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On Microlocalization of Graded and Filtered Formal Modules

Nawal M. NourEldeen[1,3,*], *A. E. Radwan*[2], *and Ahmed Aboubakr*[1,4]

[1]Department of Mathematics, College of Science, Taibah University, Madinah, Kingdom of Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt
[3]Department of Mathematics, Women's College of Arts, Sciences and Education , Ain Shams University, Cairo, Egypt
[4]Department of Mathematics, Faculty of Science, Fayoum University, 63514 Fayoum, Egypt

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Abstract: We modify and generalize the basic theory of formal completion (I-adic completion) as in [3], [8] and [6] with using a general Zariskian filtration $\mathscr{F}\mathscr{R}$ and replacing quiotient filtration $\mathscr{F}\left(\frac{\mathscr{R}}{I^n}\right)$; $n \in \mathscr{Z}$. We establish the exactness, finitness and flatness of formal completion. The formal microlocalization of $\mathscr{R}^{\wedge I}$-module $\mathscr{M}^{\wedge I}$ represents the solution of formal schemes studied on the filtered level in [10], [9] and [11].

Keywords: Formal Zariskian filtration, Microlocalization.

1 Introduction and Preliminaries

For a local ring $\mathscr{R}$, the ring $\mathscr{R}[[x]]$ in n unknowns appear as completion of $\mathscr{R}$ of a non-singular point (or prime) on a variety of dimension n. Similarly, K$[[x]]$ the ring of formal power series is the completion of $\mathscr{R} = \mathrm{K}[x]$; K field, associating with $I = (x)$.
Formal filtered and graded ring theory is a useful tool in non-commutative algebraic geometry since it forms the algebraic model for the completion of schemes along a closed subscheme on the filtered and graded levels. The graded case is the foundation for projective algebraic geometry. Therefore, they are of considerable geometric importance.
Consider $\mathscr{R}$ to be a ring that has one. Considering a chain $\mathscr{F}\mathscr{R}$, which is ascending: $\ldots \subset \mathscr{F}_{-1}\mathscr{R} \subset \mathscr{F}_0\mathscr{R} \subset \mathscr{F}_1\mathscr{R} \subset \ldots$ of $\mathscr{R}$'s additive subgroups with $1_{\mathscr{R}} \in \mathscr{F}_0\mathscr{R}$ in addition $\mathscr{F}_n\mathscr{R}\,\mathscr{F}_m\mathscr{R} \subset \mathscr{F}_{n+m}\mathscr{R}$ for every n , $m \in \mathscr{Z}$, hence $\mathscr{R}$ is called a filtered ring and $\mathscr{F}\mathscr{R}$ a filtration on $\mathscr{R}$. If $\mathscr{M}$ is a left $\mathscr{R}$ -module, it is called a filtered module if there is a chain $\mathscr{F}\mathscr{M}$, that is ascending: $\ldots \subset \mathscr{F}_{-1}\mathscr{M} \subset \mathscr{F}_0\mathscr{M} \subset \mathscr{F}_1\mathscr{M} \subset \ldots$ of $\mathscr{M}$'s additive subgroups with $\mathscr{F}_m\mathscr{R}\,\mathscr{F}_n\mathscr{M} \subset \mathscr{F}_{n+m}\mathscr{M}$ for every n , $m \in \mathscr{Z}$, $\mathscr{F}\mathscr{M}$ is called a filtration on $\mathscr{M}$. On $\mathscr{M}$, a filtration $\mathscr{F}\mathscr{M}$ is said to be exhaustive if so $\mathscr{M} = \bigcup_{n\in Z} \mathscr{F}_n\mathscr{M}$. From now on, all filtrations considered in this paper are exhaustive. On $\mathscr{M}$, a filtration $\mathscr{F}\mathscr{M}$ is said to be separated if so $\bigcap_{n\in Z} \mathscr{F}_n\mathscr{M} = 0$.
Consider I an ideal of a ring $\mathscr{R}$. By the $I-$ adic filtration on $\mathscr{R}$ we mean the filtration $\mathscr{F}\mathscr{R}$ on $\mathscr{R}$ such that $\mathscr{F}_n\mathscr{R} = \mathscr{R}$, $n \geq 0$ and $\mathscr{F}_n\mathscr{R} = I^{-n}$, $n < 0$. If $\mathscr{M}$ is a filtered $\mathscr{R}$ -module along with filtration $\mathscr{F}\mathscr{M}$, it is said to be $I-$adically filtered if so $\mathscr{F}_n\mathscr{M} = \mathscr{M}$ for $n \geq 0$ and $\mathscr{F}_n\mathscr{M} = I^{-n}\mathscr{M}$ for $n < 0$. Respect to the filtration $\mathscr{F}\mathscr{R}$, $G(\mathscr{R}) = \oplus_{n\in Z}\left(\frac{F_n\mathscr{R}}{F_{n-1}\mathscr{R}}\right)$ is the associated graded ring and similarly the associated graded module $G(\mathscr{M}) = \oplus_{n\in Z}\ (\mathscr{F}_n\mathscr{M}\,/\mathscr{F}_{n-1}\mathscr{M}) \in G(\mathscr{R})-$gr to $\mathscr{F}\mathscr{M}$.
Another graded ring $\oplus_{n\in Z}\mathscr{F}_n\mathscr{R} = \widetilde{\mathscr{R}}$ might also be associated to $\mathscr{F}\mathscr{R}$, this ring is known as Rees ring of $\mathscr{R}$. One could identify it to the subring $\sum \mathscr{F}_n\mathscr{R}\mathscr{X}^n$ in $\mathscr{R}[\mathscr{X}, \mathscr{X}^{-1}]$ here over $\mathscr{R}$, $\mathscr{X}$ is a central variable; that is first-degree homogeneous. If $\mathscr{M} \in \mathscr{R}-filt$ we may correspond a graded $\widetilde{\mathscr{R}}-module$ $\widetilde{\mathscr{M}} = \sum_{n\in Z} F_n\mathscr{M}\,\mathscr{X}^n$ in $\mathscr{M}[\mathscr{X}, \mathscr{X}^{-1}]$, it is $\mathscr{X}-$torsion free because $\mathscr{F}_n\mathscr{M} \to \mathscr{F}_{n-1}\mathscr{M}$ are injective maps. As we say, $\mathscr{F}\mathscr{R}$ is a Zariskian filtration or $\mathscr{R}$ is a Zariski ring whenever $\mathscr{X} \in \mathfrak{J}^g\left(\widetilde{\mathscr{R}}\right)$ in addition $\widetilde{\mathscr{R}}$ is Noetherian.
Throughout this paper, assuming that $\mathscr{F}\mathscr{R}$ to be

