Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/130603

Bayesian Analysis of the Kumaraswamy Distribution Based on Fuzzy Data

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Received: 3 Aug. 2024, Revised: 12 Sep. 2024, Accepted: 12 Oct. 2024
Published online: 1 Nov. 2024

Abstract: In this paper, we discussed the fuzzy Bayesian estimation method for the Kumaraswamy distribution (KD) parameters and the reliability function based on the progressive type-II fuzzy order statistics. The Bayesian estimators have been derived by Monte Carlo Integration (MCI), Markov Chain Monte Carlo (MCMC), and Tierney-Kadane (TK). These estimators are compared with the exact Bayesian estimators, via an intensive Monte Carlo simulation. The simulation results indicated that the Monte Carlo Integration and Markov Chain Monte Carlo methods provide better estimators and outperform the other estimators. Finally, two real datasets are provided to illustrate the results.

Keywords: Fuzzy data; Informative prior; Markov Chain Monte Carlo technique; Monte Carlo Integration approximation; Tierney-Kadane approximation.

1 Introduction

The two-parameter Kumaraswamy distribution on [0,1] was proposed by Kumaraswamy [1] and represented as $KD(\alpha,\beta)$. The probability density function (pdf) and cumulative distribution function (cdf) of the Kumaraswamy distribution are provided, respectively, by:

$$f(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, 0 \le x \le 1, \alpha,\beta > 0, \tag{1}$$

$$F(x;\alpha,\beta) = 1-(1-x^{\alpha})^{\beta}; \ 0 \le x \le 1, \quad \alpha,\beta > 0. \tag{2}$$

Therefore, the reliability function is given by

$$R(x;\alpha,\beta) = (1-x^{\alpha})^{\beta}; \ 0 \le x \le 1, \quad \alpha,\beta > 0 \tag{3}$$

According to Kumaraswamy and Ponnambalam [1,2], this distribution can reproduce the results of the beta distribution. In addition, it can be used to approximate a variety of distributions, including uniform, triangular, or nearly any single model distribution, depending on the parameters chosen. Jones [3] has examined in detail the properties of the Kumaraswamy distribution.

Recently, many authors have become aware of the progressive type-II censoring schemes. The most popular type-I and type-II censoring techniques restrict the experimenter's freedom by preventing them from removing units before ending the experiment. To overcome this restriction, a more general censoring system is known as the progressive type-II censoring scheme; see Balakrishnan and Aggarwala [4]. The progressive type-II censored life test is described as follows: Under this censoring scheme, n units are placed on test at time zero m failures to be observed. Suppose n units are placed on a life test, and the experimenter decides beforehand the quantity m, the number of failures to be observed. Now, at the time of the first failure, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. At the time of the second failure, R_2 of the remaining $n-R_1-2$ units are randomly removed from the experiment. Finally, at the time of the mth failure, all the remaining surviving units $R_m = n-m-\sum_{i=1}^{m-1} R_i$ are removed

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from the experiment. Therefore, a progressive type-II censoring scheme consists of m, and $R_1, R_2, .., R_m$, such that $n = m + \sum_{i=1}^{m} R_i$. The m failure times obtained from a progressive type-II censoring scheme will be denoted by $X_1, X_2, \ldots, X_m$. Thus, the likelihood function for the progressive type-scheming scheme with a predetermined number of random removal units $(R_1, R_2, .., R_m)$ can be written as follows:

$$L(\theta|X) = C\prod_{i=1}^{m} f(x_{(i)})\prod_{i=1}^{m}(1 - F(x_{(i)}))^{R_i}, \tag{4}$$

where

$$C = n(n-R_1-1)(n-R_1-R_2-2)\ldots(n-\sum_{i-1}^{m-1}(R_i+1)).$$

If $R_1 = R_2 = \ldots = R_{m-1} = 0$, then $R_m = n - m$ which corresponds to the type-II censoring sample, and if $R_1 = R_2 = \ldots = R_m = 0$, then n = m, which corresponds to the complete sample; see Wu [5].
The progressive censoring system has recently been the subject of numerous authors' analyses. Gholizadeh et al. [6] have examined the Kumaraswamy distribution under progressively type-II censored data. Feroze and El-Batal [7] computed the maximum likelihood estimate and the asymptotic variance-covariance matrix for the two parameters of the Kumaraswamy distribution under progressive type-II censoring with random removals. Eldin et al. [8] studied the parameter estimations for the Kumaraswamy distribution under general progressive type-II censoring. Erick et al. [9] studied the parameter estimations for the Kumaraswamy distribution using progressive type-II censoring, using the maximum likelihood and Bayesian methods. El-Sagheer[10] employed a progressively type-II censoring scheme to estimate test units from the Kumaraswamy distribution. They applied the EM method to get the maximum likelihood estimates for the parameters. Furthermore, Sultana et al. [11] thoroughly examined Bayesian inference for the Kumaraswamy distribution under hybrid censored samples. Among the most recent studies on progressive censoring include, but are not limited to, Sultana et al [12], Tu and Gui [13], Ghafouri and. Rastogi [14], and Kohansal and Bakouch [15].

