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Symmetric Conformable Derivative

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Abstract: Many years ago, A. Zygmund extensively studied the symmetric derivative [5, p.1001]. Over the last decade, significant properties of the conformable fractional integral have emerged. This publication bridges these two concepts, presenting results that extend the conformable fractional derivative to the symmetric derivative.

Keywords: symmetric, conformable, derivate, fractional

1 Introduction

Centuries ago, various definitions of the fractional derivative were introduced in mathematical literature by several authors, including Riemann-Liouville and Caputo, as discussed in [9]. More recently, R. Khalil introduce a new definition of the fractional derivative known as the conformable fractional derivative in [10].

In recent years, numerous results from classical calculus have been extended to conformable fractional calculus, with details available in [3], [4], [7], [10], [11], [12], [17], [18], [19], [20] and [22]. During the last century, the symmetric derivative was also extensively studied, with further insights provided in [6], [13], [15]. C. Aull extended results analogous to those in classical analysis to the symmetric derivative, as discussed in [1].

This publication is structured as follows: Section 2 presents the results related to the symmetric derivative, Section 3 introduces the definitions and properties of the conformable fractional derivative, Section 4 delves into the definition and properties of the conformable fractional symmetric derivative, and finally, Section 5, draws conclusions based on the presented results and discusses avenues for future research in this area.

2 Definitions and properties of the symmetric derivative

Definition 1. *Let p be a function on (a,b) and $m \in (a,b)$, [1, p. 708], p is said to have a symmetric derivative at m if $\lim_{h\to 0} \frac{p(m+h)-p(m-h)}{2h}$, exists.*

In this case, this limit is called the symmetric derivative of p at m, denoted as p^s, i.e.,

$$p^s(m) = \lim_{h\to 0} \frac{p(m+h)-p(m-h)}{2h}$$

In [15, p. 22], a definition of the first and second symmetric derivatives is provided, along with the relationship between the first symmetric derivative and the usual derivative.

The observation made in [1, p. 708] states that while it is evident that if the ordinary derivative exists at a point, then the symmetric derivative also exists; the reverse is not always true. The continuity of p and the existence of p^s do not guarantee the existence of an ordinary derivative.

The author of [1, p. 708-710] also presents the following results.

Lemma 1. *Let p be continuous on $[a,b]$ and suppose p^s exists in (a,b) for all m.*

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a.If $p(b) > p(a)$, then there exists $c \in (a,b)$ such that $p^s(c) \geq 0$.
b.If $p(b) < p(a)$, then there exists $c \in (a,b)$ such that $p^s(c) \leq 0$.

Lemma 2. *There exists an uncountable number of points such that $p^s(m) \geq 0$ for p satisfying the conditions of Lemma 1.*

Lemma 3. *[Quasi Rolle's Theorem] Let p be continuous on $[a,b]$, if p^s exists in (a,b) and if $p(a) = p(b) = 0$. Then there exists $m_1 \in (a,b)$ such that $p^s(m_1) \geq 0$ and $m_2 \in (a,b)$ such that $p^s(m_2) \leq 0$.*

Theorem 1. *[Quasi Mean Value Theorem] Let p be continuous on $[a,b]$, and suppose p^s exists in (a,b). Then there exist points $m_1, m_2 \in (a,b)$, such that $p^s(m_2) \leq \frac{p(b)-p(a)}{b-a} \leq p^s(m_1)$.*

Theorem 2. *Let p and p^s be continuous on (a,b). Then p is differentiable at m, and $p'(m) = p^s(m)$.*

Recent developments include the introduction of a definition of the symmetric fractional derivative in scale time in [16, p. 54]. Additionally, in (4) of Theorem 3.3 in [16, p. 55], the symmetric fractional derivative in scale time is expressed as:

$$T_\alpha(p^s(m)) = \lim_{h\to 0} \frac{p(m+h)-p(m-h)}{2h} m^{1-\alpha}, m \in \mathbb{T}, \mathbb{T} \subset \mathbb{R},$$

Furthermore, Corollary 3.4 establishes the symmetric fractional derivative in $\mathbb{R}$, as $T_\alpha(p^s)(m) = m^{1-\alpha} p^s(m)$, where p^s denotes the classical symmetric derivative.

