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Quasi-normal developable surfaces and their singularities in Euclidean 3-Space

Maryam T. Aldossary [1,*] *and Rashad A. Abdel-Baky* [2]

[1]Department of Mathematics, College of Science, Imam Abdulrahman bin Faisal University, P.O. Box-1982 Damam, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, University of Assiut, 71516 Assiut, Egypt

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Abstract: The developable surface (*DS*) is a curved surface that can be spread out on a plane without stretching or tearing, which is widely operated in much fields of engineering and industrialization. This research displays a new approach of producing developable surfaces in E^3(Euclidean 3-space). At first, we start a modified frame over a curve, named as the quasi-frame. We then initiate an exemplification of a *DS* and call it a quasi-normal *DS*. At the essence of this work, we examine the existence and uniqueness of such *DS*, then consider its categorizations via singularity theory and unfolding theory (UT). Finally, two paradigms related to our approach are presented for the purpose of clarity.

Keywords: Singularities and curvature lines, developable surface.

1 Introduction

A *DS* is a regular surface with disappearing Gaussian curvature. That is, it is a surface that can be squeezed onto a plane without deformation (i.e. it can be bending without stretching or pressure). The significances of the *DS* is lying in the fact that it is utilizing in numerous subjects of engineering and industrialization, inclusive of design of clothing, automobile ingredients, and boat hulls (see e.g. [1–6]) By the singularity theory the *DS* can be created the locomotion of Serret-Frenet frame (SFF) of a space curve [7, 8].

In [9], Izumiya et al. extracted the rectifying normal *DS* of space curve, where they showed that a regular curve is a geodesic of its rectifying *DS* and adjusted the confirmation through singularity of the rectifying *DS* and geometric invariants. Ishikawa explored the interrelation among the singularity of tangent *DS* and some of space curves. He also gave a categorization of tangent *DS* by implementation the local topology ownerships [10]. Qiming et al. researched the geometric assets of family of 1-parameter $\mathscr{D}$ surfaces related with space curves. Furthermore, they displayed that the generic singularities of this family are cuspidal edge (CE) and swallowtail (SW) [11]. There are a number of works on singularity theory of *DS* by the SFF of space curves, for model [12–14].

However, the SFF is not described for all points of any curve. A modern frame is requisite for the sort of mathematical test that is mostly done via computer graphics. Therefore, the hypothesis of rotation minimizing frame (RMF) which is further acceptable for implementations was initiated via Bishop in [15, 16]. But, it is familiar that RMF computations are not a simple task, see [17, 18]. For this reason, Coquillart [19], and Mustafa et al. [20] addressed a quasi-frame of a space curve.

In this paper, we initiate our research on the curve which including singular points in E^3. As is known that there is a great diversity among the *DS* produced by curve and the curve containing singular points. So, we provide the quasi-frame on a unit-speed curve (USC) and commence a quasi-normal *DS*. Via the UT, we address the generic assets, and define two invariants connected with the singularity of this surface. It is confirmed that the generic singularities are CE and SW, and the classes of these singularities can be defined by these invariants, respectively. Finally, models are decorated to demonstrate the implementations of the theoretical upshots.

* Corresponding author e-mail: maaldosary@iau.edu.sa

2 Basic concepts

Let $\gamma(\mathfrak{u})$ be a unit speed curve (USC) in E^3; by $\varkappa(\mathfrak{u})$ and $\tau(\mathfrak{u})$ we mean the curvature and torsion of $\gamma(\mathfrak{u})$, respectively. Let $\{\{T\}(\mathfrak{u}), N(\mathfrak{u}), B(\mathfrak{u})\}$ be the SFF on the curve $\gamma(\mathfrak{u})$, then

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \varkappa & 0 \\ -\varkappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}; \ (\frac{d}{du} ='). \tag{1}$$

Although the SFF can facilely be calculated, its alternation around the tangent of a general space curve frequently leads to unwanted twist in locomotion layout or swept surface designing. Likewise, the SFF is not constantly located for a C^1-regular space curve, and unto for a C^2-regular space curve the SFF falls unspecified at an inflection point ($\varkappa = 0$), thus lead to inadmissible discontinuity whenever utilized for surface designing [15, 16]. For that reason, Coquillart [19], and Mustafa et al. [20] addressed a quasi-frame (q-frame) of a space curve as:

$$\mathfrak{e}_1(\mathfrak{u}){=}T, \ \mathfrak{e}_2(\mathfrak{u}) = \frac{T \times \zeta}{\|T \times \zeta\|}, \ \mathfrak{e}_3(\mathfrak{u}) = \mathfrak{e}_1 \times \mathfrak{e}_2, \tag{2}$$

where ζ is the projection vector. The correlation over SFF and q-frame is:

$$\begin{bmatrix} \mathfrak{e}_1 \\ \mathfrak{e}_2 \\ \mathfrak{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \ \varphi(\mathfrak{u}) \geq 0. \tag{3}$$

Therefore,

$$\begin{bmatrix} \mathfrak{e}_1' \\ \mathfrak{e}_2' \\ \mathfrak{e}_3' \end{bmatrix} = \begin{bmatrix} 0 & \varkappa_1 & \varkappa_2 \\ -\varkappa_1 & 0 & \varkappa_3 \\ -\varkappa_2 & -\varkappa_3 & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{e}_1 \\ \mathfrak{e}_2 \\ \mathfrak{e}_3 \end{bmatrix}, \tag{4}$$

where

$$\left.\begin{aligned} \varkappa_1(\mathfrak{u}) &= \varkappa(\mathfrak{u})\cos\varphi = < \mathfrak{e}_1', \mathfrak{e}_2 >, \\ \varkappa_2(\mathfrak{u}) &= -\varkappa(\mathfrak{u})\sin\varphi = < \mathfrak{e}_1', \mathfrak{e}_3 >, \\ \varkappa_3(\mathfrak{u}) &= \tau(\mathfrak{u}) + \varphi'(\mathfrak{u}) = < \mathfrak{e}_2', \mathfrak{e}_3 > . \end{aligned}\right\} \tag{5}$$

The q-frame possess numerous features contrast to the other frames (SFF, RMF). For occasion, the q-frame can be specified even over a line ($\varkappa = 0$). Nevertheless, the q-frame is singular in all instances where T and ζ are parallel. Thus, in these cases, where T and ζ are parallel, the projection vector ζ can be pick as $\zeta{=}(0,1,0)$ or $\zeta{=}(0,0,1)$ [19, 20]. Further, we shall not write $\mathfrak{u}$ candor in our study.

A RS in E^3 is a family with one-parameter of lines L. Such a surface has a exemplification of the shape

$$\mathfrak{M} : \mathfrak{P}(\mathfrak{u}, \mathfrak{v}) = \gamma(\mathfrak{u}) + \mathfrak{v}\mathfrak{e}(\mathfrak{u}), \ \mathfrak{v} \in \mathbb{R}, \tag{6}$$

where $\gamma(\mathfrak{u})$ is its directrix curve and $\mathfrak{e}$ is the unit vector over L of $\mathfrak{M}$. The ruling of $\mathfrak{M}$ is asymptotic line. If the tangent plane of the $\mathfrak{M}$ is fixed along a fixed ruling, then $\mathfrak{M}$ is named the DS [1-3]. Tangent planes of such surfaces has only one parameter. All other ruled surfaces are coined the skew surfaces. The curve $\gamma(\mathfrak{u})$ is not unique, since each curve of the compose

$$\mathfrak{z}(\mathfrak{u}) = \gamma(\mathfrak{u}) - \sigma(\mathfrak{u})\mathfrak{e}(\mathfrak{u}), \tag{7}$$

can be taken as its directrix curve, $\sigma(\mathfrak{u})$ is a regular function. If there exists a joint orthogonal to two adjoining rulings on $\mathfrak{M}$, then the end of the joint orthogonal on the principal ruling is a central point. The trajectory of the central points is the striction curve (SC). In Eq. (7) if

$$\sigma(\mathfrak{u}) = \frac{< \gamma'(\mathfrak{u}), \mathfrak{e}'(\mathfrak{u}) >}{\left\|\mathfrak{e}'(\mathfrak{u})\right\|^2}, \tag{8}$$

then $\mathfrak{z}(\mathfrak{u})$ is the SC on $\mathfrak{M}$ and it is unique. If $\sigma = 0$ the directrix curve is the SC. The distribution parameter of $\mathfrak{M}$ is

$$\mu(\mathfrak{u}) = \frac{\det(\gamma'(\mathfrak{u}), \mathfrak{e}(\mathfrak{u}), \mathfrak{e}'(\mathfrak{u}))}{\left\|\mathfrak{e}'(\mathfrak{u})\right\|^2}. \tag{9}$$

