Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/110102

On the Λ-Fractional Rayleigh-Ritz Approximation Method

Konstantinos A. Lazopoulos[1,*], *Athanassios Pirentis*[1], *and Anastassios K. Lazopoulos*[2]

[1] Independent Researcher, Athens, Greece
[2] Mathematical Sciences Department, Hellenic Army Academy, Vari, Greece

Received: 2 Dec. 2023, Revised: 18 Feb. 2024, Accepted: 13 Mar. 2024
Published online: 1 Jan. 2025

Abstract: Approximation methods are quite important in finding solutions of various differential equations. Adapting these methods to the context of Λ-fractional calculus requires specific adjustments, which are discussed in the present study. In particular, this study introduces relevant modifications to the Rayleigh-Rich approximation method. The implementation is performed on the bending deformation of an elastic beam, elucidating the necessary modifications made to the Rayleigh-Ritz method in order to accomodate it within the framework of the Λ-fractional analysis. These modifications, along with the Λ-fractional Rayleigh-Ritz method itself, are instrumental for establishing the Λ-fractional Finite Element method.

Keywords: Λ-fractional derivative, Λ-fractional space, initial space, Λ-fractional Differential equation, Λ-fractional total energy function, Λ-fractional approximate solution.

1 Introduction

Since fractional integrals and derivatives are considered advanced mathematical tools [1,2,3,4,5,6], they have found application in the formulation of realistic responses of various phenomena in Physics and Engineering [7,8,9]. In mechanics, for example, where major factors determining material deformation include microcracks, voids and material phases, researchers in disordered (non-homogeneous) materials with microstructure, have used fractional analysis to provide elaborate descriptions of the mechanics of porous materials, colloidal aggregates, ceramics, etc. [10,11]. Additionally, viscoelasticity problems have recently also been formulated by employing fractional analysis.

In general, addressing such problems requires non-local theories. To meet this requirement, strain gradient theories were developed, in the context of which, intrinsic material lengths were introduced alongside higher order derivatives of the strain [12,13,14,15]. Many phenomena have been successfully described employing these theories, including problems concerning size effects and the lifting of various singularities, while these theories have also been used in the study of porous materials [14,16,17,18]. Another non-local approach was advanced by Kunin [19,20].

Lazopoulos introduced fractional derivatives of the strain into the strain energy density function in an effort to incorporate non-locality in the elastic response of materials [21]. Fractional calculus has been used by many researchers, not only in mechanics but also in Physics, particularly in quantum mechanics, in order to explore the concept of non-locality. In fact, the history of fractional calculus dates back to the 17th century, and some of the fields where fractional calculus has been applied include particle physics, electromagnetics, mechanics of materials, hydrodynamics, fluid flow, rheology, viscoelasticity, optics, electrochemistry and corrosion, and chemical physics.

Nevertheless, the mathematical formulation of various physical problems within the context of fractional analysis follows a procedure that may raise questions. Specifically, although the various laws of Physics have been derived by means of differentials, this is not the case for the well-known fractional derivatives, which are not directly related to differentials. Simple substitution of conventional differentials by fractional ones is not capable of expressing realistically the behaviour of various physical systems. Recently, Lazopoulos proposed the Λ-fractional derivative, along with the conjugate fractional Λ-space, wherein the Λ-fractional derivative follows conventional derivative rules [22]. Subsequently, Lazopoulos also presented the theory of Λ-fractional elastic solid mechanics [23].

The present study focuses on the formulation of the Λ-fractional Rayleigh-Ritz method, a mathematical approximation technique for solving various differential equations. The method has been used to develop solutions for both static and

* Corresponding author e-mail: kolazop@mail.ntua.gr

dynamic problems. In fact, applications of this approximation method in vibration analysis can be found in Meirovich [24], and in solutions of differential equations in Norrie and Devries [25], within the framework of Finite Element Analysis.

This work establishes the Λ-fractional Rayleigh-Ritz method, essential for the solution of static and dynamic problems within the context of Λ-fractional analysis. This work may also be considered as a step towards establishing the Λ-fractional Finite Element Analysis.

