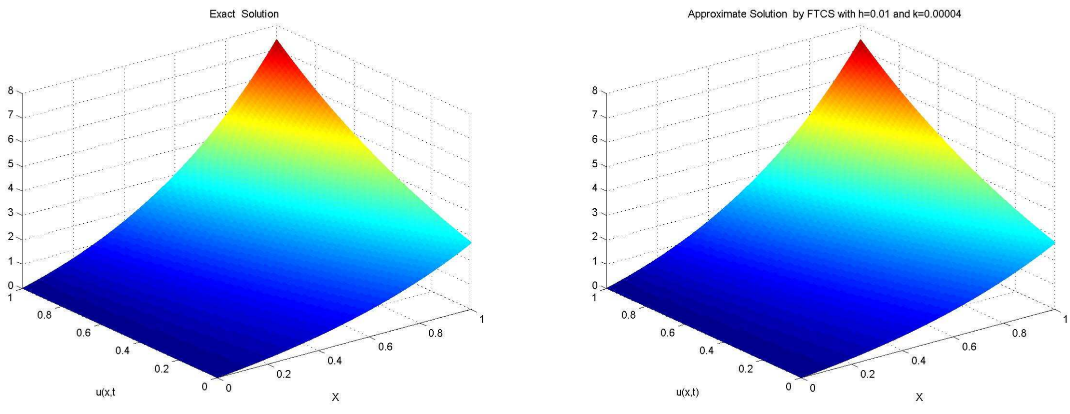

Fig. 1: Exact and approximate solution with $h = 0.01$ and $k = 0.00004$ for Example 1.

*Example 2.*Herein, we assume $a = 1$ and $b = 3$ in problem (P_1). At the same time, the functions f and k are selected so that the function $\theta(\tau,s) = \tau^2 e^{-s}$ is the exact solution of the same problem. That is, the following problem will be considered:

$$\frac{\partial\theta}{\partial s} - \frac{\partial^2\theta}{\partial\tau^2} + 3\theta = f(\tau, s, \theta, \theta_\tau), \tag{30}$$

with IC:

$$\theta(\tau,0) = \tau^2, \qquad 0 < \tau < 1, \tag{31}$$

the BC:

$$\theta(0,s) = 0, \qquad 0 < s \leq T, \tag{32}$$

and the NBC:

$$\frac{\partial\theta}{\partial\tau}(1,s) = \int_0^1 8x\theta(\tau,s)\,d\tau, \tag{33}$$

where $f(\tau,s,\theta,\theta_\tau) = 2e^{-s} + 2\theta$. However, Table 3 presents the numerical results obtained by taking $h = 1/80$, $r = 0.4$, $\tau = 0.25; 0.5; 0.75$ and $t = 1$ when using the finite difference formula. In the same regard, Table 4 displays the numerical solutions' maximum errors according to the experimental order of convergence. Besides, Figure 2 illustrates another graphical comparison between the exact and FTCS's solutions of Example 2.

Table 3: Comparison between the FTCS and exact solutions of Example 2 wit $t = 1$, $h = \frac{1}{80}$ and $r = 0.4$.

τ_i	Exact solution	FTCS's solution
0.25	0.023176788679	0.022992465073
0.5	0.092392377880	0.091969860293
0.75	0.207715796042	0.206932185659

Table 4: The experiment sequence of convergence and maximum errors for Example 2.

h	Maximum errors	Convergence order
h=1/20	1.85×10^{-2}	
h=1/40	$4.65 \cdot 10^{-3}$	1.993
h=1/80	$1.16 \cdot 10^{-3}$	1.998

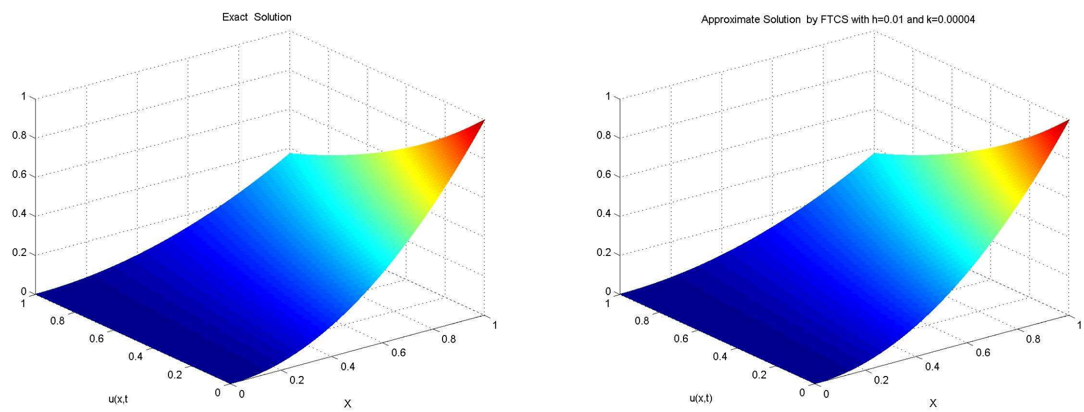

Fig. 2: Exact and approximate solution with $h = 0.01$ and $k = 0.00004$ for Example 2.

7 Conclusion

A semilinear nonlocal problem of a category of nonlinear equations of parabolic type with a 2^{nd}-type integral condition has been analyzed and studied in this study. The existence of the weak solution to the linear problem and its uniqueness have been investigated using the Fadeo-Galarkin approach. Moreover, an iterative procedure has been used to further investigate the existence of the weak solution and and its uniqueness for the semilinear problem. Following a discussion of the finite-time blow-up solution for a specific example of this kind of semilinear problem, the semilinear problem has been solved using the forward time-centered space technique.

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