* Corresponding author e-mail: neldeen@taibahu.edu.sa

Zariskian filration in which case $G(\mathscr{R}) \cong \widetilde{\mathscr{R}}/\mathscr{X}\widetilde{\mathscr{R}}$ is a Noetherian domain that is commutative .
For full details on fundamental facts and basic definitions concerning all notions and conventions for this subject, one may refer to references. [3], [5], [8], [6], [2], [10], [9], and [11].
Bassing through the Rees and the associated graded levels allow the interesting interplay on which the results in this paper depend.
The central focus of the paper encompasses two distinct sections. In the first, we modify and generalize the basic theory of I completion as in [3], [8], and [6] with a general Zariskian filteration $\mathscr{F}\mathscr{R}$. We establish exactness, finiteness, and flatness of the formal completion. In part 2, we focus on microlocalization of these formal objects It represents the solution to formal schemes studied on several levels in [10], [9], [11], [1], and [4].

2 Formal Filtered and Graded Modules

Consider $I \subset \mathscr{R}$ be a filtered ideal having good, induced filtration $\mathscr{F}I$ (as filtered submodule).Hence, the chain $\ldots \subset I^2 \subset I \subset \mathscr{R}$ is in $\mathscr{R}$ -filt. of good filtered ideals in $\mathscr{R}$. If we consider the inverse system $\{\frac{\mathscr{R}}{I^n}\}$ of induced (quotient) good filtered modules over $\mathscr{R}$ and strict filtered morphisms so that we can define the formal filtered ring $\mathscr{R}^{\wedge \mathrm{I}}$, with regard to I, of $\mathscr{R}$ by putting:

$$\mathscr{R}^{\wedge \mathrm{I}} = \varprojlim_{n}{}^{f} \frac{\mathscr{R}}{I^n}.$$

In fact, this defines a Noetherian-filtered ring such that all maps in the commutative diagram; for every n

$$\begin{array}{ccc} \mathscr{R} & \longrightarrow & \frac{\mathscr{R}}{I^n} \\ & \searrow \quad \nearrow & \\ & \mathscr{R}^{\wedge \mathrm{I}} & \end{array}$$

are strict-filtered morphisms. In a similar way, we define the formal filtered module $\mathscr{M}^{\wedge \mathrm{I}}$ of a good filtered module $\mathscr{M} \in \mathscr{R}-$ filt. with respect to I by

$$\mathscr{M}^{\wedge \mathrm{I}} = \varprojlim_{n}{}^{f} \frac{\mathscr{M}}{I^n\,\mathscr{M}}$$

as good filtered $\mathscr{R}^{\wedge \mathrm{I}}-$ module with induced filtration $\mathscr{F}\mathscr{M}^{\wedge \mathrm{I}}$. For the corresponding formal Rees objects, we have