2 The Λ-Fractional Derivative

A brief outline of Λ fractional calculus is presented in the current section. For more information, the interested reader is referred to the literature [6,7,8,9,10].

The left and right fractional integrals are defined by

$$ {}_aI_x^\gamma f(x) = \frac{1}{\Gamma(\gamma)}\int_a^x \frac{f(s)}{(x-s)^{1-\gamma}}ds, \tag{1}$$

$$ {}_xI_b^\gamma f(x) = \frac{1}{\Gamma(\gamma)}\int_x^b \frac{f(s)}{(s-x)^{1-\gamma}}ds, \tag{2}$$

respectively, where $0<\gamma\leq 1$ is the fractional order, and $\Gamma(\gamma)$ is Euler's Gamma function. Further, the left and right Riemann-Liouville (RL) fractional derivatives are defined by

$$ {}_a^{RL}D_x^\gamma f(x) = \left(\frac{d}{dx}\right)^{m+1}\left({}_aI_x^{m-\gamma}f(x)\right), \tag{3}$$

$$ {}_x^{RL}D_b^\gamma f(x) = \left(-\frac{d}{dx}\right)^{m+1}\left({}_xI_b^{m-\gamma}f(x)\right). \tag{4}$$

It is pointed out that for the left fractional integrals and derivatives, the following relation holds:

$$ {}_a^{RL}D_x^\gamma\left({}_aI_x^\gamma f(x)\right) = f(x). \tag{5}$$

A similar relation is valid for the right RL-fractional integral and derivative.

Considering only the left space, the Λ-fractional derivative (Λ-FD) is defined as

$$ {}_a^{\Lambda}D_x^\gamma f(x) = \frac{{}_a^{RL}D_x^\gamma f(x)}{{}_a^{RL}D_x^\gamma x^\gamma}. \tag{6}$$

Recalling the definition of the RL fractional derivative, Eq. (3), Λ-FD is expressed as

$$ {}_a^{\Lambda}D_x^\gamma f(x) = \frac{\frac{d}{dx}\left({}_aI_x^{1-\gamma}f(x)\right)}{\frac{d}{dx}\left({}_aI_x^{1-\gamma}x\right)} = \frac{d\,{}_aI_x^{1-\gamma}f(x)}{d\,{}_aI_x^{1-\gamma}x}. \tag{7}$$

Further, considering

$$ \begin{aligned} X &= {}_aI_x^{1-\gamma}x, \\ F(X) &= {}_aI_x^{1-\gamma}f(x), \end{aligned} \tag{8}$$

Λ-FD appears to behave as a conventional derivative with local properties in the $(X, F(X))$ Λ-fractional space (hereafter, termed simply Λ-space for brevity). In fact, a fractional differential geometry can be developed as a conventional differential geometry in Λ-space. Indeed, Eq. (8a) yields

$$ X = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)}, \tag{9}$$

In addition, Eqs. (8b), (9) suggest that

$$ F(X) = {}_aI_x{}^{1-\gamma} = \frac{1}{\Gamma(1-\gamma)}\int_a^x \frac{f(s)}{(x-s)^\gamma}ds. \tag{10}$$

Solving Eq. (9) for x yields

$$x = (\Gamma(3-\gamma)X)^{\frac{1}{2-\gamma}} = x(X). \tag{11}$$

Proceeding further to the definition of Λ-space, substituting $x(X)$ in Eq. (10), results in the function $F(x)$ being expressed as a function of X.

$$F(X) = F\left(x(X)\right). \tag{12}$$

In the above presentation, only left fractional integrals and left RL fractional derivatives were employed. In similar fashion, involving the right fractional integrals and right RL derivatives, the right Λ-FD is defined by

$$\begin{aligned} {}^{\Lambda}_{x}D^{\gamma}_{b}f(x) &= \frac{d_x I_b^{1-\gamma} f(x)}{d_x I_b^{1-\gamma} x} = \frac{dF(X)}{dX}, \\ F(x) = \frac{1}{2}\left({}_aI_x^{1-\gamma}f(x) + {}_xI_b^{1-\gamma}f(x)\right) &= \frac{1}{\Gamma(1-\gamma)}\left(\int_a^x \frac{f(s)}{(x-s)^{\gamma}}ds + \int_b^x \frac{f(s)}{(s-x)^{\gamma}}ds\right). \end{aligned} \tag{13}$$