Thus $\mathfrak{M}$ is a DS iff $\mu(\mathfrak{u}) = 0$, that is,

$$\mu = 0 \Leftrightarrow \det(\gamma', \mathfrak{e}, \mathfrak{e}') = 0. \tag{10}$$

3 q-normal developable surface

In this section, we propose a new configuration of a DS, and name it a q-normal DS: Via the presumption $(\varkappa_2(\mathfrak{u}), \varkappa_3(\mathfrak{u})) \neq (0,0)$, one register that

$$\mathfrak{M} : \mathfrak{P}(\mathfrak{u}, \mathfrak{v}) = \gamma(\mathfrak{u}) + \mathfrak{v}\mathfrak{e}(\mathfrak{u}), \ \mathfrak{v} \in \mathbb{R}, \tag{11}$$

where

$$\mathfrak{e}(\mathfrak{u}) = \frac{\varkappa_3\mathfrak{e}_1 - \varkappa_2\mathfrak{e}_2}{\sqrt{\varkappa_3^2 + \varkappa_2^2}}.$$

For $\mathfrak{M}$ we possess

$$\mathfrak{e}'(\mathfrak{u}) = \varkappa_1 + (\frac{\varkappa_2\varkappa_3' - \varkappa_3\varkappa_2'}{\varkappa_3^2 + \varkappa_2^2}) \frac{\varkappa_2\mathfrak{e}_1 + \varkappa_3\mathfrak{e}_2}{\sqrt{\varkappa_3^2 + \varkappa_2^2}}, \tag{12}$$

and so $\mu(\mathfrak{u}) = 0$. This shows that $\mathfrak{M}$ is a DRS. Further, we locate two invariants $\delta(\mathfrak{u})$, and $\sigma(\mathfrak{u})$ of $\mathfrak{M}$ as:

$$\left.\begin{aligned} \delta(\mathfrak{u}) &= \varkappa_1 + \frac{\varkappa_2\varkappa_3' - \varkappa_3\varkappa_2'}{\varkappa_3^2 + \varkappa_2^2}, \\ \sigma(\mathfrak{u}) &= \frac{\varkappa_3}{\sqrt{\varkappa_3^2 + \varkappa_2^2}} - (\frac{\varkappa_2}{\delta(\mathfrak{u})\sqrt{\varkappa_3^2 + \varkappa_2^2}})' \end{aligned}\right\} \tag{13}$$

where $\delta(\mathfrak{u}) \neq 0$. We can as well display that

$$\mathfrak{P}_\mathfrak{u} \times \mathfrak{P}_\mathfrak{v} = -(\frac{\varkappa_2}{\sqrt{\varkappa_3^2 + \varkappa_2^2}} + \mathfrak{v}\delta)\mathfrak{e}_3. \tag{14}$$

Thus the surface normal lies in the normal plane of $\gamma(\mathfrak{u})$. This is the cause why we say $\mathfrak{M}$ the q-normal *DS* along $\gamma(\mathfrak{u})$.

Theorem 3.1 (Existence and uniqueness). Via the last registrations there exists a unique q-normal *DS* located by Eq. (11).

Proof. For the existence, we have the q-normal *DS* along $\gamma(\mathfrak{u})$ explained by Eq. (11). Since $\mathfrak{M}$ is a *RS*, we may set that

$$\left.\begin{array}{l}\mathfrak{M}:\mathfrak{P}(\mathfrak{u},\mathfrak{v})=\gamma(\mathfrak{u})+\mathfrak{v}\zeta(\mathfrak{u}),\ \mathfrak{v}\in\mathbb{R},\\ \text{with }(\varkappa_3,\varkappa_2)\neq(0,0),\\ \zeta(\mathfrak{u})=\zeta_1(\mathfrak{u})\mathfrak{e}_1+\zeta_2(\mathfrak{u})\mathfrak{e}_2+\zeta_3(\mathfrak{u})\mathfrak{e}_3,\\ \|\zeta(\mathfrak{u})\|^2=\zeta_1^2+\zeta_2^2+\zeta_3^2=1,\ \zeta'(\mathfrak{u})\neq\mathbf{0}.\end{array}\right\} \quad (15)$$

Hence, $\mathfrak{M}$ is a *DRS* iff

$$\det(\gamma',\zeta,\zeta')=0\Leftrightarrow-\zeta_3\zeta_2'+\zeta_2\zeta_3'-\zeta_1(\zeta_3\varkappa_1-\zeta_2\varkappa_2)+\varkappa_3(\zeta_2^2+\zeta_3^2)=0. \quad (16)$$

Since $\mathfrak{M}$ is a *DRS* along $\gamma=\gamma(\mathfrak{u})$, we possess

$$(\mathfrak{P}_\mathfrak{u}\times\mathfrak{P}_\mathfrak{v})(\mathfrak{u},\mathfrak{v})=\psi(\mathfrak{u},\mathfrak{v})\mathfrak{e}_3, \quad (17)$$

where $\psi=\psi(\mathfrak{u},\mathfrak{v})$ is a differentiable function. The normal vector $(\mathfrak{P}_\mathfrak{u}\times\mathfrak{P}_\mathfrak{v})$ at $(\mathfrak{u},0)$ is

$$(\mathfrak{P}_\mathfrak{u}\times\mathfrak{P}_\mathfrak{v})(\mathfrak{u},0)=-\zeta_3\mathfrak{e}_2+\zeta_2\mathfrak{e}_3. \quad (18)$$

Thus, from Eqs. (17), and (18), one finds that $\zeta_3=0$, and $\zeta_2=\psi(\mathfrak{u},0)$, which displays from Eq. (16) that $\zeta_2(\zeta_1\varkappa_2+\zeta_2\varkappa_3)=0$. If $(\mathfrak{u},0)$ is a regular point $(\psi(\mathfrak{u},0)\neq0)$, then $\zeta_2(\mathfrak{u})\neq0$. Thus, we find $\zeta_1=-\frac{\varkappa_3}{\varkappa_2}\zeta_2$, with $\varkappa_2\neq0$. Then,

$$\zeta(\mathfrak{u})=-\frac{\varkappa_3}{\varkappa_2}\zeta_2\mathfrak{e}_1+\zeta_2\mathfrak{e}_2=-\left(\frac{\zeta_2}{\varkappa_2}\sqrt{\varkappa_3^2+\varkappa_2^2}\right)\mathfrak{e}(\mathfrak{u}),\varkappa_2\neq0. \quad (19)$$

This exhibitions that $\zeta(\mathfrak{u})$ has the same orientation of $\mathfrak{e}(\mathfrak{u})$. If $\varkappa_3\neq0$, we acquire the aforementioned ■.

Furthermore, we set the next results for $\delta(\mathfrak{u})$, and $\sigma(\mathfrak{u})$:

Theorem 3.2. Let $\mathfrak{M}$ be the q-normal *DS* stated by Eq. (11). Then:

(A) The ensuing are synonymous:

(1) $\mathfrak{M}$ is a cylinder,

(2) $\delta(\mathfrak{u})=0$ for all $\mathfrak{u}\in\mathfrak{I}$,

(3) $\gamma=\gamma(\mathfrak{u})$ is a contour generator due to an orthogonal projection.

(B) If $\delta(\mathfrak{u})\neq0$ for all $\mathfrak{u}\in\mathfrak{I}$, then the ensuing statement are synonymous:

(1) $\mathfrak{M}$ is a conical surface,

(2) $\sigma(\mathfrak{u})=0$ for all $\mathfrak{u}\in\mathfrak{I}$,

(3) $\gamma=\gamma(\mathfrak{u})$ is a contour generator with respect to a central projection.