$$\begin{aligned} ((\mathscr{M}^{\wedge \mathrm{I}})^{\sim})^{\wedge}g &= \varprojlim_{m}{}^{g} (\mathscr{M}^{\wedge \mathrm{I}})^{\sim} / \mathscr{X}^m (\mathscr{M}^{\wedge \mathrm{I}})^{\sim} \\ &= \varprojlim_{m}{}^{g} \left((\varprojlim_{n}{}^{f} (\frac{\mathscr{M}}{I^n \mathscr{M}}))^{\sim} / \mathscr{X}^m (\varprojlim_{n}{}^{f} (\frac{\mathscr{M}}{I^n \mathscr{M}}))^{\sim} \right) \\ &= \varprojlim_{m}{}^{g} \varprojlim_{n}{}^{g} (\frac{\mathscr{M}}{I^n \mathscr{M}})^{\sim} / \mathscr{X}^m (\frac{\mathscr{M}}{I^n \mathscr{M}})^{\sim} \\ &= \varprojlim_{n}{}^{g} \varprojlim_{m}{}^{g} \widetilde{\overline{\mathscr{M}}} / \mathscr{X}^m \widetilde{\overline{\mathscr{M}}} = \varprojlim_{n}{}^{g} \left(\widetilde{\overline{\mathscr{M}}}\right)^{\wedge g}; \end{aligned}$$

with $\overline{\mathscr{M}} = \frac{\mathscr{M}}{I^n \mathscr{M}}$. This leads to the following results:

Lemma 1. *With assumptions and conventions as before:*

i. For every n, we have that $\frac{\mathscr{M}}{I^n \mathscr{M}}$ is filtered complete.
ii. The inverse limit of the inverse system $\{\frac{\mathscr{M}}{I^n \mathscr{M}}\}$ is again filtered complete.

Theorem 1. *Under the same conventions as above*

i. $(\mathscr{R}^{\wedge \mathrm{I}})^{\sim} \cong \widetilde{\mathscr{R}}^{\wedge \tilde{I}}$ as Noetherain graded rings, additionally for each $\mathscr{M} \in \mathscr{R}-$ filt. along with good filtration $\mathscr{F}\mathscr{M}$ we have for $(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}$-module $(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}$ is finitely generated $\mathscr{X}-$torsionfree
ii. $\mathscr{F}\mathscr{M}^{\wedge \mathrm{I}}$ is separated and exhaustive filtration.

Proof:

i. It will be easy to showing that $(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}$ is a Noetherian ring. As our assumptions give that $\mathscr{F}\overline{\mathscr{M}} = \mathscr{F}(\frac{\mathscr{M}}{I^n})$; for each n, is filtration that is good, then $\mathscr{F}\mathscr{M}^{\wedge \mathrm{I}}$ is a good filtration and $(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}$is a finitely generated module. Let $\mathscr{X}\widetilde{\overline{a}} = 0$; $\widetilde{\overline{a}} \in \left(\frac{\mathscr{M}}{I^n \mathscr{M}}\right)^{\sim} \cong \frac{\widetilde{\mathscr{M}}}{(I^n \mathscr{M})^{\sim}}$. This means that $\mathscr{X}\tilde{a} \in (I^n \mathscr{M})^{\sim}$. Consequently, $\tilde{a} \in (I^n \mathscr{M})^{\sim}$ and $\widetilde{\overline{a}} = 0$. On the other hand, $\mathscr{L}_{\mathscr{X}}$ of all $\mathscr{X}-$ torsion-free $\widetilde{\mathscr{R}}-modules$ is a full subcategory in $\widetilde{\mathscr{R}}-gr$. Therefore, $(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}$ is $\mathscr{X}-$ torsion free.
ii. Keep in mind that $\mathscr{F}\mathscr{M}$ is exhaustive and separated and that the chain $\ldots \subset I^2\mathscr{M} \subset I\mathscr{M} \subset \mathscr{M}$ in $\mathscr{R}-$ filt. is a chain of induced good filtrations. Hence, for n we have

$$\bigcup \mathscr{F}_p \left(\frac{\mathscr{M}}{I^n \mathscr{M}}\right) = \bigcup \frac{\mathscr{F}_p \mathscr{M} + I^n \mathscr{M}}{I^n \mathscr{M}} = \frac{\mathscr{M}}{I^n \mathscr{M}}$$

As well as $\mathscr{F}\mathscr{M}^{\wedge \mathrm{I}}$ is exhaustive filtration. Similarly, we can verify that $\bigcap \mathscr{F}_p \mathscr{M}^{\wedge \mathrm{I}} = 0$ and $\mathscr{F}\mathscr{M}^{\wedge \mathrm{I}}$ is separated.