In the following, it will be demonstrated how Λ-space $(X, F(X))$ is defined from the initial space $(x, f(x))$. Furthermore, the pull back of the results in the initial space will also be established.

3 The Λ-Fractional Rayleigh-Rich Approximation Method

As mentioned previously, derivatives in Λ-space behave like conventional ones, satisfying the prerequisites of the Differential Topology and corresponding to differentials creating a Differential Geometry. Therefore, the Rayleigh-Ritz method is applied in Λ-space exactly as in the conventional case. Subsequently, the results are transferred into the initial space.

The potential energy in Λ-space is defined by

$$\Pi(X) = \int_{\Omega} U d\mathbf{X} - \int_{\Omega} \mathbf{Q} d\mathbf{X}. \tag{14}$$

Let $Y(X)$ be the one-dimensional approximate displacement in Λ-space, which can be expanded according to

$$Y(X) = \sum_{i=0}^{k} A_i X^i, \tag{15}$$

resulting in the potential energy being expressed as

$$\Pi(X) = \sum_{i=0}^{k} \mu_i A_i + \sum_{i=0, j=0}^{i=k, j=k} \mu_i A_i A_j. \tag{16}$$

Then, the minimization of Π is defined through the $(k+1)\times(k+1)$ system of equations:

$$\frac{\partial \Pi}{\partial A_i} = 0, \quad i = 0, \ldots, k. \tag{17}$$

The solution of the equilibrium system, defined by Eq. (17) above, is valid only in Λ-space. The final solution is obtained by transferring the Λ-space solution into the initial space, by means of Eq. (13b).

The presented Λ-fractional Rayleigh-Ritz method is implemented to the bending deformation of an elastic beam under constant, continuous loading. In the case of the cantilever beam of Fig. 1, the following parameters were considered: beam length of $l = 2m$, continuous loading $q = -125\,kp/m$, and stiffness $EI = 60,000\,kp\cdot m^2$. Furthermore, it was assumed that $\gamma = 0.5$. In Λ-space, the bending deformation of the beam has both left and right contributions. In the following, we elaborate these contributions individually, and subsequently determine the total deformation of the elastic beam.

3.1 The left Λ-fractional beam bending

In the case of the left Λ-fractional beam bending, the pole of fractional integration was at $x = 0$, and the beam's edges were located at $x = 0.25\,m$ and $x = 2.25\,m$, respectively.

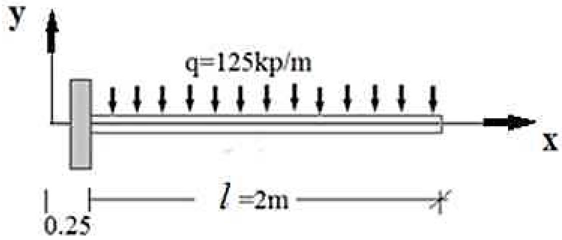

Fig. 1: The elastic cantilever beam in the (x,y) initial space.

The left Λ-space is defined with (X,Y) coordinates given by

$$X = \frac{1}{\Gamma(1-0.5)} \int_{0.25}^{x} \frac{s}{(x-s)^{0.5}} ds = 0.75\sqrt{x-0.25}(x+0.125), \tag{18}$$

$$Y(X) = \frac{1}{\Gamma(1-0.5)} \int_{0.25}^{x} \frac{y(s)}{(x-s)^{0.5}} ds. \tag{19}$$

Transferring the problem from the initial space (x,y) to Λ-space (X,Y), the cantilever beam is defined in $0 < X < 2.53$. Therefore, the length of the beam in Λ-space is equal to $L = 2.53\,m$ (Fig. 2).