Only precise data can be used in the previous studies to estimate the parameters of various lifetime distributions under progressive type-II censoring. However, experiments don't give precise knowledge in real-world situations. For instance, it is impossible to identify a person's reaction time to a particular stimulus during a psychological experience; instead, the psychologist used the following imprecise data to estimate it: About 25 to 35 seconds are needed for the reaction time. The fuzzy idea must be incorporated into statistical approaches to address the data's lack of precision. In these situations, the imperfection of the given data might be modeled using fuzzy numbers, see Dubois [16].

In recent years, many papers have extended the statistical methods to analyze fuzzy data for many distributions. Denœux [17] demonstrated how the EM algorithm may be applied to statistical problems using fuzzy data for parametric statistical models. Pak et al. [18,19,20,21,22] carried out several studies to create inferential methods when the available data is in the form of fuzzy numbers. Makhdoom et al. [23] used a type-II censoring scheme to estimate the exponential distribution's parameter. Khoolenjani and Shahsanaie [24] have addressed various approaches to find the exponential mean parameter under type-II censoring from fuzzy data. Shafiq [25] studied Bayesian and classical inferences for the Pareto distribution of lifetime fuzzy data. Chaturvedi et al. [26] introduced the classical and Bayesian approaches for estimating the parameters of the Rayleigh distribution based on type-II progressively hybrid censored fuzzy lifetime data. AL-Sultany [27] and Mabrouk [28] have established inferences for inverse Rayleigh and inverse Lindley distributions, respectively. Maswadah [29,30] derived the Bayes inference for fuzzy data based on progressive type-II and dual generalized order statistics. Moreover, Ghosh [31], Seham and Amira Younis [32], and Alharbi and Kamel [33] for the censored sample of fuzzy numbers obtained different classical and Bayes estimates.

The rest of this paper is planned as follows: Section 1 presents the maximum likelihood estimates for the Kumaraswamy distribution parameters based on a progressive type-II censoring scheme from fuzzy lifetime data. In Section 2, the Bayes estimates of the parameters α, β, and the reliability function R(t) are obtained by using the approximation forms of exact Bayes, Tierney-Kadane, Monte Carlo Integration, as well as the Markov Chain Monte Carlo technique under the squared error loss function. Monte Carlo simulation results are presented in Section 3, which compares the different estimation methods. Numerical examples are presented in Section 4. Finally, we make some concluding remarks in Section 5.

We present the likelihood function formula under the progressive type-II censoring scheme. To drive the Bayesian estimates of KD parameters. Now, assume that n independent units are put on a test and that the lifetime distribution of each unit is given by $f(x;\theta)$. Consider the problem where, under a progressive type-II censoring scheme, failure times are not observed precisely, and only partial information about them is available, so we can represent them as fuzzy numbers $\tilde{x}_i = (a_i, c_i, b_i), i = 1, \ldots, m$ and its corresponding membership functions $(\mu_{\tilde{x}_1}, \mu_{\tilde{x}_2}, \ldots, \mu_{\tilde{x}_m})$. Let $c_{(1)} \leq c_{(2)} \leq \cdots \leq c_{(m)}$

denote the ordered values of the means of these fuzzy numbers. The lifetime of R_i surviving units, which are removed from the test after the ith failure, can be encoded as fuzzy numbers, $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_m)$ where $\tilde{z}_i = (\tilde{z}_{i_1}, \tilde{z}_{i2}, \ldots, \tilde{z}_{iRi})$, for $i = 1, \ldots, m$ with the membership functions as follows:

$$\mu_{\tilde{z}_{ij}} = \begin{cases} 0 & z \le c_i, \\ 1 & z > c_i, \end{cases} \qquad j, \ldots, R_i$$

The fuzzy data $\tilde{w} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_m)$ is the set of observed lifetimes. Then the corresponding likelihood function for observed data $\tilde{w}$ can be obtained using Zadeh's definition of the probability of a fuzzy event, see Zadeh [34], as

$$\begin{aligned} L_o(\widetilde{w}; \alpha, \beta) &= \prod_{i=1}^m \int f(x) \mu_{\widetilde{x}_i}(x) dx \prod_{i=1}^m \prod_{j=1}^{R_i} \int f(z) \mu_{\widetilde{z}_{ij}}(z) dz \\ &= \alpha^m \beta^m e^{\beta \sum_{i=1}^m R_i log(1-c_{(i)}^{\alpha})} \prod_{i=1}^m \int x^{\alpha-1}(1-x^{\alpha})^{(\beta-1)} \mu_{\widetilde{x}_i}(x) dx \end{aligned} \tag{5}$$