Theorem 2. *Under the same conventions as above*

i. $G(\mathscr{M}^{\wedge \mathrm{I}}) = (G(\mathscr{M}))^{\wedge G(I)}$ for each $\mathscr{M} \in \mathscr{R}-$ filt. along with good filtration $\mathscr{F}\mathscr{M}$.
ii. $\left(\mathscr{M}^{\wedge \mathrm{I}}\right)^{\sim} / (1-\mathscr{X})\left(\mathscr{M}^{\wedge \mathrm{I}}\right)^{\sim} \cong \mathscr{M}^{\wedge \mathrm{I}}$ for each $\mathscr{M} \in \mathscr{R}-$ filt. along with good filtration $\mathscr{F}\mathscr{M}$. Moreover, there are isomorphisms

$$\mathscr{F}_p(\mathscr{M}^{\wedge \mathrm{I}}) \cong \frac{((\mathscr{M}^{\wedge \mathrm{I}})^{\sim})_p + (1-\mathscr{X})(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}}{(1-\mathscr{X})(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}}, p \in \mathbb{Z}$$

Proof:

i. Given the sequence, that is exact

$$0 \to I^n \mathscr{M} \to \mathscr{M} \to \frac{\mathscr{M}}{I^n \mathscr{M}} \to 0.$$

Here, all morphisms are strict, and filtrations are good. Hence, get the following exact sequence:

$$0 \to G(I^n\mathscr{M}) \to G(\mathscr{M}) \to G(\frac{\mathscr{M}}{I^n\mathscr{M}}) \to 0$$

of finitely generated modules and

$$\begin{aligned} G(\mathscr{M}^{\wedge \mathrm{I}}) &\cong (\mathscr{M}^{\wedge \mathrm{I}}))^{\sim}/\mathscr{X}(\mathscr{M}^{\wedge \mathrm{I}})^{\sim} \\ &= (\varprojlim_n{}^{f}(\frac{\mathscr{M}}{I^n\mathscr{M}}))^{\sim}/\mathscr{X}(\varprojlim_n{}^{f}(\frac{\mathscr{M}}{I^n\mathscr{M}}))^{\sim} \\ &\cong \varprojlim_n{}^{g}(\widetilde{\overline{\mathscr{M}}}/\mathscr{X}^m\widetilde{\overline{\mathscr{M}}}) = \varprojlim_n{}^{g} G(\frac{\mathscr{M}}{I^n\mathscr{M}}) \\ &\cong \varprojlim_n{}^{g}\frac{G(\mathscr{M})}{G(I^n\mathscr{M})} \cong \varprojlim_n{}^{g}\frac{G(\mathscr{M})}{(G(I))^n G(\mathscr{M})} \\ &= G(\mathscr{M})^{\wedge \mathrm{G(I)}}; \overline{\mathscr{M}} = \frac{\mathscr{M}}{I^n\mathscr{M}}. \end{aligned}$$

Clearly , $\mathscr{M}$ is an $I-$ adic complete iff $G(\mathscr{M})$ is $G(I)-$adic complete.

ii.In a similar way, we get

$$\begin{aligned} &(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}/(1-\mathscr{X})(\mathscr{M}^{\wedge \mathrm{I}})^{\sim} \\ &= (\varprojlim_n{}^{f}\overline{\mathscr{M}})^{\sim}/(1-\mathscr{X})(\varprojlim_n{}^{f}\overline{\mathscr{M}})^{\sim} \\ &\cong \varprojlim_n{}^{g}(\widetilde{\overline{\mathscr{M}}}/(1-\mathscr{X})\widetilde{\overline{\mathscr{M}}}) \\ &\cong \varprojlim_n{}^{f}\overline{\mathscr{M}} = \mathscr{M}^{\wedge \mathrm{I}}. \end{aligned}$$

and if $p \in \mathrm{Z}$ then we have

$$\begin{aligned} \mathscr{F}_p\mathscr{M}^{\wedge \mathrm{I}} &\cong \mathscr{F}_p\left((\mathscr{M}^{\wedge \mathrm{I}})^{\sim}/(1-\mathscr{X})(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}\right) \\ &= \varprojlim_n \mathscr{F}_p\left(\widetilde{\overline{\mathscr{M}}}/(1-\mathscr{X})\widetilde{\overline{\mathscr{M}}}\right) \\ &= \varprojlim_n\left(\frac{\widetilde{\overline{\mathscr{M}}}_p+(1-\mathscr{X})\widetilde{\overline{\mathscr{M}}}}{(1-\mathscr{X})\widetilde{\overline{\mathscr{M}}}}\right) \\ &\cong \frac{((\mathscr{M}^{\wedge \mathrm{I}})^{\sim})_p+(1-\mathscr{X})(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}}{(1-\mathscr{X})(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}} \\ &; \ \overline{\mathscr{M}} = \frac{\mathscr{M}}{I^n\mathscr{M}}. \end{aligned}$$

*Remark.*One may define and construct formal Rees-level or formal associated-level using $\tilde{I}$ or $G(I)$, respectively, depending on the approach we need. Here in this work, the results (some of them) depending on the compatibility of the $I-$adic filtration and $\mathscr{F}\mathscr{R}$. Also, as we've not forgotten the connection between the Ress functor $\sim$ and the associated functor G and the descent functor D from $\widetilde{\mathscr{R}}-gr$. to $\mathscr{R}-$filt; see [2] and [6].