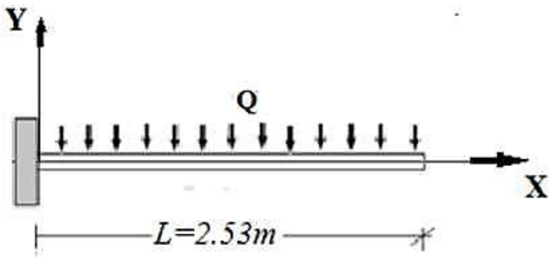

Fig. 2: The elastic cantilever beam in the left Λ-space.

The constant load q in the initial space corresponds to a variable, continuous load $Q(X)$ in the left Λ-space, shown in Fig. 3. The functional expression of Q with respect to the initial space variable x is given by

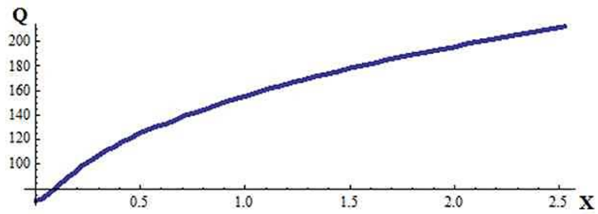

Fig. 3: The variable continuous load Q versus the axial coordinate X in Λ-space.

$$Q(x) = \frac{1}{\Gamma(1-0.5)} \int_{0.25}^{x} \frac{q}{(x-s)^{0.5}} ds = \frac{0.56419(62.5-250x)}{(x-0.25)^{0.5}}. \tag{20}$$

Following the same procedure, the constant stiffness EI of the beam in the initial space becomes the variable stiffness $EI^{\Lambda}(x)$ in Λ-space, expressed as

$$EI^{\Lambda}(x) = \frac{1}{\Gamma(1-0.5)} \int_{0.25}^{x} \frac{EI}{(x-s)^{0.5}} ds = \frac{16925.7(4x-1)}{(x-0.25)^{0.5}}. \tag{21}$$

Combining $EI^{\Lambda}(x)$ with Λ-space variable X, Eq. (18), results in the fractional stiffness being expressed with respect to X, that is, $EI^{\Lambda}(X)$ (Fig. 4).

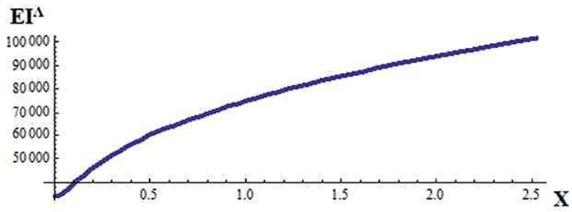

Fig. 4: The variable stiffness EI^{Λ} versus the axial coordinate X in Λ-space.

The total energy of the beam is defined by

$$\Omega = \int_{0.25}^{2.53} \left(\frac{EI^{\Lambda}}{2} \left(\frac{d^2Y(X)}{dX^2} \right)^2 - Q(X)Y(X) \right) dX. \tag{22}$$

The approximate solution is expressed as

$$Y_{app}(X) = a_0 + a_1 X + a_2 X^2 + a_3 X^3 + o(X^3). \tag{23}$$

The boundary conditions at $X = 0$ are:

$$X = \frac{dY(X)}{dX} = 0 \quad \text{with} \quad a_0 = a_1 = 0. \tag{24}$$

Further, performing the calculus in Eq. (22) results in

$$\Omega = 951a_2 + 342\,629 a_2^2 + 1\,856 a_3 + 3.07 \cdot 10^6 a_2 a_3 + 8.22 \cdot 10^6 a_3^2. \tag{25}$$

The minimization of the total energy Ω is defined by the conditions:

$$\frac{\partial \Omega}{\partial a_2} = \frac{\partial \Omega}{\partial a_3} = 0. \tag{26}$$