Proof (A): From Eq. (12), it is evident that $\mathfrak{M}$ is a cylinder iff $\mathfrak{e}(\mathfrak{u})$ is fixed, i.e. $\delta(\mathfrak{u})=0$. Accordingly, the case (1) is comparable to the case (2). Assume that the case (3) holds. Then there exists a fixed unit vector $\mathfrak{x}\in E^3$ such that $<\mathfrak{e}_3,\mathfrak{x}>=0$. So there exist $a,b\in\mathbb{R}$ such that $\mathfrak{x}=a\mathfrak{e}_1+b\mathfrak{e}_2$. Since $<\mathfrak{e}_3',\mathfrak{x}>=0$, we possess $a\varkappa_2+b\varkappa_3=0$, so that we find $\mathfrak{x}=\pm\mathfrak{e}(\mathfrak{u})$. Specifically, the case (1) holds. Assume that $\mathfrak{e}(\mathfrak{u})$ is fixed. Then we take $\mathfrak{x}=\mathfrak{e}(\mathfrak{u})$.

(B) The case (1) shows that the set of singular value of $\mathfrak{M}$ is a fixed vector. Thus, in view of Eqs. (7), (8), and Eq. (11), We can locate that

$$\begin{aligned}\mathfrak{z}'(\mathfrak{u})&=\left[\frac{\varkappa_3}{\sqrt{\varkappa_3^2+\varkappa_2^2}}-\left(\frac{\varkappa_2}{\delta(\mathfrak{u})\sqrt{\varkappa_3^2+\varkappa_2^2}}\right)'\right]\mathfrak{e}(\mathfrak{u})\\&=\sigma(\mathfrak{u})\mathfrak{e}(\mathfrak{u}).\end{aligned}$$

Then $\mathfrak{M}$ is a cone iff $\sigma(\mathfrak{u})=0$. It follows that (1) and (2) are analogous. By the establishment of the central projection shows that there exists a fixed point $\mathfrak{c}\in\mathbb{R}^3$ such that $<\mathfrak{e}_3,\gamma-\mathfrak{c}>=0$. If (1) holds, then $\mathfrak{c}(\mathfrak{u})$ is fixed. For the fixed point $\mathfrak{c}$, we find

$$\begin{aligned}\langle\mathfrak{e}_3,\gamma-\mathfrak{c}\rangle&=\langle\mathfrak{e}_3,\gamma-\mathfrak{c}\rangle\\&=\langle\mathfrak{e}_3,\frac{\langle\gamma',\mathfrak{e}'\rangle}{\|\mathfrak{e}'\|^2}\mathfrak{e}\rangle\\&=\frac{\langle\gamma',\mathfrak{e}'\rangle}{\|\mathfrak{e}'\|^2}\langle\mathfrak{e}_3,\mathfrak{e}\rangle=0.\end{aligned}$$

This shows that (3) holds. For the contrary, by (3), there exists a fixed point $\mathfrak{c}\in\mathbb{R}^3$ such that $<\mathfrak{e}_3,\gamma-\mathfrak{c}>=0$. Making the derivation of the both sides, we see

$$0=<\mathfrak{e}_3,\gamma-\mathfrak{c}>'=<\varkappa_2\mathfrak{e}_1+\varkappa_3\mathfrak{e}_2,\gamma-\mathfrak{c}>,$$

so we may write $\gamma-\mathfrak{c}=f(\mathfrak{u})\mathfrak{e}(\mathfrak{u})$, where $f(\mathfrak{u})$ is a differentiable function. Making the derivations more, we find:

$$\begin{aligned}0=<\mathfrak{e}_3,\gamma-\mathfrak{c}>''&=<\varkappa_2\mathfrak{e}_1+\varkappa_3\mathfrak{e}_2,\mathfrak{e}_1>\\&+<(\varkappa_2\mathfrak{e}_1+\varkappa_3\mathfrak{e}_2)',\gamma-\mathbf{c}>,\end{aligned}$$

or likewise,

$$0=<\mathfrak{e}_3,\gamma-\mathfrak{c}>''=\varkappa_2-f\delta\sqrt{\varkappa_3^2+\varkappa_2^2}.$$

It locates that

$$\mathfrak{c}=\gamma(\mathfrak{u})-\frac{\varkappa_2}{\delta\sqrt{\varkappa_3^2+\varkappa_2^2}}\mathfrak{e}(\mathfrak{u})=\gamma-\frac{<\gamma',\mathfrak{e}'>}{\|\mathfrak{e}'\|^2}\mathfrak{e}(\mathfrak{u})=\mathfrak{z}(\mathfrak{u}).$$

Therefore, $\mathfrak{c}(\mathfrak{u})$ is fixed, so that (1) holds ■

As a outcome the following corollaries can be addressed.

Corollary 3.1. The q-normal *DS* $\mathfrak{M}$ is a non-cylindrical iff $\delta(\mathfrak{u}) \neq 0$.

Corollary 3.2. The q-normal *DS* $\mathfrak{M}$ is a tangential developable iff $\delta(\mathfrak{u}) \neq 0$, and $\sigma(\mathfrak{u}) \neq 0$.

Proof. Via the proof of Theorem 3.1, when $\delta(\mathfrak{u}) \neq 0$, and $\sigma(\mathfrak{u}) \neq 0$, we possess $\mathfrak{e}' \neq \mathbf{0}$, and $\mathfrak{c}' \neq \mathbf{0}$. Via $\det(\gamma', \mathfrak{e}, \mathfrak{e}') = 0$, $<\mathfrak{c}', \mathfrak{e}'>=0$ and $<\mathfrak{e}, \mathfrak{e}'>=0$, we can get $\mathfrak{c}' \| \mathfrak{e}$. It shows that $\mathfrak{M}$ is a tangent surface $\blacksquare$.

We here specify connections through the singularity of $\mathfrak{M}$ and the two invariants $\delta(\mathfrak{u})$, and $\sigma(\mathfrak{u})$, as follows:

Theorem 3.3. Let $\gamma : I \subseteq \mathbb{R} \to E^3$ be a *USC* with $\varkappa_2^2 + \varkappa_3^2 \neq 0$. Then:
(1) $(\mathfrak{u}_0, \mathfrak{v}_0)$ is a regular point of $\mathfrak{M}$ iff

$$\frac{\varkappa_2(\mathfrak{u}_0)}{\sqrt{\varkappa_3^2(\mathfrak{u}_0) + \varkappa_2^2(\mathfrak{u}_0)}} + \mathfrak{v}_0 \delta(\mathfrak{u}_0) \neq 0$$

(2) Let $(\mathfrak{u}_0, \mathfrak{v}_0)$ be a singular point of $\mathfrak{M}$, then $\mathfrak{M}$ is locally diffeomorphic (*LD*) to *CE* at $(\mathfrak{u}_0, u_0)$ if

(*i*) $\delta(\mathfrak{u}_0) \neq 0$, $\sigma(\mathfrak{u}_0) \neq 0$, and

$$\mathfrak{v}_0 = -\frac{\varkappa_2(\mathfrak{u}_0)}{\delta(\mathfrak{u}_0)\sqrt{\varkappa_3^2(\mathfrak{u}_0) + \varkappa_2^2(\mathfrak{u}_0)}},$$

or

(*ii*) $\delta(\mathfrak{u}_0) = \varkappa_2(\mathfrak{u}_0) = 0$, $\delta'(\mathfrak{u}_0) \neq 0$, and

$$\mathfrak{v}_0 \neq -\frac{\varkappa_2(\mathfrak{u}_0)}{\delta(\mathfrak{u}_0)\sqrt{\varkappa_3^2(\mathfrak{u}_0) + \varkappa_2^2(\mathfrak{u}_0)}},$$

or

(*iii*) $\delta(\mathfrak{u}_0) = \varkappa_2(\mathfrak{u}_0) = 0$, $\varkappa_2'(\mathfrak{u}_0) \neq 0$. Clearly, if $\delta'(\mathfrak{u}_0) \neq 0$ then

$$2\varkappa_1(\mathfrak{u}_0)\varkappa_3'(\mathfrak{u}_0) + \varkappa_1'(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0) - \varkappa_2''(\mathfrak{u}_0) \neq 0.$$

(3) Let $(\mathfrak{u}_0, \mathfrak{v}_0)$ be a singular point of $\mathfrak{M}$, then $\mathfrak{M}$ is *LD* to *SW* at $(\mathfrak{u}_0, \mathfrak{v}_0)$ if $\delta(\mathfrak{u}_0) \neq 0$, $\sigma(\mathfrak{u}_0) = 0$, $\sigma'(\mathfrak{u}_0) \neq 0$, and

$$\mathfrak{v}_0 = -\frac{\varkappa_2(\mathfrak{u}_0)}{\delta(\mathfrak{u}_0)\sqrt{\varkappa_3^2(\mathfrak{u}_0) + \varkappa_2^2(\mathfrak{u}_0)}}$$

Here,

$$CE = \{(p_{1,2}, p_3) \mid p_1 = u, p_2 = v^2, p_3 = v^3\},$$
$$SW = \{(p_1, p_2, p_3) \mid p_1 = u,$$
$$p_2 = 3\mathrm{v}^2 + u\mathrm{v}^2,\ p_3 = 4\mathrm{v}^3 + 2u\mathrm{v}\}$$

The graphs of the *CE*, and *SW* are plotted in Figure 1.