Theorem 3.*With notations as in the previous: The formal functor $(\ )^{\wedge}$ on the filtered scale (on the Ress-scale, on the associated graded scale) is exact on modules with good filtrations (on finitely generated Ress-modules, on finitely generated associated graded modules). Moreover,*

$$\mathscr{R}^{\wedge \mathrm{I}}/(I^{\wedge I})^n \cong \mathscr{R}/I^n \quad , \ \widetilde{\mathscr{R}}^{\wedge \tilde{I}}/\left(\tilde{I}^{\wedge \tilde{I}}\right)^{\sim} \cong \widetilde{\mathscr{R}}/\tilde{I}^n$$

and

$$(G(\mathscr{R}))^{\wedge G(I)}/\left((G(I))^{\wedge G(I)}\right)^n \cong \frac{G(\mathscr{R})}{(G(I))^n}$$

Proof: Similar proofs can be written as in [4] and [5]. Since we have good filtration and strict morphisms. Given the following sequence of good filtered modules, which is strict and exact:

$$o \to I^n\mathscr{R} \to \mathscr{R} \to \frac{\mathscr{R}}{I^n\mathscr{R}} \to o$$

leading to a sequence of good $\mathscr{R}^{\wedge \mathrm{I}}$ -modules, which is exact:

$$0 \to (I^n)^{\wedge I} \to \mathscr{R}^{\wedge \mathrm{I}} \to (\mathscr{R}/I^n)^{\wedge I} \to 0$$

Then it is enough to note that$(\frac{\mathscr{R}}{I^n})^{\wedge I} \cong \frac{\mathscr{R}}{I^n}$. Based on Remark 2, we may derive the other isomorphisms.

It is also possible to continue the formal theory and establish modified results of these in [8] and [6]. This has some interest because we can apply these results to formal, filtered, and graded schemes.

Theorem 4.*With notations as above:*

i.For every $\mathscr{M} \in \mathscr{R}-$filt. along with filtration $\mathscr{F}\mathscr{M}$, which is good we have $\mathscr{R}^{\wedge \mathrm{I}}\bigotimes_{\mathscr{R}}\mathscr{M} \cong \mathscr{M}^{\wedge \mathrm{I}}$
ii.$\mathscr{R}^{\wedge \mathrm{I}}$ is a flat $\mathscr{R}-$ module.

Proof:

i.Since $\mathscr{F}\mathscr{M}$ is good filtration on $\mathscr{M} \in \mathscr{R}-$ filt. Hence, there is a sequence:
$0 \to \mathrm{K} \to \mathrm{T} \to \mathscr{M} \to 0$ in $\mathscr{R}-$ filt. which is strict and exact with T is filt-free. So we get the commutative and exact diagram

$$\begin{array}{ccccccc} \mathscr{R}^{\wedge \mathrm{I}}\bigotimes_{\mathscr{R}}\mathrm{K} & \to & \mathscr{R}^{\wedge \mathrm{I}}\bigotimes_{\mathscr{R}}\mathrm{T} & \to & \mathscr{R}^{\wedge \mathrm{I}}\bigotimes_{\mathscr{R}}\mathscr{M} & \to & 0 \\ \downarrow\gamma & & \downarrow\beta & & \downarrow\alpha & & \\ 0 \to \mathrm{K}^{\wedge I} & \to & \mathrm{T}^{\wedge I} & \to & \mathscr{M}^{\wedge \mathrm{I}} & \to & 0 \end{array}$$

Since the finite direct sums of good filtrations commute with the completion functor. Then β is an isomorphism, hence α is surjective. It follows that α is injective and α is an isomorphism.

ii.Obviously, (i) $\Rightarrow$ (ii).

Theorem 5.*For moment, let $\mathscr{R}$ be $I-$ adic (not necessarily commutative) Zariski filtered ring along with the $I-$ adic filtration $\mathscr{F}_I R$. Then the following assertions are true:*

i.$\mathscr{R}$ is a subring of $\mathscr{R}^{\wedge \mathrm{I}} = \mathscr{R}^{\wedge \mathscr{F}_I}$ as filtered rings such that $\mathscr{R}^{\wedge \mathscr{F}_I}$ is Zariski filtered ring.
ii.In addition, if $\mathscr{R}$ is strongly filtered, then $\mathscr{R}^{\wedge \mathrm{I}} = \mathscr{R}^{\wedge \mathscr{F}_I}$ is also strongly filtered.