Solving Eqs. (26) yields

$$a_2 = -0.00541, \quad a_3 = 0.00090. \tag{27}$$

Hence, Eq. (23) can be expressed in the form:

$$Y_{app}(X) = -0.00541 \left(0.75\sqrt{x-0.25}(x+0.125)\right)^2 + 0.00090X^3. \tag{28}$$

The deflection $Y(X)$ of the bending beam in Λ-space is shown in Fig. 5. In turn, using Eq. (18), the deflection of the beam in Λ-space with respect to the initial space variable x is expressed as

$$Y_{app}(x) = -0.00541 \left(0.75\sqrt{x-0.25}(x+0.125)\right)^2 + 0.00090 \left(0.75\sqrt{x-0.25}(x+0.125)\right)^3. \tag{29}$$

The true deflection of the beam in the initial space is subsequently defined by

$$y_{app}^{l}(x) = \frac{1}{\Gamma(0.5)} \frac{d}{dx} \int_{0.25}^{x} \frac{Y_{app}(s)}{(x-s)^{0.5}} ds. \tag{30}$$

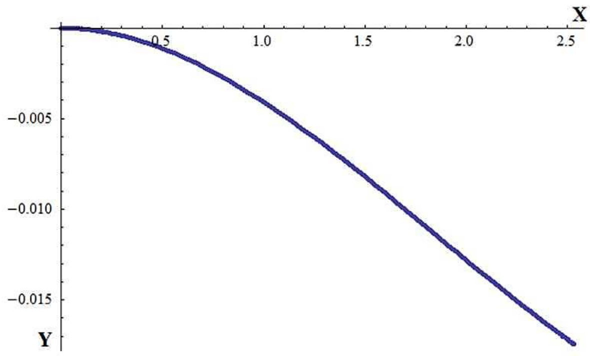

Fig. 5: The deflection of the beam in Λ-space.

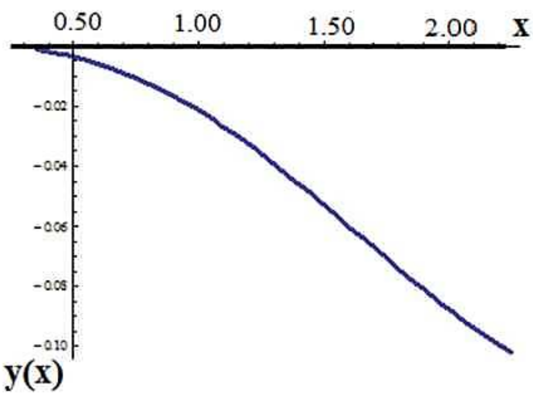

Fig. 6: The left Λ-fractional deflection in the initial space.

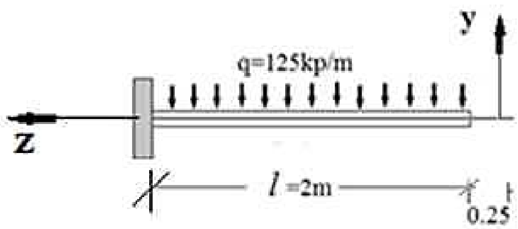

Fig. 7: The elastic cantilever beam in the (z,y) initial space.

The true deflection in the initial space is shown in Fig. 6.

3.2 The right Λ-fractional beam bending

In the case of the right Λ-fractional beam bending, the initial space is considered in the (z,y) coordinate system, as illustrated in Fig. 7. Now, the right fractional pole is located at $z=0$, and the beam's edges at $z=0.25\,m$ and $z=2.25\,m$, respectively. In addition, at the clamped end of the beam the following boundary conditions are satisfied:

$$y(2.25)=\left.\frac{dy(z)}{dz}\right|_{z=2.25}=0. \tag{31}$$

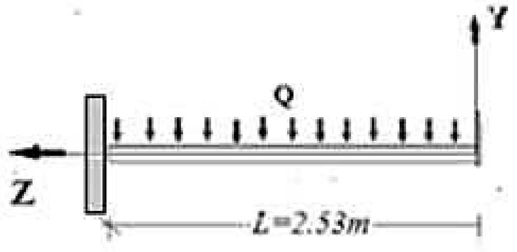

Fig. 8: The elastic cantilever beam in the right Λ-space.