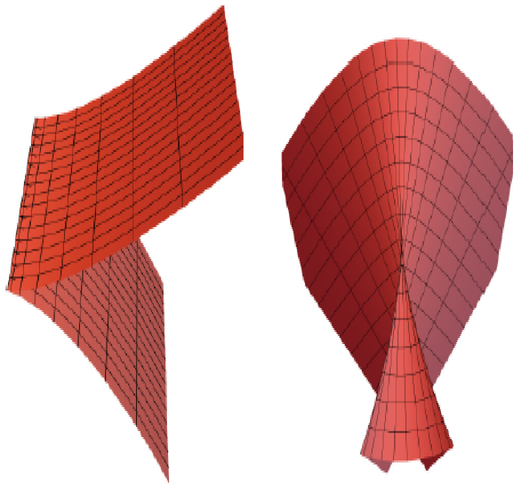

Fig. 1: Figure 1. $\mathscr{CE}$ (left) and $\mathscr{SW}$ (right).

3.1 Support functions

For a *USC* γ: $I \to E^3$, we set $\omega : I \times E^3 \to \mathbb{R}$, by $\omega(\mathfrak{u}, \mathfrak{x}) =< \mathfrak{e}_3(\mathfrak{u}), \mathfrak{x} - \gamma(\mathfrak{u}) >$. We say it support function on $\gamma(\mathfrak{u})$ with respect to $\mathfrak{e}_3$. We put $\omega_{\mathfrak{x}_0}(\mathfrak{u}) = \omega(\mathfrak{u}, \mathfrak{x}_0)$ for any fixed $\mathfrak{x}_0 \in \mathbb{R}^3$.

Proposition 3.1. Let γ: $I \to E^3$ be a *USC* with $\varkappa_2^2 + \varkappa_3^2 \neq 0$, and $\omega_{\mathfrak{x}_0}(\mathfrak{u}) =< \mathfrak{e}_3(\mathfrak{u}), \mathfrak{x} - \gamma(\mathfrak{u}) >$. Then:

(1)- $\omega_{\mathfrak{x}_0}(\mathfrak{u}) = 0$ iff there exists u, $\mathfrak{v} \in \mathbb{R}$, such that $\mathfrak{x}_0 - \gamma(\mathfrak{u}_0) = u\mathfrak{e}_1(\mathfrak{u}_0) + \mathfrak{v}\mathfrak{e}_2(\mathfrak{u}_0)$.

(2)- $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0) = \omega_{x_0}'(\mathfrak{u}_0) = 0$ iff there exists $\mathfrak{v} \in \mathbb{R}$ such that

$$\mathfrak{x}_0 - \gamma(\mathfrak{u}_0) = \mathfrak{v}\left(\frac{\varkappa_3\mathfrak{e}_1 - \varkappa_2\mathfrak{e}_2}{\sqrt{\varkappa_3^2 + \varkappa_2^2}}\right)(\mathfrak{u}_0).$$

(A). Suppose that $\delta(\mathfrak{u}_0) \neq 0$. Then:

(1)- $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0) = \omega_{x_0}'(\mathfrak{u}_0) = \omega_{\mathfrak{x}_0}''(\mathfrak{u}_0) = 0$ iff

$$\mathfrak{x}_0 - \gamma(\mathfrak{u}_0) = -\frac{\varkappa_2}{\delta\sqrt{\varkappa_3^2 + \varkappa_2^2}} \frac{\varkappa_3\mathfrak{e}_1 - \varkappa_2\mathfrak{e}_2}{\sqrt{\varkappa_3^2 + \varkappa_2^2}}(\mathfrak{u}_0). \quad (1)$$

(2)-$\omega_{\mathfrak{x}_0}(\mathfrak{u}_0) = \omega_{x_0}'(\mathfrak{u}_0) = \omega_{\mathfrak{x}_0}''(\mathfrak{u}_0) = \omega_{\mathfrak{x}_0}^{(3)}(\mathfrak{u}_0) = 0$ iff $\sigma(\mathfrak{u}_0) = 0$, and (1).

(3)- $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0) = \omega_{x_0}'(\mathfrak{u}_0) = \omega_{\mathfrak{x}_0}''(\mathfrak{u}_0) = \omega_{\mathfrak{x}_0}^{(3)}(\mathfrak{u}_0) = \omega_{\mathfrak{x}}^{(4)}(\mathfrak{u}_0) = 0$ iff $\sigma(\mathfrak{u}) = \sigma'(\mathfrak{u}) = 0$, and (1).

(B). Suppose that $\delta(\mathfrak{u}_0) = 0$. Then we specify that:

(1)- $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0) = \omega_{x_0}'(\mathfrak{u}_0) = \omega_{\mathfrak{x}_0}''(\mathfrak{u}_0) = 0$ iff $\varkappa_2(\mathfrak{u}_0) = 0$, the is, $\varkappa_2(\mathfrak{u}_0) = 0$, $\varkappa_2'(\mathfrak{u}_0) - \varkappa_1(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0) = 0$, and there exists $\mathfrak{v} \in \mathbb{R}$ such that $\mathfrak{x}_0 - \gamma(\mathfrak{u}_0) = \mathfrak{v}\mathfrak{e}_1(\mathfrak{u}_0)$.

(2)- $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0) = \omega_{x_0}'(\mathfrak{u}_0) = \omega_{\mathfrak{x}_0}''(\mathfrak{u}_0) = \omega_{\mathfrak{x}_0}^{(3)}(\mathfrak{u}_0) = 0$ iff one of the following situations holds (a)- $\delta'(\mathfrak{u}) \neq 0$,

$\varkappa_2(\mathfrak{u}_0)$, that is, $\varkappa_2(\mathfrak{u}_0)=0$, $\varkappa_2^{'}(\mathfrak{u}_0)=\varkappa_1(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)=0$,

$$2\varkappa_1(\mathfrak{u}_0)\varkappa_3^{'}(\mathfrak{u}_0)+\varkappa_1^{'}(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)-\varkappa_2^{''}(\mathfrak{u}_0)\neq 0$$

and

$$\mathfrak{x}_0-\gamma(\mathfrak{u}_0=$$

$$-\frac{\varkappa_2(\mathfrak{u}_0)}{2\varkappa_1(\mathfrak{u}_0)\varkappa_3^{'}(\mathfrak{u}_0)+\varkappa_1^{'}(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)-\varkappa_2^{''}(\mathfrak{u}_0)}\mathfrak{e}_1(\mathfrak{u}_0).$$

(b) $\delta^{'}(\mathfrak{u})=0$, $\varkappa_2(\mathfrak{u}_0)=\varkappa_2^{'}(\mathfrak{u}_0)$, that is,

$$\varkappa_1(\mathfrak{u}_0)=\varkappa_2(\mathfrak{u}_0)=0,\ \varkappa_1^{'}(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)-\varkappa_2^{''}(\mathfrak{u}_0)=0,$$

and there exists $u\in\mathbb{R}$ such that $\mathfrak{x}_0-\gamma(\mathfrak{u}_0)=u\mathfrak{e}_1(\mathfrak{u}_0)$.
Proof. Since $\omega_{\mathfrak{x}_0}(\mathfrak{u})=<\mathfrak{e}_3(\mathfrak{u}),\mathfrak{x}_0-\gamma(\mathfrak{u})>$, we determine the following:

$$\omega_{\mathfrak{x}_0}=\begin{cases}\text{(i) }\omega_{\mathfrak{x}_0}=\langle\mathfrak{e}_3,\mathfrak{x}_0-\gamma\rangle,\\ \text{(ii) }\omega_{\mathfrak{x}_0}'=\langle-\varkappa_2\mathfrak{e}_1-\varkappa_3\mathfrak{e}_2,\mathfrak{x}_0-\gamma\rangle,\\ \text{(iii) }\omega_{\mathfrak{x}_0}''=\varkappa_2+\langle(-\varkappa_2'+\varkappa_1\varkappa_3)\mathfrak{e}_1-(\varkappa_3'+\varkappa_1\varkappa_2)\mathfrak{e}_2\\ \quad-(\varkappa_3^2+\varkappa_2^2)\mathfrak{e}_3,\mathfrak{x}_0-\gamma\rangle,\\ \text{(iv) }\omega_{\mathfrak{x}_0}^{(3)}=2\varkappa_2'-\varkappa_1\varkappa_3+\langle(\varkappa_2(\varkappa_1^2+\varkappa_2^2+\varkappa_3^2)\\ \quad+\varkappa_1'\varkappa_3+2\varkappa_1\varkappa_3'-\varkappa_2'')\mathfrak{e}_1\\ \quad+(\varkappa_3(\varkappa_1^2+\varkappa_2^2+\varkappa_3^2)-\varkappa_1'\varkappa_2-2\varkappa_1\varkappa_2'-\varkappa_3'')\mathfrak{e}_2\\ \quad-3(\varkappa_2\varkappa_2'+\varkappa_3\varkappa_3')\mathfrak{e}_3,\mathfrak{x}_0-\gamma\rangle,\\ \text{(v) }\omega_{\mathfrak{x}_0}^{(4)}=3\varkappa_2''-3\varkappa_1\varkappa_3'+\varkappa_2(\varkappa_1^2+\varkappa_2^2+\varkappa_3^2)\\ \quad+\langle\ldots,\mathfrak{x}_0-\gamma\rangle.\end{cases}$$

By definition of support height functions, we include $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0)=0$ iff $\mathfrak{x}_0-\gamma(\mathfrak{u}_0)=\mathfrak{v}\mathfrak{e}_1(\mathfrak{u}_0)+a\mathfrak{e}_2(\mathfrak{u}_0)+b\mathfrak{e}_3(\mathfrak{u}_0)$, and $<\mathfrak{x}_0-\gamma(\mathfrak{u}_0),\mathfrak{e}_3(\mathfrak{u}_0)>=0$. Then, we guaranty $\mathfrak{x}_0-\gamma(\mathfrak{u}_0)=\mathfrak{v}\mathfrak{e}_1(\mathfrak{u}_0)+a\mathfrak{e}_2(\mathfrak{u}_0)$. Therefore, (1) holds. By (ii),$\omega_{\mathfrak{x}_0}(\mathfrak{u}_0)=\omega_{\mathfrak{x}_0}^{'}(\mathfrak{u}_0)=0$ iff $\mathfrak{x}_0-\gamma(\mathfrak{u}_0)=\mathfrak{v}\mathfrak{e}_1(\mathfrak{u}_0)+a\mathfrak{e}_2(\mathfrak{u}_0)$, and $-\mathfrak{v}\varkappa_2(\mathfrak{u}_0)-a\varkappa_3(\mathfrak{u}_0)=0$. If $\varkappa_2(\mathfrak{u}_0)\neq 0$, and $\varkappa_3(\mathfrak{u}_0)\neq 0$, then we include

$$\mathfrak{v}=-a\frac{\varkappa_3(\mathfrak{u}_0)}{\varkappa_2(\mathfrak{u}_0)},\text{ and }a=-\mathfrak{v}\frac{\varkappa_2(\mathfrak{u}_0)}{\varkappa_3(\mathfrak{u}_0)}.$$

Then there exists $c\in\mathbb{R}$ such that

$$\mathfrak{x}_0-\gamma(\mathfrak{u}_0)=c\frac{\varkappa_3\mathfrak{e}_1-\varkappa_2\mathfrak{e}_2}{\sqrt{\varkappa_3^2+\varkappa_2^2}}(\mathfrak{u}_0).$$

Suppose that $\varkappa_2(\mathfrak{u}_0)=0$. Then we locate $\varkappa_3(\mathfrak{u}_0)\neq 0$; so that $\varkappa_3(\mathfrak{u}_0)a=0$. Accordingly, we realize

$$\mathfrak{x}_0-\gamma(\mathfrak{u}_0):=\mathfrak{v}\mathbf{e_1}(\mathfrak{u}_0)=c\frac{\varkappa_3\mathfrak{e}_1-\varkappa_2\mathfrak{e}_2}{\sqrt{\varkappa_3^2+\varkappa_2^2}}(\mathfrak{u}_0).$$

If $\varkappa_3(\mathfrak{u}_0)=0$; then we possess $\mathfrak{x}_0-\gamma(\mathfrak{u}_0)=a\mathbf{e_2}(\mathfrak{u}_0)$. For that reason, (2) holds. By (iii) $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0)=\omega_{\mathfrak{x}_0}^{'}(\mathfrak{u}_0)=\omega_{\mathfrak{x}_0}^{''}(\mathfrak{u}_0)=0$ iff

$$\mathfrak{x}_0-\gamma(\mathfrak{u}_0)=c\frac{\varkappa_3\mathfrak{e}_1-\varkappa_2\mathfrak{e}_2}{\sqrt{\varkappa_3^2+\varkappa_2^2}}(\mathfrak{u}_0),$$

and

$$\varkappa_2(\mathfrak{u}_0)+c\frac{\varkappa_3\left(\varkappa_1\varkappa_3-\varkappa_2^{'}\right)+\varkappa_2\left(\varkappa_1\varkappa_2+\varkappa_3^{'}\right)}{\sqrt{\varkappa_3^2+\varkappa_2^2}}(\mathfrak{u}_0)=0.$$

It exhibitions that

$$\frac{\varkappa_2}{\sqrt{\varkappa_3^2+\varkappa_2^2}}(\mathfrak{u}_0)+c(\varkappa_1+\frac{\varkappa_2\varkappa_3^{'}-\varkappa_3\varkappa_2^{'}}{\sqrt{\varkappa_3^2+\varkappa_2^2}})(\mathfrak{u}_0)=0.$$

Then,

$$\delta(\mathfrak{u}_0)=\varkappa_1(\mathfrak{u}_0)+\frac{\varkappa_2\varkappa_3^{'}-\varkappa_3\varkappa_2^{'}}{\sqrt{\varkappa_3^2+\varkappa_2^2}}(\mathfrak{u}_0),\text{ and }\neq 0$$

$$c=-\frac{\varkappa_2}{\sqrt{\varkappa_3^2+\varkappa_2^2}}(\mathfrak{u}_0)$$

or $\delta(\mathfrak{u}_0)=0$, $\varkappa_2(\mathfrak{u}_0)=0$. This completes the proof of (A), (3) and (B), (1). Suppose that $\delta(\mathfrak{u}_0)\neq 0$. By (iv), $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0)=\omega_{\mathfrak{x}_0}^{'}(\mathfrak{u}_0)$
$=\omega_{\mathfrak{x}_0}^{''}(\mathfrak{u}_0)=\omega_{\mathfrak{x}_0}^{(3)}(\mathfrak{u}_0)=0$ iff

$$\begin{aligned}0=&2\varkappa_2'-\varkappa_1\varkappa_3\\&-\frac{\varkappa_2}{\delta\sqrt{\varkappa_3^2+\varkappa_2^2}}(\varkappa_2(\varkappa_1^2+\varkappa_2^2+\varkappa_3^2)\\&+\varkappa_1'\varkappa_3+2\varkappa_1\varkappa_3'-\varkappa_2'')\\&-\frac{\varkappa_2}{\sqrt{\varkappa_3^2+\varkappa_2^2}}(\varkappa_2(\varkappa_1^2+\varkappa_2^2+\varkappa_3^2)\\&-\varkappa_1'\varkappa_2-2\varkappa_1\varkappa_2'-\varkappa_3'')\end{aligned}$$

at $\mathfrak{u}=\mathfrak{u}_0$. It offer that

$$\begin{aligned}&2\varkappa_2'(\mathfrak{u}_0)-\varkappa_1(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)\\&-\frac{\varkappa_2}{\delta}\left(\varkappa_1'+\frac{2\varkappa_1(\varkappa_2'\varkappa_2+\varkappa_3'\varkappa_3)}{\varkappa_3^2+\varkappa_2^2}\right)\\&-\frac{\varkappa_2}{\delta}\left(\frac{\varkappa_3''\varkappa_2-\varkappa_2''\varkappa_3}{\varkappa_3^2+\varkappa_2^2}\right)(\mathfrak{u}_0).\end{aligned}$$