Proof: The statements are easily verified by using [8] and [6].

3 Algebraic Formal Microlocalization

Consider a subset $\mathscr{S} \subset \mathscr{R}$ of $\mathscr{R}$, which is multiplicatively closed, then $\sigma(\mathscr{S})$, $\eta(\mathscr{S}) = \mathscr{S}^{\wedge I}$ and $\widetilde{\mathscr{S}}$ are subsets of $G(\mathscr{R})$, $\widetilde{\mathscr{R}}$ and $\mathscr{R}^{\wedge \mathrm{I}}$, consequently, which are multiplicatively closed.

Given the diagram, that is a commutative

$$\begin{array}{ccc} & \mathscr{R} & \\ & \downarrow \eta & \\ & (\mathscr{R}^{\wedge \mathrm{I}})^{\sim}/(1-\mathscr{X})(\mathscr{R}^{\wedge \mathrm{I}})^{\sim} \cong \mathscr{R}^{\wedge \mathrm{I}} & \\ D \nearrow & & \searrow G \\ \left(\mathscr{R}^{\wedge \mathrm{I}}\right)^{\sim} & \xrightarrow{\pi_1} & \left(\mathscr{R}^{\wedge \mathrm{I}}\right)^{\sim}/\mathscr{X}\left(\mathscr{R}^{\wedge \mathrm{I}}\right)^{\sim} \cong G(\mathscr{R}^{\wedge \mathrm{I}}) \\ \pi_n \searrow & & \nearrow \varphi_n \\ & \dfrac{(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}}{\mathscr{X}^n(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}} & \end{array}$$

Since $\ker(\varphi_n) = \left\{ \tilde{r} + \mathscr{X}^n\left(\mathscr{R}^{\tilde{\wedge} I}\right)^{\sim} : \tilde{r} \in \mathscr{X}\left(\mathscr{R}^{\tilde{\wedge} I}\right)^{\sim} \right\}$ is nilpotent, it follows that $\pi_n((\mathscr{S}^{\wedge I})^{\sim})$ is an Ore set as soon as $\sigma(\mathscr{S}^{\wedge I}) = \sigma(\eta(\mathscr{S})) = \varphi_n\, \pi_n((\mathscr{S}^{\wedge I})^{\sim})$ is an Ore set. Hence for every n, we may construct:

$$\left(\pi_n((\mathscr{S}^{\wedge I})^{\sim}) \right)^{-1} \frac{(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}}{\mathscr{X}^n(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}} = Q^g_{(\mathscr{S}^{\wedge I})^{\sim}(n)} \frac{(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}}{\mathscr{X}^n(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}}$$

the ring of quotients of $(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}/\ \mathscr{X}^n(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}$ with respect to $(\mathscr{S}^{\wedge I})^{\sim}(n)$. In general, for graded quotient rings, we refer to several references, like [7] and [8].

There is an inverse system in $(\mathscr{R}^{\wedge \mathrm{I}})^{\sim} - gr.$, as we take the canonical homomorphism.

$$Q^g_{(\mathscr{S}^{\wedge I})^{\sim}(n)} \frac{(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}}{\mathscr{X}^n(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}} \to Q^g_{(\mathscr{S}^{\wedge I})^{\sim}(n-1)} \frac{(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}}{\mathscr{X}^{n-1}(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}}$$

It's inverse limite, in $(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}$-gr., is indicated by $\widetilde{Q}^{\mu}_{(\mathscr{S}^{\wedge I})^{\sim}}((\mathscr{R}^{\wedge \mathrm{I}})^{\sim})$. The latter defines the microlocalization of $(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}$ with respect to $(\mathscr{S}^{\wedge I})^{\sim}$. Similarly, we can define microlocalization $\tilde{Q}^{\mu}_{(\mathscr{S}^{\wedge I})^{\sim}}((\mathscr{M}^{\wedge \mathrm{I}})^{\sim})$ of a graded $(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}$−module $\mathscr{M}^{\wedge \mathrm{I}\sim}$ and obtain:

$$\begin{aligned} \tilde{Q}^{\mu}_{(\mathscr{S}^{\wedge I})^{\sim}}((\mathscr{M}^{\wedge \mathrm{I}})^{\sim}) &= \varprojlim_n{}^{g}\ Q^g_{(\mathscr{S}^{\wedge I})^{\sim}(n)}\left(\frac{(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}}{\mathscr{X}^n(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}} \right) \\ &= \varprojlim_n{}^{g}\ Q^g_{(\mathscr{S}^{\wedge I})^{\sim}(n)}\left[\left(\varprojlim_m{}^{f}\left(\frac{\mathscr{M}}{I^m\mathscr{M}}\right)\right)^{\sim} / \mathscr{X}^n \left(\varprojlim_m{}^{f}\left(\frac{\mathscr{M}}{I^m\mathscr{M}}\right)\right)^{\sim} \right] \\ &= \varprojlim_n{}^{g}\ \varprojlim_m{}^{g}\ Q^g_{\tilde{\overline{S}}(n,m)}\left(\widetilde{\overline{\mathscr{M}}} / \mathscr{X}^n \widetilde{\overline{\mathscr{M}}} \right) \\ &= \varprojlim_m{}^{g}\ \varprojlim_n{}^{g}\ Q^g_{\tilde{\overline{S}}(n,m)}\left(\widetilde{\overline{\mathscr{M}}} / \mathscr{X}^n \widetilde{\overline{\mathscr{M}}} \right) \\ &= \varprojlim_m{}^{g}\ Q^g_{\tilde{\overline{S}}(m)}\left(\widetilde{\overline{\mathscr{M}}} \right) \end{aligned}$$

Where $\overline{\mathscr{M}} = \frac{\mathscr{M}}{I^m\ \mathscr{M}}$ is a filtered $\frac{\mathscr{R}}{I^m}$−module along with good filtration $\mathscr{F}\overline{\mathscr{M}}$. It's actually easy to verify that $\tilde{Q}^{\mu}_{(\mathscr{S}^{\wedge I})^{\sim}}((\mathscr{M}^{\wedge \mathrm{I}})^{\sim})$ defines a graded $\mathscr{X}$-torsion free $\tilde{Q}^{\mu}_{(\mathscr{S}^{\wedge I})^{\sim}}((\mathscr{R}^{\wedge \mathrm{I}})^{\sim})$ −module.

The graded $(\mathscr{R}^{\wedge \mathrm{I}})^{\sim}$− homomorphisms:

$$(\mathscr{M}^{\wedge \mathrm{I}})^{\sim} \to \frac{(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}}{\mathscr{X}^n(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}} \to Q^g_{(\mathscr{S}^{\wedge I})^{\sim}(n)}\left(\frac{(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}}{\mathscr{X}^n(\mathscr{M}^{\wedge \mathrm{I}})^{\sim}} \right)$$

are compatible with the corresponding inverse system, hence producing a unique morphism in $(\mathscr{R}^{\wedge \mathrm{I}})^{\sim} - gr.$: $(\mathscr{M}^{\wedge \mathrm{I}})^{\sim} \to \tilde{Q}^{\mu}_{(\mathscr{S}^{\wedge I})^{\sim}}((\mathscr{R}^{\wedge \mathrm{I}})^{\sim})$.

Now, describe

$$Q^{\mu}_{\mathscr{S}^{\wedge I}}(\mathscr{R}^{\wedge \mathrm{I}}) = \frac{\tilde{Q}^{\mu}_{(\mathscr{S}^{\wedge I})^{\sim}}((\mathscr{R}^{\wedge \mathrm{I}})^{\sim})}{(1-\mathscr{X})\,\tilde{Q}^{\mu}_{(\mathscr{S}^{\wedge I})^{\sim}}((\mathscr{R}^{\wedge \mathrm{I}})^{\sim})}$$

and

$$Q^{\mu}_{\mathscr{S}^{\wedge I}}(\mathscr{M}^{\wedge \mathrm{I}}) = \frac{\tilde{Q}^{\mu}_{(\mathscr{S}^{\wedge I})^{\sim}}((\mathscr{M}^{\wedge \mathrm{I}})^{\sim})}{(1-\mathscr{X})\,\tilde{Q}^{\mu}_{(\mathscr{S}^{\wedge I})^{\sim}}((\mathscr{M}^{\wedge \mathrm{I}})^{\sim})}$$

i.e., by dehomogenization of the construction at the Rees-module scale. Also, $\mathscr{M}^{\wedge \mathrm{I}} \to Q^{\mu}_{\mathscr{S}^{\wedge I}}(\mathscr{M}^{\wedge \mathrm{I}})$ is a unique filtered morphism. $Q^{\mu}_{\mathscr{S}^{\wedge I}}(\mathscr{M}^{\wedge \mathrm{I}})$ is said to be the filtered microlocalization of the formal $\mathscr{R}^{\wedge \mathrm{I}}$−module $\mathscr{M}^{\wedge \mathrm{I}}$ at $\mathscr{S}^{\wedge I}$.

Lemma 2. *With conventions as above:*

i. $Q^{\mu}_{f^{\wedge}}(\mathscr{M}^{\wedge \mathrm{I}}) \cong \left(Q^{\mu}_{f}(\mathscr{M}^{\wedge \mathrm{I}}) \right)^{\wedge \mathscr{J}}$; $\mathscr{J} = Q^{\mu}_{f}(I)$.

ii. $\tilde{Q}^{\mu}_{(f^{\wedge})^{\sim}}((\mathscr{M}^{\wedge \mathrm{I}})^{\sim}) \cong \left(\tilde{Q}^{\mu}_{\tilde{f}}\left(\widetilde{\mathscr{M}} \right) \right)^{\wedge \widetilde{\mathscr{J}}}$; $\widetilde{\mathscr{J}} = (Q^{\mu}_{f}(I))^{\sim}$.