Finally, [16, p. 57] provides some calculus results for the symmetric fractional derivative in Theorem 3.5.

3 Conformable fractional derivative

In [10, p. 66], the conformable fractional derivative is defined as follows:

Definition 2. *Given a function $p : [0,\infty] \to \mathbb{R}$, the "conformable fractional derivative" of p of order α is defined as follows:*

$$T_\alpha(p)(m) = \lim_{\varepsilon\to 0} \frac{p(m+\varepsilon m^{1-\alpha})-p(m)}{\varepsilon},$$
for all $m \to 0, \alpha \in (0,1)$

If p is α-differentiable on some interval $(0,a)$, where $a > 0$ and $\lim_{m\to 0^+} p^{(\alpha)}(m)$ exists, then it is defined:

$$p^{(\alpha)}(0) = \lim_{m\to 0^+} p^{(\alpha)}(m)$$

Several results concerning this α-derivative can be found in [10, p. 66-67].

Theorem 3. *If a function $p : [0,\infty) \to \mathbb{R}$ is α-differentiable at $m_0 > 0$ and $\alpha \in (0,1]$ then p is continuous at m_0.*

Theorem 4. *Let $\alpha \in (0,1]$, p and q be α-differentiable functions at a point $m > 0$, then:*

a.$T_\alpha(ap+bq) = aT_\alpha(p) + bT_\alpha(q)$, for all $a,b \in \mathbb{R}$.
b.$T_\alpha(m^x) = xm^{x-\alpha}$.
c.$T_\alpha(\lambda) = 0$.
d.$T_\alpha(p \cdot q) = pT_\alpha(q) + qT_\alpha(p)$
e.$T_\alpha(\frac{p}{q}) = \frac{qT_\alpha(p)-pT_\alpha(q)}{[q]^2}, q \neq 0$.
f.If, in addition, p is differentiable, then $T_\alpha(p)(m) = m^{1-\alpha}\frac{dp(m)}{dm}$.

Also, in [10, p. 68], these results can be found.

Theorem 5. *[Rolle's Theorem for functions with conformable fractional derivative] Let $\alpha > 0$ and $p : [a,b] \to \mathbb{R}$ be a function continuos on $[a,b]$ that satisfies:*

i.p is α-differentiable for some $\alpha \in (0,1)$.
ii.$p(a) = p(b)$.

Then, there exits $c \in (a,b)$ such that $p^{(\alpha)}(c) = 0$.

Theorem 6. *[Mean Value Theorem for α-differentiable functions] Let $\alpha > 0$ and $p : [a,b] \to \mathbb{R}$ be a function that satisfies:*

i.p is continuous on $[a,b]$.
ii.p is α-differentiable for some $\alpha \in (0,1)$.

Then, there exits $c \in (a,b)$ such that $p^{(\alpha)}(c) = \frac{p(b)-p(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}$.

4 Conformable symmetric fractional derivative

The definition presented in this publication considers the definitions by R. Khalil [10, p. 66], C. Aull [1, p. 708], and the results by Zhao Da-Fang [16, p. 56-57], for work in $\mathbb{R}$.

Definition 3. *Let $p : \mathbb{R} \to \mathbb{R}$ be a function. The conformable symmetric fractional derivative of order α is defined as:*

$$T_\alpha p^s(m) = \lim_{\varepsilon\to 0} \frac{p(m+\varepsilon m^{1-\alpha})-p(m-\varepsilon m^{1-\alpha})}{2\varepsilon},$$

for all $m \in \mathbb{R}$, $\alpha = \frac{1}{2l+1}$, $l \in \mathbb{Z}^+$.

If p has a conformable symmetric fractional derivative in $(0,a)$ and $\lim_{m\to 0^+} T_\alpha p^s(m)$ exists, then $T_\alpha p^s(0) = \lim_{m\to 0^+} T_\alpha p^s(m)$. We will refer to this derivative as the α-symmetric fractional derivative. The relationship between the α-symmetric fractional derivative and the classical derivative is presented in the following theorem.