Since

$$\begin{aligned}\delta'=&\varkappa_1'-2\frac{(\varkappa_2'\varkappa_2+\varkappa_3'\varkappa_3)(\varkappa_3'\varkappa_2-\varkappa_2'\varkappa_3)}{\varkappa_3^2+\varkappa_2^2}\\&+\frac{\varkappa_3''\varkappa_2-\varkappa_2''\varkappa_3}{\varkappa_3^2+\varkappa_2^2}.\end{aligned}$$

and

$$2\varkappa_2'(\mathfrak{u}_0)-\varkappa_1(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0) -\varkappa_2(\mathfrak{u}_0)\frac{\delta'(\mathfrak{u}_0)}{\delta(\mathfrak{u}_0)} -2\varkappa_2\frac{\varkappa_2'\varkappa_2+\varkappa_3'\varkappa_3}{\varkappa_3^2+\varkappa_2^2}(\mathfrak{u}_0)=0.$$

Moreover, we stratify the association

$$\left(\frac{\varkappa_2}{\sqrt{\varkappa_3^2+\varkappa_2^2}}\right)'=-\frac{\varkappa_3}{\sqrt{\varkappa_3^2+\varkappa_2^2}}\frac{\varkappa_3'\varkappa_2-\varkappa_2'\varkappa_3}{\varkappa_3^2+\varkappa_2^2} =-\frac{\varkappa_3}{\sqrt{\varkappa_3^2+\varkappa_2^2}}(\delta-\varkappa_1)$$

to the above equations. Then, we conform

$$\delta(\mathfrak{u}_0)\sqrt{\varkappa_3^2(\mathfrak{u}_0)+\varkappa_2^2(\mathfrak{u}_0)}\left(\frac{\varkappa_3}{\sqrt{\varkappa_3^2(\mathfrak{u}_0)+\varkappa_2^2(\mathfrak{u}_0)}}-\frac{\varkappa_2}{\delta\sqrt{\varkappa_3^2(\mathfrak{u}_0)+\varkappa_2^2(\mathfrak{u}_0)}}\right)'(\mathfrak{u}_0) =-\delta(\mathfrak{u}_0)\sigma(\mathfrak{u}_0)\sqrt{\varkappa_3^2(\mathfrak{u}_0)+\varkappa_2^2(\mathfrak{u}_0)} =0.$$

so that $\sigma(\mathfrak{u}_0)$. The converse emphasis also holds. Suppose that $\delta(\mathfrak{u}_0)=0$. Then, by (iv), $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0)=\omega'_{\mathfrak{x}_0}(\mathfrak{u}_0)=\omega''_{\mathfrak{x}_0}(\mathfrak{u}_0)=\overset{(3)}{\omega_{\mathfrak{x}_0}}(\mathfrak{u}_0)=0$ iff $\varkappa_2(\mathfrak{u}_0)=0$, that is, $\varkappa_2(\mathfrak{u}_0)=0$, $\varkappa_2'(\mathfrak{u}_0)-\varkappa_1(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)=0$, there exists $\mathfrak{v}\in\mathbb{R}$ such that $\mathfrak{x}_0-\gamma(\mathfrak{u}_0)=\mathfrak{v}\mathfrak{e}_1(\mathfrak{u}_0)$, and

$$2\varkappa_2'(\mathfrak{u}_0)-\varkappa_1(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0) +\mathfrak{v}\left(2\varkappa_1(\mathfrak{u}_0)\varkappa_3'(\mathfrak{u}_0) +\varkappa_1'(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)-\varkappa_3''(\mathfrak{u}_0)\right)=0.$$

Since $\delta(\mathfrak{u}_0)=0$, and $\varkappa_2(\mathfrak{u}_0)$, we find $\varkappa_1'(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)-\varkappa_2''(\mathfrak{u}_0)=0$, so that

$$\varkappa_2'(\mathfrak{u}_0)+\mathfrak{v}(2\varkappa_1(\mathfrak{u}_0)\varkappa_3'(\mathfrak{u}_0)+\varkappa_1'(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)-\varkappa_3''(\mathfrak{u}_0))=0.$$

It outlines that $2\varkappa_1(\mathfrak{u}_0)\varkappa_3'(\mathfrak{u}_0)+\varkappa_1'(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)-\varkappa_3''(\mathfrak{u}_0)\neq 0$, and

$$\mathfrak{v}=-\frac{\varkappa_2'(\mathfrak{u}_0)}{2\varkappa_1(\mathfrak{u}_0)\varkappa_3'(\mathfrak{u}_0)+\varkappa_1'(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)-\varkappa_3''(\mathfrak{u}_0)}$$

or

$$2\varkappa_1(\mathfrak{u}_0)\varkappa_3'(\mathfrak{u}_0)+\varkappa_1'(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0)-\varkappa_3''(\mathfrak{u}_0)=0,\text{ and } \varkappa_2'(\mathfrak{u}_0)=0.$$

Therefore we dominate (B), (2), (a) or (b). By the equivalent pretexts to the last, we locate (A), (5) ■.

3.2 UT of functions by one-variable

Now, we employ somewhat generic outcomes on the singularities as in [9, 17]. Set F: $(\mathbb{R}\times\mathbb{R}^r,(\mathfrak{u}_0,\mathfrak{x}_0))\to\mathbb{R}$ be a differentiable function, and $f(\mathfrak{u})=F_{x_0}(\mathfrak{u},\mathfrak{x}_0)$. Thus F is coined an r-parameter $\mathscr{U}\mathscr{T}$ of $f(\mathfrak{u})$. We talk that $f(\mathfrak{u})$ has A_k-singularity at $\mathfrak{u}_0$ if $f^{(p)}(\mathfrak{u}_0)=0$ for all $1\leq p\leq k$, and $f^{(k+1)}(\mathfrak{u}_0)\neq 0$. We also say that f has $A_{\geqslant k}$-singulaity $(k\geqslant 1)$ at $\mathfrak{s}_0$. Let the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at $\mathfrak{u}_0$ be $j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(\mathfrak{u},\mathfrak{x}_0)\right)(\mathfrak{u}_0)=\Sigma_{j=0}^{k-1}L_{ji}(\mathfrak{u}-\mathfrak{u}_0)^j$ (without the fixed term); $i=1,...,r$. Then $F(\mathfrak{u})$ is coined a p-versal UT if the $k\times r$ matrix of coefficients (L_{ji}) has rank k $(k\leq r)$. So, we write serious set on the UT relative to the last registrations. We now confirm substantial set on the $\mathscr{U}$ relative to the above registrations. The discriminant set of F is the set

$$\mathfrak{D}_F=\left\{\mathfrak{x}\in\mathbb{R}^r \mid \text{ there exists } \mathfrak{u} \text{ with } F(\mathfrak{u},\mathfrak{x}) =\frac{\partial F}{\partial\mathfrak{u}}(\mathfrak{u},\mathfrak{x})=0 \text{ at } (\mathfrak{u},\mathfrak{x})\right\}. \tag{20}$$

As in [10-13], we set the well-known categorization as follows:

Theorem 3.4. Let F: $(\mathbb{R}\times\mathbb{R}^r,(\mathfrak{u}_0,\mathfrak{x}_0))\to\mathbb{R}$ be an r-parameter UT of $f(\mathfrak{u})$, which has the A_k singularity at $\mathfrak{u}_0$. Suppose that F is a p-versal UT, then: (a) As $k=2$, then $\mathfrak{D}_F$ is LD to $\mathbf{C}\times\mathbb{R}^{r-1}$; (b) As $k=3$, then $\mathfrak{D}_F$ is LD to $SW\times\mathbb{R}^{r-2}$.
For the proof of Theorem 3.3, we use the following

Proposition 3.2. Let γ: $I\to\mathbb{R}^3$ be a USC with $\varkappa_2^2+\varkappa_3^2\neq 0$, and.$\omega_{\mathfrak{x}_0}(\mathfrak{u})=<\mathfrak{e}_3(\mathfrak{u}),\mathfrak{x}-\gamma(\mathfrak{u})>$. If $\omega_{\mathfrak{x}_0}$ has an A_k-singularity $(k=2,\ 3)$ at $\mathfrak{u}_0\in\mathbb{R}$, then ω is a $p-$versal U of $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0)$.
Proof. Let $\mathfrak{x}=(x_1,x_2,x_3)$, $\gamma=(\alpha_1,\alpha_2,\alpha_3)$ and $\mathfrak{e}_3=(l_1,l_2,l_3)$. Then,

$$\omega(\mathfrak{u},\mathfrak{x})=(x_1-\alpha_1(\mathfrak{u}))l_1(\mathfrak{u}) +(x_2-\alpha_2(\mathfrak{u}))l_2(\mathfrak{u})+(x_3-\alpha_3(\mathfrak{u}))l_3(\mathfrak{u}). \tag{21}$$

and

$$\frac{\partial\omega}{\partial x_i}(\mathfrak{u},\mathfrak{x})=l_i(\mathfrak{u}),\ (\text{i=1, 2, 3}).$$