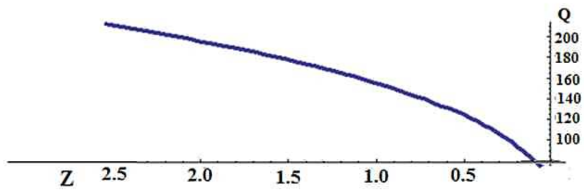

Fig. 9: The variable continuous load Q versus the axial coordinate Z in Λ-space.

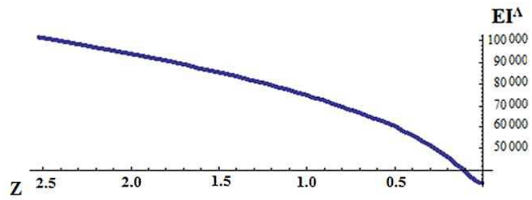

Fig. 10: The variable stiffness EI^{Λ} versus the axial coordinate Z in Λ-space.

The left Λ-space is defined by the (Z,Y) coordinates according to:

$$Z = \frac{1}{\Gamma(1-0.5)} \int_{0.25}^{z} \frac{s}{(z-s)^{0.5}} ds = 0.75\sqrt{z-0.25}(z+0.125), \tag{32}$$

$$Y(Z) = \frac{1}{\Gamma(1-0.5)} \int_{0.25}^{z} \frac{y(s)}{(z-s)^{0.5}} ds. \tag{33}$$

Transferring the problem from the initial space to Λ-space (Z,Y), the cantilever beam is defined in $0 < Z < 2.53$, that is, the length of the beam in Λ-space is equal to $L = 2.53\, m$ (Fig. 8).

The constant load q in the initial space corresponds to the $Q(Z)$ variable continuous load in the right Λ-space, shown in Fig. 9. The constant stiffness EI of the beam in the initial space becomes the variable stiffness $EI^{\Lambda}(Z)$, shown in Fig. 10.

The total energy of the beam is now defined by

$$\Omega = \int_{0.25}^{2.53} \left(\frac{EI^{\Lambda}}{2} \left(\frac{d^2Y(Z)}{dZ^2} \right)^2 - Q(Z)Y(Z) \right) dZ. \tag{34}$$

The approximate solution is expressed as

$$Y_{app}(Z) = b_0 + b_1(Z-2.53) + b_2(Z-2.53)^2 + b_3(Z-2.53)^3 + o(Z^3). \tag{35}$$

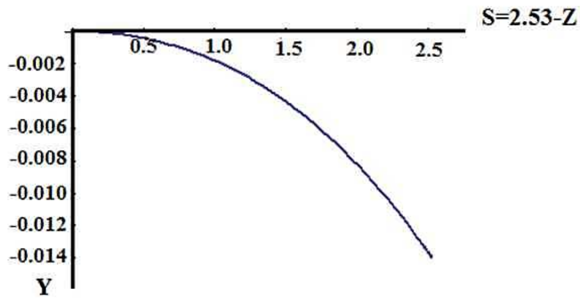

Fig. 11: The right Λ-fractional elastic curve in Λ-space.

The boundary conditions at $Z = L = 2.53$ are

$$b_0 = b_1 = 0. \tag{36}$$

Further, performing the calculus in Eq. (34) results in

$$\Omega = 515.125\, b_2 + 342\,629\, b_2^2 - 856.613\, b_3 - 2.064 \cdot 10^6\, b_2 b_3 + 4.45 \cdot 10^6\, b_3^2. \tag{37}$$

The minimization of the total energy Ω is defined by the conditions

$$\frac{\partial \Omega}{\partial b_2} = \frac{\partial \Omega}{\partial b_3} = 0. \tag{38}$$

The solution to Eqs. (38) yields

$$b_2 = -0.00153, \quad b_3 = -0.00025. \tag{39}$$

Hence, Eq. (35) yields the approximate solution

$$Y_{app}(Z) = -0.00153\,(Z - 2.53)^2 - 0.00025\,(Z - 2.53)^3. \tag{40}$$

The deflection $Y(Z)$ of the bending beam in Λ-space is shown in Fig. 11.