Where, as in [2], we denote the multiplicative set $\{1, f, f^2, \ldots\}^{\sim}$ by $\tilde{f}$ and its image by $\tilde{f}^{\wedge}$.

Proof: The statement in (i) is easily verified by using the definition. For the statement in (ii) we have

$$\begin{aligned} \tilde{Q}^{\mu}_{(f^{\wedge})^{\sim}}((\mathscr{M}^{\wedge \mathrm{I}})^{\sim}) &= \tilde{Q}^{\mu}_{(f^{\wedge})^{\sim}}\left(\left(\varprojlim_n{}^{f} \frac{\mathscr{M}}{I^n\ \mathscr{M}}\right)^{\sim} \right) \\ &= \tilde{Q}^{\mu}_{(f^{\wedge})^{\sim}}\left(\varprojlim_n\left(\frac{\mathscr{M}}{I^n\ \mathscr{M}}\right)^{\sim} \right) \\ &= \tilde{Q}^{\mu}_{\tilde{f}^{\wedge}}\left(\varprojlim_n{}^{g} \frac{\widetilde{\mathscr{M}}}{(I^n\mathscr{M})^{\sim}} \right) \\ &= \varprojlim_n{}^{g}\ \tilde{Q}^{\mu}_{(f^{\wedge})^{\sim}}\left(\frac{\widetilde{\mathscr{M}}}{(I^n\mathscr{M})^{\sim}} \right) \\ &= \varprojlim_n{}^{g}\ \tilde{Q}^{\mu}_{\tilde{f}}\left(\widetilde{\mathscr{M}} \right) / \tilde{Q}^{\mu}_{\tilde{f}}\left((I^n\mathscr{M})^{\sim} \right) \\ &= \varprojlim_n{}^{g}\ \tilde{Q}^{\mu}_{\tilde{f}}\left(\widetilde{\mathscr{M}} \right) / (\tilde{Q}^{\mu}_{\tilde{f}}(\tilde{I}))^{n}\ \tilde{Q}^{\mu}_{\tilde{f}}\left(\widetilde{\mathscr{M}} \right) \\ &= \left(\tilde{Q}^{\mu}_{\tilde{f}}\left(\widetilde{\mathscr{M}} \right) \right)^{\wedge \widetilde{\mathscr{J}}} ;\ \widetilde{\mathscr{J}} = \tilde{Q}^{\mu}_{\tilde{f}}(\tilde{I}). \end{aligned}$$

Theorem 6.*Let $\mathscr{M}$ be a good filtered $\mathscr{R}$-module and $I \subset \mathscr{R}$ good filtered ideal, then the microlocalization of the filtered formal $\mathscr{M}^{\wedge I}$ that of $\mathscr{M}$ is the filtered formal of microlocalization of $\mathscr{M}$ along with respect to $\mathscr{J} = Q_f^{\mu}(I)$. Therefore, they represent solutions to formal schemes at the filtered level.*

Proof: The statements will be clear by using lemma 2 and refs [9] and [10].

*Remark.*From the viewpoint of the theorem 6, the author observed that it is not necessary here to study the algebraic properties and applications of these objects. One may follow this geometrically in [2], [10], [9], [11] and [4]. We hope to come back to another application for them in the forthcoming work.

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Nawal M. NourEldeen is Assistant Professor of abstract algebra at Women's College of Arts, Sciences and Education, Ain Shams University, Egypt. Dr. Nawal received her MS. and Ph.D. degrees in abstract algebra in 2002 and 2008, respectively from Ain Shams University, and now she works at College of Science, Taibah University, Madinah, Kingdom of Saudi Arabia.

A. E. Radwan received the PhD degree in pure mathematics (algebraic geometry) at Ain-Shams University., Cairo, Egypt (1991). Now, he is a full Prof. Dr. at Ain-shams University. His research interests are in the areas of algebraic geometry and topological structure with its applications. He has published many more articles in mathematical journals. He is referee and evaluator of more research articles and thesis. He is a member of the board of directors of the Egyptian Mathematical Society.

Ahmed Aboubakr Assistant Professor in Mathematics at College of Science, Taibah University, Madinah, Kingdom of Saudi Arabia. Attained a Doctor of Philosophy (Ph.D.) in Mathematics with the highest honors from the University of Oviedo in Spain (2017). Served as a Postdoctoral Fellow at the University of Oviedo in Spain, demonstrating substantial expertise in this academic discipline.