Theorem 7. *Let p be a differentiable function at $m \in \mathbb{R}$, and $\alpha = \frac{1}{2l+1}$, where $l \in \mathbb{Z}^+$. Then $T_\alpha p^s(m) = \frac{m^{1-\alpha}}{2}[p'_+(m) + p'_-(m)]$, where $p'_+(m)$ and $p'_-(m)$ are the right and left-hand side derivatives of p at m.*

*Proof.*Let $m \in \mathbb{R}$ and $\alpha = \frac{1}{2l+1}$, where $l \in \mathbb{Z}^+$. If we take

$$\lim_{\varepsilon\to 0^+} \frac{p(m+\varepsilon m^{1-\alpha})-p(m-\varepsilon m^{1-\alpha})}{2\varepsilon} =$$

$$\lim_{\varepsilon\to 0^+} \frac{p(m+\varepsilon m^{1-\alpha})-p(m)-p(m-\varepsilon m^{1-\alpha})+p(m)}{2\varepsilon} =$$

$$\frac{1}{2}\left[\lim_{\varepsilon\to 0^+} \frac{p(m+\varepsilon m^{1-\alpha})-p(m)}{\varepsilon} + \frac{p(m+(-\varepsilon)m^{1-\alpha})-p(m)}{-\varepsilon}\right]$$

If we let h = $\varepsilon m^{1-\alpha}$, then

$$T_\alpha p^s(m) =$$

$$\frac{m^{1-\alpha}}{2}\left[\lim_{h\to 0^+} \frac{p(m+h)-p(m)}{h} + \lim_{h\to 0^+} \frac{p(m+(-h))-p(m)}{-h}\right]$$

$$= \frac{m^{1-\alpha}}{2}\left[p'_+(m) + p'_-(m)\right]$$

Corollary 1. *Let p be an α-fractional derivative function at $m \in \mathbb{R}$ and $\alpha = \frac{1}{2l+1}$, $l \in \mathbb{Z}^+$, then p has α-symmetric fractional derivative, and also $T_\alpha p(m) = T_\alpha p^s(m)$ for every $m \in \mathbb{R}$.*

As a consequence of Definition 3 and the results of Theorem 3.5 in [6, p. 57], we present the following theorems without proof.

Theorem 8. *Let p and q be functions with α-symmetric fractional derivative for $m \in I, I \subset \mathbb{R}, \lambda \in \mathbb{R}$ and $\alpha = \frac{1}{2l+1}$ for $l \in \mathbb{Z}^+$. Then:*

a.$p \pm q$ has α-symmetric fractional derivatives for $m \in I$, and $T_\alpha(p \pm q)^s(m) = T_\alpha p^s(m) + T_\alpha q^s(m)$.
b.λp has α-symmetric fractional derivatives for $m \in I$, and $T_\alpha(\lambda p)^s(m) = \lambda T_\alpha p^s(m)$.

The product rule does not hold for functions with α-symmetric fractional derivatives.

However, if we add the condition that the functions are continuous at every point, we obtain the following theorem.

Theorem 9. *Let p and q be continuous functions with α-symmetric fractional derivatives for $m \in I, I \subset \mathbb{R}$, and $\alpha = \frac{1}{2l+1}$ for $l \in \mathbb{Z}^+$. Then:*

a.pq has α-symmetric fractional derivatives for $m > 0, m \in I$, and $T_\alpha(pq)^s(m) = [T_\alpha p^s(m)]q(m) + p(m)[T_\alpha q^s(m)]$.
b.$\frac{p}{q}$ has α-symmetric fractional derivative for $m > 0, m \in I$, and $T_\alpha(\frac{p}{q})^s(m) = \frac{[T_\alpha p^s(m)]q(m)-p(m)[T_\alpha q^s(m)]}{[q(m)]^2}$, provided $q(m) \neq 0$.