Then, using the expression (5), the observed data log-likelihood function $L(\tilde{w}; \alpha, \beta) = logL_o(\widetilde{w}; \alpha, \beta)$ can be calculated as follows:

$$\begin{aligned} L(\widetilde{w}; \alpha, \beta) = m(log\alpha + log\beta) + \sum_{i=1}^m log \int x^{\alpha-1}(1-x^{\alpha})^{\beta-1} \mu_{\widetilde{x}}(x) dx \\ + \beta \sum_{i=1}^m R_i log(1 - c_{(i)}^{\alpha}). \end{aligned} \tag{6}$$

Maximizing (6) for α and β, we obtain

$$\frac{\partial L(\widetilde{w}; \alpha, \beta)}{\partial \alpha} = \frac{m}{\alpha} - \beta \sum_{i=1}^m R_i \frac{c_{(i)}^{\alpha} logx}{1 - c_{(i)}^{\alpha}} + \sum_{i=1}^m \frac{\int [1-(\beta-1)x^{\alpha}(1-x^{\alpha})^{-1}] x^{\alpha-1} logx (1-x^{\alpha})^{(\beta-1)} \mu_{\widetilde{x}}(x) dx}{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1} \mu_{\widetilde{x}}(x) dx}$$

$$\frac{\partial L(\widetilde{w}; \alpha, \beta)}{\partial \beta} = \frac{m}{\beta} + \sum_{i=1}^m R_i log(1 - c_{(i)}^{\alpha}) + \sum_{i=1}^m \frac{\int x^{\alpha-1} log(1-x^{\alpha})(1-x^{\alpha})^{(\beta-1)} \mu_{\widetilde{x}_i}(x) dx}{\int x^{\alpha-1}(1-x^{\alpha})^{(\beta-1)} \mu_{\widetilde{x}_i}(x) dx}.$$

We can find the MLEs for the unknown parameters α and β using an iterative finite difference method.

2 Fuzzy Bayesian Estimation

2.1 Exact Bayes (EB)

In this section, we derive the exact Bayes estimates for the KD parameters α, β, and the reliability function based on the progressive type-II censored fuzzy data. We assume the prior distributions of α, β are Gamma (a, b) and Gamma (c, d), respectively; thus, we have

$$\begin{aligned} \pi_1(\alpha) &= \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-\alpha b}, \quad a, b \ge 0, \\ \pi_2(\beta) &= \frac{d^c}{\Gamma(c)} \beta^{c-1} e^{-d\beta}, \quad c, d \ge 0 \end{aligned}$$

Thus, the joint prior density on α and β is given by

$$P(\alpha, \beta) \propto \alpha^{a-1} \beta^{c-1} e^{-b\alpha - d\beta}. \tag{7}$$

Based on the likelihood function (5) and the joint prior (7). Therefore, the joint posterior of (α, β) can be expressed as follows:

$$\pi(\alpha, \beta | \widetilde{w}) = k\alpha^{(m+a-1)} \beta^{(m+c-1)} exp[-\alpha b - \beta [d - \sum_{i=1}^m R_i log(1 - {c_i}^{\alpha})]] J_{11} \tag{8}$$

where

$$J_{11} = \prod_{i=1}^m \int x^{\alpha-1}(1-x^{\alpha})^{\beta-1} \mu_{\widetilde{x}}(x) dx.$$

We can derive the exact Bayes estimators of a function g(α, β) of the unknown parameters α, β, and reliability R(t) under the squared error loss function as follows:

$$\hat{g}(\alpha, \beta) = E(g(\alpha, \beta) | \widetilde{w}) = \frac{\int_0^{\infty} \int_0^{\infty} g(\alpha, \beta) \alpha^{m+a-1} \beta^{m+c-1} \phi J_{11} d\alpha d\beta}{\int_0^{\infty} \int_0^{\infty} \alpha^{m+a-1} \beta^{m+c-1} \phi J_{11} d\alpha d\beta}, \tag{9}$$

$$\hat{R}(t) = E(g(\alpha, \beta) | \tilde{w}) = E((1 - t^{\alpha})^{\beta} | \tilde{w}) = \frac{\int_0^{\infty} \int_0^{\infty} (1-t^{\alpha})^{\beta} \alpha^{m+a-1} \beta^{m+c-1} \phi J_{11} d\alpha d\beta}{\int_0^{\infty} \int_0^{\infty} \alpha^{m+a-1} \beta^{m+c-1} \phi J_{11} d\alpha d\beta}. \tag{10}$$

Using Simpson's rule method for solving the above two equations, the exact Bayes estimators of α, β, and reliability R(t) can be obtained.