For that reason, the 2-jets of $\frac{\partial\omega}{\partial x_i}$ at $\mathfrak{u}_0$ is:

$$j^2\frac{\partial\omega}{\partial x_0}(\mathfrak{u}_0,\mathfrak{x}_0)=l_i(\mathfrak{u}_0)+l_i'(\mathfrak{u}_0)(\mathfrak{u}-\mathfrak{u}_0)+\frac{1}{2}l_i''(\mathfrak{u}_0)(\mathfrak{u}-\mathfrak{u}_0)^2.$$

We address the matrix

$$\mathfrak{A}=\begin{bmatrix} l_1(\mathfrak{u}_0) & l_2(\mathfrak{u}_0) & l_3(\mathfrak{u}_0)\\ l_1'(\mathfrak{u}_0) & l_2'(\mathfrak{u}_0) & l_3'(\mathfrak{u}_0)\\ l_1''(\mathfrak{u}_0) & l_2''(\mathfrak{u}_0) & l_3''(\mathfrak{u}_0)\end{bmatrix}=\begin{bmatrix}\mathfrak{e}_3(\mathfrak{u}_0)\\ \mathfrak{e}_3'(\mathfrak{u}_0)\\ \mathfrak{e}_3''(\mathfrak{u}_0)\end{bmatrix}. \tag{22}$$

By the formula in Eq. (4), we locate

$$\mathfrak{A}(\mathfrak{u}_0) = \begin{bmatrix} \mathfrak{e}_3 \\ -\varkappa_2\mathfrak{e}_1 - \varkappa_3\mathfrak{e}_2 \\ \left(\varkappa_1\varkappa_3 - \varkappa_2'\right)\mathfrak{e}_1 - \left(\varkappa_1\varkappa_2 + \varkappa_3'\right)\mathfrak{e}_2 + \left(\varkappa_2^2 + \varkappa_3^2\right)\mathfrak{e}_3 \end{bmatrix}. \quad (23)$$

Since the orthonormal frame $\{\mathfrak{e}_1(\mathfrak{u}), \mathfrak{e}_2(\mathfrak{u}), \mathfrak{e}_3(\mathfrak{u})\}$ is a basis of $\mathscr{E}^3$, then the rank of $\mathfrak{A}(\mathfrak{u}_0)$ is egalitarian to the rank of

$$\begin{bmatrix} 0 & 0 & 1 \\ -\varkappa_2(\mathfrak{u}_0) & -\varkappa_3(\mathfrak{u}_0) & 0 \\ \left(\varkappa_1\varkappa_3 - \varkappa_2'\right)(\mathfrak{u}_0) & -\left(\varkappa_1\varkappa_2 + \varkappa_3'\right)(\mathfrak{u}_0) & \left(\varkappa_2^2 + \varkappa_3^2\right)(\mathfrak{u}_0) \end{bmatrix}. \quad (24)$$

This displays rank $\mathfrak{A} = 3$, iff

$$\varkappa_2\left(\varkappa_1\varkappa_2 + \varkappa_3'\right) + \varkappa_3\left(\varkappa_1\varkappa_3 - \varkappa_2'\right) = \varkappa_1\left(\varkappa_2^2 + \varkappa_3^2\right) + \left(\varkappa_2\varkappa_3' - \varkappa_2'\varkappa_3\right) \neq 0.$$

The last situation is identical to $\delta(\mathfrak{u}_0) \neq 0$ Also, the rank of

$$\begin{bmatrix} \mathfrak{e}_3(\mathfrak{u}_0) \\ \mathfrak{e}_3'(\mathfrak{u}_0) \end{bmatrix} = \begin{bmatrix} \mathfrak{e}_3(\mathfrak{u}_0) \\ -\varkappa_2'(\mathfrak{u}_0)\mathfrak{e}_1(\mathfrak{u}_0) - \varkappa_3'(\mathfrak{u}_0)\mathfrak{e}_2(\mathfrak{u}_0) \end{bmatrix}$$

is constantly two. If $\omega_{\mathfrak{x}_0}$ has an A_k-singularity (k = 2, 3) at $\mathfrak{u}_0$, then ω is p-versal $\mathscr{U}\mathscr{T}$ of $\omega_{\mathfrak{x}_0}$. This completes the proof ■

Proof of Theorem 3.3. Via immediate computation, we locate

$$\mathfrak{P}_\mathfrak{u} \times \mathfrak{P}_\mathfrak{v} = -\left(\frac{\varkappa_2}{\sqrt{\varkappa_3^2 + \varkappa_2^2}} + \mathfrak{v}\delta\right)\mathfrak{e}_3.$$

Consequently, $(\mathfrak{u}_0, \mathfrak{v}_0)$ is non-singular point iff $\mathfrak{P}_\mathfrak{u} \times \mathfrak{P}_\mathfrak{v} \neq$ **0**. This situation is corresponding to

$$\frac{\varkappa_2(\mathfrak{u}_0)}{\sqrt{\varkappa_3^2(\mathfrak{u}_0) + \varkappa_2^2(\mathfrak{u}_0)}} + \mathfrak{v}_0\delta(\mathfrak{u}_0) \neq 0.$$

This completes the proof of (1). By Proposition 3.1-(2), $\mathfrak{D}_\omega$ is the image of the q-normal $\mathscr{D}\mathscr{S}$. Suppose $\delta(\mathfrak{u}_0) \neq 0$. By Proposition 3.1-(A)-(1), (2), and (3), $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0)$ has an A_2-type singularity (respectively, an A_3-type singularity) at $\mathfrak{u}=\mathfrak{u}_0$ iff

$$\mathfrak{v}_0 = -\frac{\varkappa_2(\mathfrak{u}_0)}{\delta(\mathfrak{u}_0)\sqrt{\varkappa_3^2(\mathfrak{u}_0) + \varkappa_2^2(\mathfrak{u}_0)}}$$

and $\sigma(\mathfrak{u}_0) \neq 0$ (respectively, $\sigma(\mathfrak{u}_0) = 0$ and $\sigma'(\mathfrak{u}_0) \neq 0$). By Theorem 3.4 and Proposition 3.1, we gain (2)-(i) and (3). Suppose $\delta(\mathfrak{u}_0) = 0$. By Proposition 3.1-(B)-(1) and (2), $\omega_{\mathfrak{x}_0}(\mathfrak{u}_0)$ has an A_2-type singularity iff $\varkappa_2(\mathfrak{u}_0) = 0$, and

$\varkappa_2'(\mathfrak{u}_0) \neq 0$ or
$\varkappa_2'(\mathfrak{u}_0) + \mathfrak{v}_0\left(2\varkappa_1(\mathfrak{u}_0)\varkappa_3'(\mathfrak{u}_0) + \varkappa_1'(\mathfrak{u}_0)\varkappa_3(\mathfrak{u}_0) - \varkappa_3''(\mathfrak{u}_0)\right) \neq 0$

In view of Theorem 3.4 and Proposition 3.2, we find (2)-(iii) ■.

3.3 Examples

In this subsection, we allocate two epitomes.