Theorem 10. *If p is differentiable for $m \in I, I \subset \mathbb{R}, \lambda \in \mathbb{R}$, and $\alpha = \frac{1}{2l+1}$ for $l \in \mathbb{Z}^+$. Then:*

a.$T_\alpha(\lambda)^s = 0$, where λ is a constant.
b.$T_\alpha p^s(m^x) = xm^{x-\alpha}$.
c.$T_\alpha p^s(m) = m^{1-\alpha}p^s(m)$.

The Mean Value Theorem does not hold for α-symmetric fractional derivatives.

An analogous result to the Mean Value Theorem is presented for functions with α-symmetric fractional derivatives.

Hereafter, we present results that allow us to prove this theorem.

Theorem 11. *Let p be continuous on $[a,b]$ and have a α-symmetric fractional derivative on $[a,b]$.*

a.If $p(a) < p(b)$, then there exists a point m, $a \leq m \leq b$ such that $T_\alpha p^s(m) \geq 0$.
b.If $p(a) > p(b)$, then there exists a point m, $a \leq m \leq b$ such that $T_\alpha p^s(m) \leq 0$.

*Proof.*To prove $a)$, we consider m such that $p(a) < \kappa < p(b)$, allowing us to define the set $M = \{n : p(n) > \kappa, a < n < b\}$. Since M is bounded below by a, it is not-empty and contains a point m distinct from a and b due to the continuity of p on $[a,b]$.

For each neighborhood of m, there exist points $n_2 > m$ implying $p(n_2) > \kappa$. Specifically, if we chosen $n_2 = m + \varepsilon m^{1-\alpha}$ then $p(m + \varepsilon m^{1-\alpha}) > \kappa$. Similarly, for n values such that $a \leq n_1 \leq m$, we have $p(n_1) \leq \kappa$, and if $n_1 = m - \varepsilon m^{1-\alpha}$, then $p(m - \varepsilon m^{1-\alpha}) \leq \kappa$.

Therefore,

$$p(m + \varepsilon m^{1-\alpha}) - p(m - \varepsilon m^{1-\alpha}) \geq 0,$$

which implies,

$$T_\alpha p^s(m) = \lim_{\varepsilon\to 0} \frac{p(m + \varepsilon m^{1-\alpha}) - p(m - \varepsilon m^{1-\alpha})}{2\varepsilon} \geq 0.$$

The proof for b) follows a similar argument.

Theorem 12. *[Quasi Rolle's Theorem] If p is continuous on [a,b], has a α-symmetric fractional derivative on (a,b), and $p(a) = p(b)$, then there exists a point m_1, $a < m_1 < b$, such that $T_\alpha p^s(m_1) \geq 0$ and a point m_2, $a < m_2 < b$, such that $T_\alpha p^s(m_2) \leq 0$.*

*Proof.*If $p(n) = 0$, the result is obvious. When $p(n) \neq 0$, there exist points c such that $p(c) > 0$ or d such that $p(d) < 0$, or both.

By Theorem 11, there exists a m_1, $a < m_1 < c$ such that $T_\alpha p^s(m_1) \geq 0$ and a m_2, $c < m_2 < b$ such that $T_\alpha p^s(m_2) \leq 0$; but also, by the same theorem, there exists a m_1, $d \leq m_1 \leq b$ such that $T_\alpha p^s(m_1) \geq 0$ and a m_2, $a < m_2 < d$ such that $T_\alpha p^s(m_2) \leq 0$.

*Example 1.*Let $p : \mathbb{R} \to \mathbb{R}$ be defined as $p(m) = |m|$ and $\alpha = \frac{1}{3}$. It follows that p is continuos on $[-2,2]$, has α-symmetric fractional derivative on $(-2,2)$ and $p(-2) = p(2)$. Considering the symmetric derivative of $p(m) = |m|$ as the function

$$T^s_{\frac{1}{3}}|m| = \begin{cases} \frac{|m|}{m} m^{1-\frac{1}{3}} & ,\text{if } m \neq 0 \\ 0 & ,\text{if } m = 0 \end{cases}$$

We have that,

$$m_1 = -1, \quad T^s_{\frac{1}{3}}|m_1| = \frac{|-1|}{-1}(-1)^{1-\frac{1}{3}} = -1 \leq 0$$

$$m_2 = 1, \quad T^s_{\frac{1}{3}}|m_2| = \frac{|1|}{1}(1)^{1-\frac{1}{3}} = 1 \geq 0$$