2.2 Tierney-Kadane Approximation (TK)

The Bayes estimate of $g(\alpha,\beta)$ in Eq. (9) for any function of α, β, and R(t) can be written in the following expression.

$$\hat{g}(\alpha,\beta) = E(g(\alpha,\beta)|\tilde{w}) = \frac{\int_0^\infty \int_0^\infty g(\alpha,\beta)e^{Q(\alpha,\beta)}d\alpha d\beta}{\int_0^\infty \int_0^\infty e^{Q(\alpha,\beta)}d\alpha d\beta}, \quad (11)$$

where,

$$Q(\alpha,\beta) = (m+a-1)log\alpha + (m+c-1)log\beta - \alpha b - \beta[d - \sum_{i=1}^m R_i log(1-c_i{}^\alpha)] + \sum_{i=1}^m \log \int x^{\alpha-1}(1-x^\alpha)^{\beta-1}\mu_{\tilde{x}}(x)\,dx,$$

$$H(\alpha,\beta) = \frac{Q(\alpha,\beta)}{n},$$

and

$$H^*(\alpha,\beta) = H(\alpha,\beta) + \frac{\log(g(\alpha,\beta))}{n}.$$

Then, the expression in (11) can be written as follows:

$$\hat{g}(\alpha,\beta) = \frac{\int_0^\infty \int_0^\infty e^{nH^*(\alpha,\beta))}d\alpha d\beta}{\int_0^\infty \int_0^\infty e^{nH(\alpha,\beta)}d\alpha d\beta}. \quad (12)$$

According to Tierney-Kadane [35], the Bayes estimate of the function $g(\alpha,\beta)$ can be obtained by approximating Eq. (12) as follows:

$$\hat{g}(\alpha,\beta) = \left[\frac{det\Sigma^*}{det\Sigma}\right]^{1/2} exp\{n[H^*\left(\hat{\alpha}^*,\hat{\beta}^*\right) - H\left(\hat{\alpha},\hat{\beta}\right)]\}, \quad (13)$$

where $\left(\hat{\alpha}^*,\hat{\beta}^*\right)$and $\left(\hat{\alpha},\hat{\beta}\right)$ maximize $H^*(\alpha,\beta)$ and $H(\alpha,\beta)$, respectively, and Σ^* and Σ are the negatives of the inverse Hessian of $H^*(\alpha,\beta)$ and $H(\alpha,\beta)$ at $\left(\hat{\alpha}^*,\hat{\beta}^*\right)$and $\left(\hat{\alpha},\hat{\beta}\right)$, respectively.
As a result, Appendix A contains the derivatives of $H^*(\alpha,\beta)$ and $H(\alpha,\beta)$. From the second derivative of $H(\alpha,\beta)$, the determinant of the negative of the inverse Hessian of $H(\alpha,\beta)$ at $\hat{\alpha}$ and $\hat{\beta}$ is given by $\det\Sigma = (H_{11}H_{22} - H_{12}{}^2)^{-1}$.
Similarly, from the second derivative of $H^*(\alpha,\beta)$, the determinant of the negative of the inverse Hessian of $H^*(\alpha,\beta)$ at $\hat{\alpha}^*$ and $\hat{\beta}^*$ is given by $\det\Sigma^* = (H_{11}^* * H_{22}^* - H_{12}^*{}^2)^{-1}$.
Thus, we can find the Bayes estimates for the parameters α and β, and R(t) from (13).

2.3 Monte Carlo Integration (MCI)

The Monte Carlo Integration method is useful for obtaining solutions to problems involving integration that are too complicated to solve analytically or with other numerical techniques. Consider $\pi(\theta)$ is the prior function and $L(\theta|X)$ is the likelihood function, then the Bayes estimate of the parameter θ is given by

$$E[\theta|X_1,X_2,\ldots,X_n] = \frac{\int \theta L(\theta|X)\pi(\theta)d\theta}{\int L(\theta|X)\pi(\theta)d\theta} = \frac{\int h_1(\theta|X)q(\theta)d\theta}{\int h_2(\theta|X)q(\theta)d\theta}$$

where $h_1(\theta|X) = \frac{\theta L(\theta|X)\pi(\theta)}{q(\theta)}$, $h_2(\theta|X) = \frac{L(\theta|X)\pi(\theta)}{q(\theta)}$, and $q(\theta)$ be an important sampling distribution.
Let $\theta_1,\ldots,\theta_N$ be a sample from $q(\theta)$. Then

$$E[\theta|X_1,X_2,\ldots,X_n] \approx \frac{\frac{1}{N}\sum_{i=1}^N h_1(\theta_i|X)}{\frac{1}{N}\sum_{i=1}^N h_2(\theta_i|X)}. \quad (14)$$

Based on the Monte Carlo Integration approximation technique (14), we can find the Bayesian estimators from the joint posterior density function for α and β, which is given by

$$\pi(\alpha,\beta|\widetilde{w}) = K\alpha^{(m+a-1)}\beta^{(m+c-1)}.exp[-\alpha b - \beta[d - \sum_{i=1}^m R_i log(1-c_i{}^\alpha)]] \cdot\prod_{i=1}^m \int x^{\alpha-1}(1-x^\alpha)^{\beta-1}\mu_{\widetilde{x}}(x)dx$$