Example 1. Let $\gamma(\mathfrak{u}) = (\mathfrak{u}, \frac{1}{2}\mathfrak{u}^2, \mathfrak{u}^3)$, and $(0,0,1)$ is the projection vector. Then,

$$\left.\begin{aligned} \mathfrak{e}_1(\mathfrak{u}) &= \tfrac{1}{\sqrt{1+\mathfrak{u}^2+\mathfrak{u}^4}}(1, \mathfrak{u}, \mathfrak{u}^2), \\ \mathfrak{e}_2(\mathfrak{u}) &= \tfrac{1}{\sqrt{1+\mathfrak{u}^2}}(\mathfrak{u}, -1, 0), \\ \mathfrak{e}_2(\mathfrak{u}) &= \tfrac{1}{\sqrt{1+\mathfrak{u}^2}\sqrt{1+\mathfrak{u}^2+\mathfrak{u}^4}}(\mathfrak{u}^2, \mathfrak{u}^3, -1-\mathfrak{u}^2). \end{aligned}\right\}$$

Therefore,

$$\mathfrak{e}_1'(\mathfrak{u}) = \frac{1}{(1+\mathfrak{u}^2+\mathfrak{u}^4)^{\frac{3}{2}}}(-\mathfrak{u}-2\mathfrak{u}^3, 1-\mathfrak{u}^4, 2\mathfrak{u}+\mathfrak{u}^3),$$

$$\mathfrak{e}_2'(\mathfrak{u}) = \frac{1}{(1+\mathfrak{u}^2)^{\frac{3}{2}}}(1, \mathfrak{u}, 0).$$

Thus,

$$\varkappa_1(\mathfrak{u}) = <\mathfrak{e}_1', \mathfrak{e}_2> = -\frac{1}{\sqrt{1+\mathfrak{u}^2}\sqrt{1+\mathfrak{u}^2+\mathfrak{u}^4}},$$

$$\varkappa_2(\mathfrak{u}) = <\mathfrak{e}_1', \mathfrak{e}_3> = -\frac{2\mathfrak{u}+\mathfrak{u}^3}{\sqrt{1+\mathfrak{u}^2}\sqrt{1+\mathfrak{u}^2+\mathfrak{u}^4}},$$

$$\varkappa_3(\mathfrak{u}) = <\mathfrak{e}_2', \mathfrak{e}_3> = -\frac{\mathfrak{u}^2}{(1+\mathfrak{u}^2)\sqrt{1+\mathfrak{u}^2+\mathfrak{u}^4}},$$

and

$$\mathfrak{e}(\mathfrak{u}) = \Big(\frac{3\mathfrak{u}+\mathfrak{u}^3}{\sqrt{4+9\mathfrak{u}^2+6\mathfrak{u}^4+2\mathfrak{u}^6}}, -\frac{2}{\sqrt{4+9\mathfrak{u}^2+6\mathfrak{u}^4+2\mathfrak{u}^6}}, \frac{\mathfrak{u}^3}{\sqrt{4+9\mathfrak{u}^2+6\mathfrak{u}^4+2\mathfrak{u}^6}}\Big).$$

Hence, the q-normal $\mathscr{D}\mathscr{S}$ along $\gamma(\mathfrak{u})$ is

$$\mathfrak{M} : \mathfrak{P}(\mathfrak{u}, \mathfrak{v}) = \begin{bmatrix} \mathfrak{u} + \mathfrak{v}\frac{3\mathfrak{u}+\mathfrak{u}^3}{\sqrt{4+9\mathfrak{u}^2+6\mathfrak{u}^4+2\mathfrak{u}^6}} \\ \frac{1}{2}\mathfrak{u}^2 - \mathfrak{v}\frac{2}{\sqrt{4+9\mathfrak{u}^2+6\mathfrak{u}^4+2\mathfrak{u}^6}} \\ \mathfrak{u}^3 + \mathfrak{v}\frac{\mathfrak{u}^3}{\sqrt{4+9\mathfrak{u}^2+6\mathfrak{u}^4+2\mathfrak{u}^6}} \end{bmatrix}, \ \mathfrak{v} \in \mathbb{R}.$$

The graphs of the curve $\gamma(\mathfrak{u})$ and q-normal *DS* are plotted in Figure 2. **Example 2**. Let $\gamma(\mathfrak{u}) = (\sin\mathfrak{u}, \cos\mathfrak{u}, \mathfrak{u})$, with the projection vector $(0,0,1)$. Then we have:

$$\mathfrak{e}_1(\mathfrak{u}) = \frac{1}{2}\left(\sqrt{2}\cos\mathfrak{u}, -\sqrt{2}\sin\mathfrak{u}, \sqrt{2}\right),$$

$$\mathfrak{e}_2(\mathfrak{u}) = (-\sin\mathfrak{u}, \cos\mathfrak{u}, 0),$$

$$\mathfrak{e}_3(\mathfrak{u}) = \frac{1}{2}\left(\sqrt{2}\cos\mathfrak{u}, -\sqrt{2}\sin\mathfrak{u}, -\sqrt{2}\right).$$

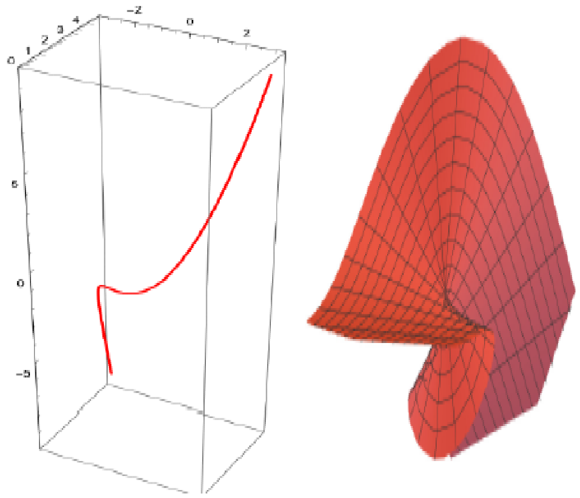

Fig. 2: The curve $\gamma(\mathfrak{u})$ (left) and the q-normal DS (CE right).

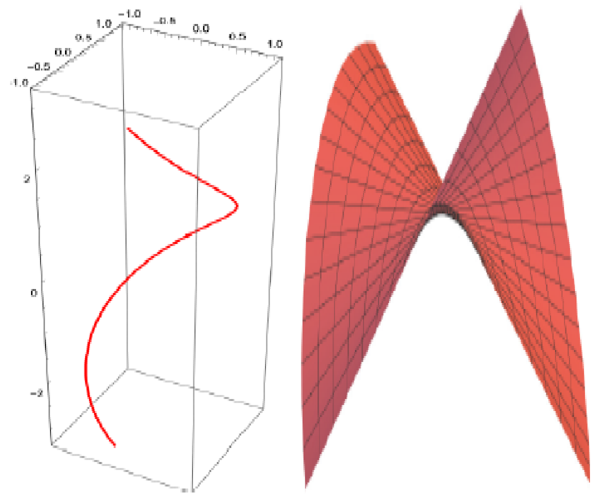

Fig. 3: The curve $\gamma(\mathfrak{u})$ (left) and the Q-normal DS (SW right).

and

$$\left.\begin{array}{l}\mathfrak{e}_1^{'}(\mathfrak{u})=\frac{1}{\sqrt{2}}(-\sin u,-\cos u,0),\\ \mathfrak{e}_2^{'}(\mathfrak{u})=(-\cos u,\sin u,0).\end{array}\right\}$$

Hence, we obtain

$$\varkappa_1(\mathfrak{u})=<\mathfrak{e}_1^{'},\mathfrak{e}_2>=\frac{1}{\sqrt{2}},\ \varkappa_2(\mathfrak{u})=<\mathfrak{e}_1^{'},\mathfrak{e}_3>=0,$$
$$\varkappa_3(\mathfrak{u})=<\mathfrak{e}_2^{'},\mathfrak{e}_3>=-\frac{1}{\sqrt{2}},$$

Therefore, we attain

$$\mathfrak{e}(\mathfrak{u})=(-\cos u,\sin u,1).$$

The q-normal *DS* along $\gamma(\mathfrak{u})$ is

$$\mathfrak{M}:\mathfrak{P}(\mathfrak{u},\mathfrak{v})=(\sin\mathfrak{u}-\mathfrak{v}\cos\mathfrak{u},\cos\mathfrak{u}+v\sin\mathfrak{u},u-\mathfrak{v});\ ,\mathfrak{v}\in\mathbb{R}.$$

The graphs of the curve $\gamma(\mathfrak{u})$ and q-normal *DS* are appeared in Figure 3.

4 Conclusion

In this work, we address a novel form of *DR* surfaces in E^3. We set the q-frame on a unit-speed curve and insert a Quasi-normal *DS*. Expanding the *UT*, we distinguish the public assets, and display novel two invariants connected to the singularities of this surface. It is proved that the public singularities are *CE* and *SW*, and the styles of these singularities can be distinguished by these invariants, respectively. Lastly, two epitomes are explained to clarify the executions of the theoretical outcomes.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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