Theorem 13. *[Quasi Theorem of Mean Value Cauchy for functions with α-symmetric fractional derivative] Let p and q be continuous functions on $[a,b]$ with α-symmetric fractional derivative on $[a,b]$. If $q^s(m) \neq 0$ for every $m \in (a,b)$, then there exist points m_1 and m_2 between a and b, such that $a < m_1 < b$ and $a < m_2 < b$, satisfying:*

$$\frac{T_\alpha p^s(m_2)}{T_\alpha q^s(\frac{1}{\alpha}m_2^\alpha)} \leq \frac{p(b)-p(a)}{q(\frac{1}{\alpha}b^\alpha) - q(\frac{1}{\alpha}a^\alpha)} \leq \frac{T_\alpha p^s(m_1)}{T_\alpha q^s(\frac{1}{\alpha}m_1^\alpha)}$$

*Proof.*Given r defined as:

$$r(m) =$$

$$p(m)\left[q\left(\frac{1}{\alpha}b^\alpha\right) - q\left(\frac{1}{\alpha}a^\alpha\right)\right] - q\left(\frac{1}{\alpha}m^\alpha\right)[p(b)-p(a)],$$

we have that r is continuous on $[a,b]$ and has α-symmetric fractional derivative on (a,b).

Furthermore,

$$r(a) = p(a)q\left(\frac{1}{\alpha}b^\alpha\right) - p(b)q\left(\frac{1}{\alpha}a^\alpha\right),$$

and

$$r(b) = -p(b)q\left(\frac{1}{\alpha}a^\alpha\right) + p(a)q\left(\frac{1}{\alpha}b^\alpha\right).$$

We know that r satisfies the hypotheses of Theorem 12, thus there exist m_1, $a < m_1 < b$, and m_2, $a < m_2 < b$, such that $T_\alpha r^s(m_2) \leq 0$ and $T_\alpha r^s(m_1) \geq 0$.

Since

$$T_\alpha r^s(m) =$$

$$T_\alpha p^s(m)\left[q\left(\frac{1}{\alpha}b^\alpha\right) - q\left(\frac{1}{\alpha}a^\alpha\right)\right] - T_\alpha q^s\left(\frac{1}{\alpha}m^\alpha\right)[p(b)-p(a)],$$

we have,

$$T_\alpha p^s(m_2)\left[q\left(\frac{1}{\alpha}b^\alpha\right) - q\left(\frac{1}{\alpha}a^\alpha\right)\right] \leq T_\alpha q^s\left(\frac{1}{\alpha}m_2^\alpha\right)[p(b)-p(a)],$$

and

$$T_\alpha p^s(m_1)\left[q\left(\frac{1}{\alpha}b^\alpha\right) - q\left(\frac{1}{\alpha}a^\alpha\right)\right] \leq T_\alpha q^s\left(\frac{1}{\alpha}m_1^\alpha\right)[p(b)-p(a)].$$

Hence,

$$\frac{T_\alpha p^s(m_2)}{T_\alpha q^s\left(\frac{1}{\alpha}m_2^\alpha\right)} \leq \frac{p(b)-p(a)}{q\left(\frac{1}{\alpha}b^\alpha\right) - q\left(\frac{1}{\alpha}a^\alpha\right)} \leq \frac{T_\alpha p^s(m_1)}{T_\alpha q^s\left(\frac{1}{\alpha}m_1^\alpha\right)}.$$