The normalizing constant in this case is k.
Consequently, the conditional marginal densities for α and β can be written as

$$f(\alpha|\widetilde{w},\beta_0) = K_1\alpha^{m+a-1}.exp[[-\alpha b - \beta_0[d - \sum_{i=1}^m R_i log(1-c_i{}^\alpha)]] \cdot\left[\prod_{i=1}^m \int x^{\alpha-1}(1-x^\alpha)^{\beta_0-1}\mu_{\widetilde{x}}(x)dx\right]$$

$$f(\beta|\widetilde{w},\alpha_0) = K_2\beta^{m+c-1}.exp\left[-\beta\left[d-\sum_{i=1}^{m} R_i log(1-c_i{}^{\alpha_0})\right]\right]$$
$$.\prod_{i=1}^{m}\int x^{\alpha_0-1}(1-x^{\alpha_0})^{\beta-1}\mu_{\widetilde{x}}(x)dx$$

We assume that $q(\theta)$ is the gamma density function with the hyperparameters a and b for the conditional posterior density for α, and the gamma density function with the hyperparameters c and d for the conditional posterior density for β. As a result, we can determine the Bayes estimators for α and β. Additionally, we use the mean of the generated samples from the posterior densities to determine the Bayes estimates of R(t).

2.4 Markov Chain Monte Carlo (MCMC)

The MCMC technique is a general simulation method for sampling from posterior distributions and computing the Bayes estimators for each parameter. The conditional posterior distributions of α and β are obtained from (8), which cannot be reduced analytically to well-known distributions. So, we use the Markov Chain Monte Carlo method to simulate the sample from the posterior distribution $\pi(\vartheta|\widetilde{w})$ to calculate the Bayes estimate of $\vartheta(\alpha,\beta)$. Using a Metropolis-Hastings, see Hastings [36]. We carry out the Metropolis-Hastings algorithm in the following steps:

Step1: Start with an initial guess value $\vartheta^{(0)}$

Step2: Set j=1 and generate a new candidate parameter value ϑ^* from proposal density $q\left(\vartheta^{(1)}|\vartheta^{(0)}\right)$.

Step3: Evaluate the acceptance probability as

$$\rho(\vartheta^*,\vartheta^{(j-1)}) = \min\left\{\frac{\pi(\vartheta^*|\widetilde{w})q\left(\vartheta^{(j-1)}|\vartheta^*\right)}{\pi\left(\vartheta^{(j-1)}|\widetilde{w}\right)q\left(\vartheta^*|\vartheta^{(j-1)}\right)},1\right\}.$$

Step4: Generate u from a uniform (0,1) distribution.

Step5: If $u \leq \rho(\vartheta^*,\vartheta^{(j-1)})$, accept the proposal and set $\vartheta^{(j)} = \vartheta^*$, else, set$\vartheta^{(j)} = \vartheta^{(j-1)}$.

Step6: Set j=j+1.

Step7: Repeat steps 2-6, N times and obtain $\vartheta^{(j)}, R^{(j)}(t), j = 1,2,\ldots,N$.

Step8: Obtain the Bayes estimates of the parameter $\vartheta(\alpha,\beta)$ and R(t) under the squared loss function as the mean of generated samples from the posterior densities, i.e.

$$\hat{\vartheta} = E_\pi(\vartheta|\widetilde{w}) = \frac{1}{N}\sum_{j=1}^{N}\vartheta^{(j)}$$
$$\hat{R}(t) = E_\pi(R(t)|\widetilde{w}) = \frac{1}{N}\sum_{j=1}^{N}R^{(j)}(t).$$

3 Simulation Study

In this section, we present some experimental results based on a Monte Carlo simulation study to compare the performance of the Bayes method based on the other four methods based on the progressive type-II censored fuzzy lifetime data. The estimators' performance has been compared based on the estimate and mean squared error (MSE). For this purpose,

$$MSE(\theta^*) = \frac{1}{M}\sum_{i=1}^{M}(\theta-\theta^*)^2,$$

in this case, M is the number of replications and θ^* is the point estimate for the unknown parameter θ. The performance of these estimates and the MSE values for each parameter have been evaluated using Monte Carlo simulations with 1000 replications for each sample size under the squared error loss function. The general fuzzy information system, shown in Fig. 1, was used to fuzzify each realization of X, see Maswadah [29], and encode the simulation data of size m with the appropriate membership functions:

$$\mu_{\tilde{x}_i}(x) = \begin{cases} 1, & x \leq h \; or \;\; x \geq mh, \text{ h=0.05} \\ \frac{x-ih}{h}, & ih < x \leq (i+1)h, \;\; i = 1(1)(m-2). \\ \frac{mh-x}{h}, & (m-1)h \leq x < mh \end{cases}$$