Combining Eqs. (32) and (40), we obtain the deflection $Y(z)$ of the beam in Λ-space with respect to the initial space variable z:

$$Y_{app}(z) = -0.00153\left(0.75\sqrt{z - 0.25}(z + 0.125) - 2.53\right)^2 + 0.00090\left(0.75\sqrt{z - 0.25}(z + 0.125) - 2.53\right)^3. \tag{41}$$

The true deflection of the beam in the initial space is then defined by

$$y^r_{app}(z) = \frac{1}{\Gamma(0.5)} \frac{d}{dz} \int_{0.25}^{z} \frac{Y_{app}(s)}{(z - s)^{0.5}} ds. \tag{42}$$

The true deflection in the initial space is shown in Fig. 12.

3.3 The total Λ-fractional beam bending

The total bending deflection of the beam is the average of the left and right Λ-fractional beam deflections, that is, the average of the quantities defined in Eqs. (30) and (42). Hence,

$$y_{app}(x) = \frac{1}{2}\left(y^l_{app}(x) + y^r_{app}(2.25 - x)\right). \tag{43}$$

The total deflection of the beam is shown in Fig. 13, whereas the conventional elastic curve for the considered elastic beam is presented in Fig. 14. Comparing the Λ-fractional and conventional elastic curves, it is evident that the former is significantly larger than latter.

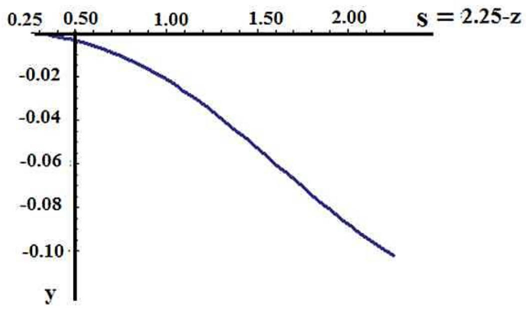

Fig. 12: The right Λ-fractional deflection in the initial space.

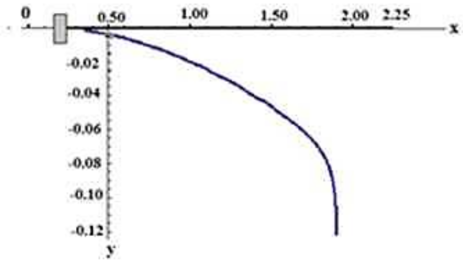

Fig. 13: The Λ-fractional elastic curve of the beam.

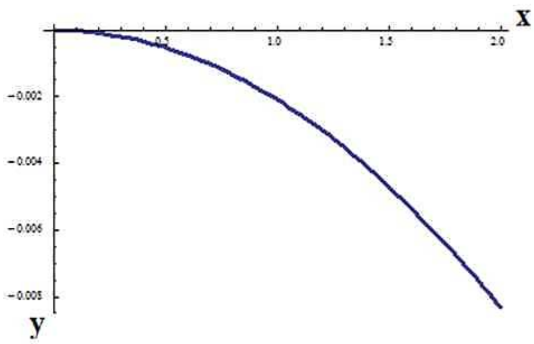

Fig. 14: The elastic curve for the conventional beam.

4 Conclusion

This study presented a method for establishing approximation methods for solving various differential equations. Specifically, the Rayleigh-Ritz approximation method, applied to a conservative system, was presented for solving Λ-fractional differential equations. The presented procedure can be employed to provide approximate solutions in Λ-space in the context of other approximation methods as well, like the virtual displacement method, the collocation method, and the weighted residuals or Galerkin method. The final goal is the establishment of the Λ-fractional finite element analysis for the solution of Λ-fractional differential equations.

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