Example 2. Given the functions $p(m) = |m|$ and $q(m) = m^2+1$, since both functions are continuous on the interval $[-1,2]$, they have an α-symmetric fractional derivative on $(-1,2)$ and $q(m) \neq 0$ for all $m \in [-1,2]$. Therefore, the functions satisfy the Quasi Theorem of Mean Value Cauchy for functions with α-symmetric fractional derivative, and we have:

$$\frac{p(2)-p(-1)}{q\left(\frac{1}{\frac{1}{3}}2^{\frac{1}{3}}\right) - q\left(\frac{1}{\frac{1}{3}}(-1)^{\frac{1}{3}}\right)} = \frac{|2|-|-1|}{(3\cdot 2^{\frac{1}{3}})^2+1-(3(-1)^{\frac{1}{3}})^2-1}$$

$$= \frac{1}{5.286609} = 0.189157$$

Therefore, $d = \frac{T_{\frac{1}{3}}|c|^s}{T_{\frac{1}{3}}q^s(3c^{\frac{1}{3}})} = \frac{1}{2(3c^{\frac{1}{3}})} = 0.189157.$

Hence, $c^{\frac{1}{3}} = 0.88110$, implying $c = 0.684031$.

For values $m_2 < c$, for example $m_2 = 0.68402$, we have:

$$\frac{T_{\frac{1}{3}}|m_2|^s}{T_{\frac{1}{3}}q^s\left(\frac{1}{\frac{1}{3}}m_2^{\frac{1}{3}}\right)} = \frac{1}{6\cdot(0.68402)^{\frac{1}{3}}} = \frac{1}{5.2865724}$$

$$= 0.189158 \geq d$$

And for values $m_1 > c$, for example $m_1 = 0.68404$, we obtain:

$$\frac{T_{\frac{1}{3}}|m_1|^s}{T_{\frac{1}{3}}q^s\left(\frac{1}{\frac{1}{3}}m_1^{\frac{1}{3}}\right)} = \frac{1}{6\cdot(0.68404)^{\frac{1}{3}}} = \frac{1}{5.2866239}$$

$$= 0.189156 \leq d$$

Theorem 14. *[Quasi Theorem of Mean Value for functions with α-symmetric fractional derivative] Let p be continuous on $[a,b]$ and have a α-symmetric fractional derivative on (a,b). Then there exist m_1 and m_2 such that $a < m_1 < b$ and $a < m_2 < b$, satisfying*

$$T_\alpha p^s(m_2) \leq \frac{p(b)-p(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha} \leq T_\alpha p^s(m_1)$$

*Proof.*We define a function r that satisfies the hypotheses of Theorem 12 as follows:

$$r(m) = p(m) - p(a) - \frac{p(b)-p(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}\left[\frac{1}{\alpha}m^\alpha - \frac{1}{\alpha}a^\alpha\right]$$

Since p is continuous on $[a,b]$ and has an α-symmetric fractional derivative on (a,b), then r is also continuous. Furthermore,

$$r(a) = p(a) - p(a) - \frac{p(b)-p(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}\left[\frac{1}{\alpha}a^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right] = 0,$$

and

$$r(b) = p(b) - p(a) - \frac{p(b)-p(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}\left[\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}\right] = 0.$$

By Theorem 12, for r there exists m_1, $a < m_1 < b$, such that

$$T_{\alpha}r^{s}(m_1) =$$

$$T_{\alpha}p^{s}(m_1) - \frac{p(b)-p(a)}{\frac{1}{\alpha}b^{\alpha}-\frac{1}{\alpha}a^{\alpha}}\left[T_{\alpha}(\tfrac{1}{\alpha}m_1^{\alpha}) - T_{\alpha}(\tfrac{1}{\alpha}a^{\alpha})\right] \geq 0.$$

This implies

$$T_{\alpha}p^{s}(m_1) - \frac{p(b)-p(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}} \geq 0,$$

and there exists m_2, $a < m_2 < b$, such that

$$T_{\alpha}r^{s}(m_2) = T_{\alpha}p^{s}(m_2) - \frac{p(b)-p(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}} \leq 0.$$

Therefore,

$$T_{\alpha}p^{s}(m_1) \geq \frac{p(b)-p(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}} \quad \text{and} \quad T_{\alpha}p^{s}(m_2) \leq \frac{p(b)-p(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}}.$$

Example 3. Considering the function $p(x) = |x|$ on the interval $[-2,3]$ and $\alpha = \frac{1}{3}$, this function has an α-symmetric fractional derivative.

$$T^{s}_{\frac{1}{3}}(m) = \begin{cases} \frac{|m|}{m}m^{1-\alpha} & ,\text{if } m \neq 0 \\ 0 & ,\text{if } m = 0 \end{cases}$$

We have that,

$$\frac{|3| - |-2|}{\frac{1}{\frac{1}{3}}(3)^{\frac{1}{3}} - \frac{1}{\frac{1}{3}}(-2)^{\frac{1}{3}}} = \frac{1}{8.10651195} = 0.12335762$$

Then, $T_{\frac{1}{3}}p^{s}(c) = c^{1-\frac{1}{3}} = 0.12335762$, wich implies $c = (0.12335762)^{\frac{3}{2}} = 0.04332615$.