The progressively type-II censored samples from the KD were generated using the method proposed by Balakrishnan and Sandhu [37]. In the simulation study, we generated several data sets for samples from the KD of sizes, namely n = 25, 50, 75, and 100, with 1000 replications. These samples have been generated with the true values of three different values of α=(1.001, 2.007) and two different values of β =(2.007, 3.083) based on progressive type-II fuzzy censored

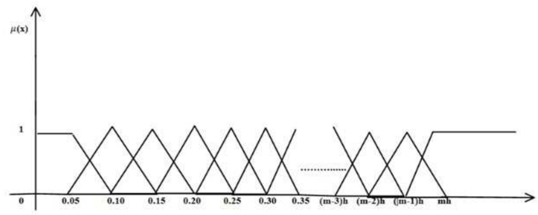

Fig. 1: Simulation data of size m is encoded using the general fuzzy information system.

data with uncensored levels m equal to [n/2] and [3n/4]. The hyperparameters (a, b, c, d) are taken for informative prior means so that exactly equal to the true values of (α; β) are taken as follows: (1.5, 3.9, 1.8, 2.2), (1.8, 2.2, 1.8, 2.2), (1.5, 3.9, 2.71, 1.9), and (1.8, 2.2, 2.71, 1.9). To compute the Bayes estimates using the MCMC algorithm, using the number of iterations N = 10000, the ML estimates for unknown parameters α and β have been used as initial values for running the MCMC algorithm and choosing the gamma distribution as a proposal distribution. Tables [1–4] provide the results of the simulation. Based on these estimations, some of the points have been summarized in the following main points:

1. The estimated MSE values for the parameter α, increase as the value of α,increases and decrease as the value of the parameter β increases, and vice versa for the parameter β.
2. The estimated MSE values decrease as the hyperparameters of the informative prior increase and vice versa.
3. For the parameters α and β, in general, the estimated MSE values based on the MCMC and the Monte Carlo Integration methods are smaller than those based on the other methods.
4. The estimated MSE values of all the parameters decrease as the sample size n increases.
5. For the reliability function, the estimated MSE values decrease as the value of lifetime t increases and decrease as the value of α and β increases.
6. The MSE values for the reliability function based on the Tierney-Kadane approximation and Monte Carlo Integration methods are smaller than those based on the other methods.

4 Data Analysis

In this section, we analyze two real data sets, representing the monthly water capacity of the Shasta Reservoir in California, USA, to illustrate the procedures developed in this paper.
The first dataset was reported by Query Monthly CDEC Senser Values [38] and included February 1991–2010.
The second dataset found in Kohansal [39], is related to the monthly water capacity of the Shasta Reservoir in California, USA, between August and December from 1975 to 2016. These datasets are good fit for the Kumaraswamy distribution. However, determining the lifetime of the Shasta Reservoir dataset could not provide an exact result. Therefore, such a dataset could be presented as imprecise numbers. Assume that the membership functions are used for the imprecision of the failure times based on fuzzy numbers $\widetilde{x}_i = (x_i, h_i)$, where $h_i = 0.05x_i$ for $i = 1(1)n$, with the membership functions:

$$\mu_{\tilde{x}_i}(x) = \begin{cases} \frac{x-(x_i-h_i)}{h_i}, & x_i - h_i \leq x \leq x_i \quad i = 1,..,n. \\ \frac{(x_i+h_i)-x}{h_i}, & x_i \leq x \leq x_i + h_i \end{cases} \tag{15}$$

Table (1): The estimates and MSEs values for the parameter α, based on Bayes methods for $\alpha = 1,2$ and $\beta = 2,3$ based on the progressive type-II censored samples at $(m = n/2, 3n/4)$.

Methods				Exact Bayes		Monte-Carlo Integration		Markov Chain Monte Carlo		Tierney-Kadane	
n	m	α	β	$\hat{\alpha}_{EB}$	MSE	$\hat{\alpha}_{MCI}$	MSE	$\hat{\alpha}_{MCMC}$	MSE	$\hat{\alpha}_{TK}$	MSE
25	12	1.001	2.007	0.8424	0.0332	0.9489	0.0271	0.9520	0.0206	0.8701	0.0205
			3.083	0.9102	0.0232	1.0116	0.0217	0.9701	0.0162	0.9406	0.0095
		2.007	2.007	1.5028	0.2890	1.9312	0.0851	1.8684	0.0882	1.5174	0.2787
			3.083	1.5483	0.2534	1.9520	0.0662	1.9074	0.0725	1.5237	0.2625
	18	1.001	2.007	0.8792	0.0256	0.9514	0.0250	0.9701	0.0162	1.0169	0.0146
			3.083	0.9383	0.0184	1.0028	0.0217	0.9702	0.0150	0.9561	0.0072
		2.007	2.007	1.6016	0.2184	1.9380	0.0782	1.8947	0.0757	1.5950	0.2282
			3.083	1.6016	0.2126	1.9581	0.0596	1.9287	0.0601	1.5690	0.2348
50	25	1.001	2.007	0.9770	0.0093	0.9576	0.0167	0.9754	0.0119	0.9906	0.0034
			3.083	1.0305	0.0064	0.9938	0.0139	0.9862	0.0086	1.0497	0.0070
		2.007	2.007	1.7447	0.1222	1.9619	0.0494	1.9315	0.0489	1.6669	0.1489
			3.083	1.7351	0.1139	1.9728	0.0376	1.9588	0.0420	1.6524	0.1492
	37	1.001	2.007	0.9770	0.0093	0.9651	0.0156	0.9750	0.0106	1.1129	0.0246
			3.083	1.0225	0.0088	1.0063	0.0124	0.9797	0.0083	1.0658	0.0094
		2.007	2.007	1.8064	0.1021	1.9732	0.0483	1.9567	0.0423	1.8098	0.0997
			3.083	1.7874	0.0933	1.9825	0.0358	1.9793	0.0324	1.7806	0.0954
75	37	1.001	2.007	1.0352	0.0101	0.9636	0.0133	0.9846	0.0084	1.0500	0.0059
			3.083	1.0844	0.0081	1.0043	0.0103	1.0031	0.0071	1.1007	0.0143
		2.007	2.007	1.8348	0.0816	1.9798	0.0397	1.9724	0.0345	1.7350	0.1121
			3.083	1.8224	0.0701	1.9869	0.0298	1.9771	0.0281	1.7235	0.1075
	56	1.001	2.007	1.0352	0.0101	0.9642	0.0112	0.9882	0.0070	1.1285	0.0255
			3.083	1.0844	0.0807	1.0126	0.0787	0.9960	0.0054	1.1028	0.0152
		2.007	2.007	1.8540	0.0691	1.9784	0.0338	1.9630	0.0334	1.8611	0.0674
			3.083	1.8408	0.0623	1.9854	0.0247	1.9861	0.0214	1.8417	0.0618
100	50	1.001	2.007	1.0621	0.0124	0.9583	0.0105	0.9922	0.0062	1.0865	0.0109
			3.083	1.1128	0.0114	0.9954	0.0083	1.0026	0.0046	1.0964	0.0164
		2.007	2.007	1.8748	0.0671	1.9742	0.0312	1.9610	0.0311	1.7750	0.0926
			3.083	1.8634	0.0563	1.9813	0.0237	1.9903	0.0222	1.7670	0.0863
	75	1.001	2.007	1.1306	0.0173	0.9627	0.0086	0.9924	0.0055	1.1459	0.0291
			3.083	1.0874	0.0086	1.0076	0.0061	1.0018	0.0043	1.1287	0.0208
		2.007	2.007	1.9083	0.0501	1.9798	0.0242	1.9774	0.0236	1.9183	0.0499
			3.083	1.8940	0.0437	1.9864	0.0177	1.9835	0.0174	1.9002	0.0435

Table (2): The estimates and MSEs values for the parameter β, based on Bayes methods for $\alpha = 1,2$ and $\beta = 2,3$ based on the progressive type-II censored samples at $(m = n/2, 3n/4)$.