For $m_1 < c, m_1 = 0.04332$, we have $T_{\frac{1}{3}}p^{s}(m_1) = (0.04332)^{\frac{2}{3}} = 0.123346$.

For $m_2 > c, m_2 = 0.04333$, we have $T_{\frac{1}{3}}p^{s}(m_2) = (0.04333)^{\frac{2}{3}} = 0.123365$.

Theorem 15. *Let p and $T_{\alpha}p^{s}$ be continuous on (a,b). Then $T_{\alpha}p$ exists, and furthermore, $T_{\alpha}p = T_{\alpha}p^{s}$.*

Proof. For $\varepsilon > 0$ sufficiently small, there exists m such that $a < m < m + \varepsilon m^{1-\alpha} < b$. Since p and $T_{\alpha}p^{s}$ are continuous on $(m, m + \varepsilon m^{1-\alpha})$, by Theorem 14, there exist m_1, $m < m_1 < m + \varepsilon m^{1-\alpha}$, and m_2, $m < m_2 < m + \varepsilon m^{1-\alpha}$, such that

$$T_{\alpha}p^{s}(m_2) \leq \frac{p(m+\varepsilon m^{1-\alpha}) - p(m)}{\varepsilon} \leq T_{\alpha}p^{s}(m_1)$$

Since $T_{\alpha}p^{s}$ is continuous on $(m, m + \varepsilon m^{1-\alpha})$, by the Mean Value Theorem for continuous functions, there exists at least one m_3, $m < m_3 < m + \varepsilon m^{1-\alpha}$, such that

$$T_{\alpha}p^{s}(m_3) = \frac{p(m+\varepsilon m^{1-\alpha}) - p(m)}{\varepsilon}$$

Thus,

$$\lim_{\varepsilon \to 0} T_{\alpha}p^{s}(m_3) = \lim_{\varepsilon \to 0} \frac{p(m+\varepsilon m^{1-\alpha}) - p(m)}{\varepsilon}$$

and we obtain

$$T_{\alpha}p^{s}(m) = T_{\alpha}p(m)$$

5 Conclusion

The possibility remains open to continue research in order to find analogous definitions of the "conformable symmetric second derivative" and other higher-order derivatives. Furthermore, it would be interesting to explore the existence of "conformable symmetric partial derivatives" for functions of multiple variables.

These theoretical developments could further expand the practical applications, providing more robust and precise tools for the analysis of complex phenomena.

The exploration of these lines of research represents a stimulating opportunity for future studies in the field of applied mathematics.

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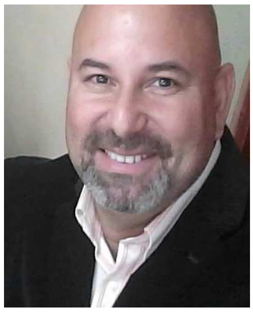

Miguel Vivas-Cortez born in Acarigua, Portuguesa State, Venezuela, graduated in Mathematics from Universidad Centroccidental Lisandro Alvarado (UCLA) and earned his Master's degree in Pure Mathematics. His Master's thesis, which examined a competitive Lotka-Volterra system with infinite delay, was published in the journal Nonlinear Analysis: Real World Applications (https://doi.org/10.1016/j.nonrwa.2005.09.005).

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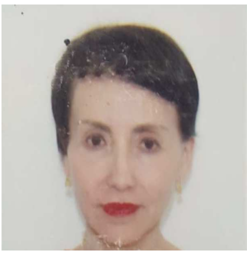

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