Methods				Exact Bayes		Monte-Carlo Integration		Markov Chain Monte Carlo		Tierney-Kadane	
n	m	α	β	$\hat{\beta}_{EB}$	MSE	$\hat{\beta}_{MCI}$	MSE	$\hat{\beta}_{MCMC}$	MSE	$\hat{\beta}_{TK}$	MSE
25	12	1.001	2.007	1.4297	0.3956	1.7774	0.1959	1.6217	0.2239	1.6243	0.2536
			3.083	2.1468	0.9638	2.6935	0.3713	2.4242	0.5648	2.1294	1.0723
		2.007	2.007	1.4085	0.4001	1.7778	0.1944	1.6349	0.2165	1.8408	0.1140
			3.083	1.9254	1.4317	2.6566	0.4450	2.4305	0.5535	2.5605	0.3996
	18	1.001	2.007	1.6217	0.2370	1.8547	0.1535	1.7627	0.1368	1.7936	0.1883
			3.083	2.4335	0.5470	2.8135	0.3061	2.6174	0.3690	2.4125	0.7182
		2.007	2.007	1.5431	0.2816	1.8570	0.1516	1.7399	0.1548	1.9170	0.1194
			3.083	2.1632	0.9717	2.7900	0.3396	2.5859	0.3964	2.7118	0.3124
50	25	1.001	2.007	1.7837	0.1543	1.9155	0.1155	1.8194	0.1118	1.9068	0.1683
			3.083	2.7109	0.2295	2.8715	0.2132	2.7244	0.2823	2.5779	0.5570
		2.007	2.007	1.6478	0.2150	1.9185	0.1142	1.8137	0.1244	2.0353	0.1304
			3.083	2.3328	0.7184	2.8949	0.2544	2.7264	0.2827	2.8871	0.2464
	37	1.001	2.007	1.9309	0.1233	1.9483	0.0942	1.8761	0.0880	2.0497	0.1762
			3.083	2.9310	0.1963	2.9564	0.1915	2.8313	0.2039	2.8898	0.3898
		2.007	2.007	1.7562	0.1623	1.9522	0.0932	1.8899	0.0817	2.0871	0.1446
			3.083	2.5566	0.4538	2.9603	0.2090	2.8417	0.1952	3.0394	0.2585
75	37	1.001	2.007	1.9452	0.1251	1.9464	0.0940	1.8801	0.0892	2.0337	0.1718
			3.083	2.4973	0.5611	2.9525	0.1915	2.8397	0.2112	2.8857	0.3772
		2.007	2.007	1.7626	0.1627	1.9507	0.0932	1.8702	0.0897	2.0768	0.1429
			3.083	2.5692	0.4448	2.9579	0.2091	2.8440	0.1962	3.0052	0.2543
	56	1.001	2.007	2.0237	0.1109	1.9674	0.0650	1.9032	0.0661	2.0825	0.1509
			3.083	3.1269	0.1870	3.0592	0.1790	2.9273	0.1316	3.0231	0.3426
		2.007	2.007	1.8146	0.1280	1.9725	0.0643	1.9313	0.0636	2.0895	0.1258
			3.083	2.6921	0.3284	3.0024	0.1463	2.9031	0.1391	3.1009	0.2485
100	50	1.001	2.007	2.0217	0.1182	1.9557	0.0646	1.9074	0.0651	2.0849	0.1612
			3.083	2.6587	0.4029	3.0059	0.1743	2.8798	0.1754	3.0492	0.3126
		2.007	2.007	1.8128	0.1358	1.9611	0.0640	1.9139	0.0686	2.0660	0.1236
			3.083	2.6780	0.3483	2.9827	0.1462	2.9181	0.1602	3.0285	0.2469
	75	1.001	2.007	2.0983	0.1092	1.9952	0.0582	1.9410	0.0469	2.1286	0.1423
			3.083	3.2729	0.2141	3.0313	0.1247	2.9593	0.1057	3.1447	0.3050
		2.007	2.007	1.8774	0.0983	2.0008	0.0578	1.9399	0.0468	2.1127	0.1159
			3.083	2.8104	0.2402	3.0508	0.1315	2.9583	0.1103	3.1686	0.2387

Table (3): The estimates and MSEs values for the reliability, based on Bayes methods for $\alpha = 1,2$ and $\beta = 2,3$at $t = 0.5$ based on the progressive type-II censored samples at $(m = n/2, 3n/4)$.

Methods				Exact Bayes		Monte-Carlo Integration		Markov Chain Monte Carlo		Tierney-Kadane	
n	m	α	β	$\hat{R}_{EB}$	MSE	$\hat{R}_{MCI}$	MSE	$\hat{R}_{MCMC}$	MSE	$\hat{R}_{TK}$	MSE
25	12	1.001	2.007	0.3265	0.0093	0.3039	0.0086	0.3590	0.0150	0.2276	0.0031
			3.083	0.2083	0.0105	0.1753	0.0049	0.2279	0.0141	0.1310	0.0011
		2.007	2.007	0.5706	0.0041	0.6062	0.0059	0.5870	0.0038	0.4511	0.0172
			3.083	0.4592	0.0064	0.4743	0.0082	0.4712	0.0067	0.3276	0.0113
	18	1.001	2.007	0.2969	0.0054	0.2876	0.0059	0.3159	0.0071	0.2295	0.0031
			3.083	0.1788	0.0057	0.1631	0.0039	0.1963	0.0078	0.1291	0.0012
		2.007	2.007	0.5554	0.0040	0.5930	0.0043	0.5612	0.0034	0.4706	0.0132
			3.083	0.4353	0.0045	0.4572	0.0058	0.4417	0.0040	0.3394	0.0094
50	25	1.001	2.007	0.2924	0.0045	0.2746	0.0040	0.3132	0.0065	0.2053	0.0041
			3.083	0.1693	0.0042	0.1548	0.0024	0.1866	0.0061	0.1048	0.0010
		2.007	2.007	0.5680	0.0030	0.5822	0.0031	0.5751	0.0028	0.4643	0.0131
			3.083	0.4409	0.0038	0.4432	0.0041	0.4486	0.0036	0.3342	0.0092
	37	1.001	2.007	0.2757	0.0029	0.2678	0.0031	0.2882	0.0035	0.2199	0.0029
			3.083	0.1514	0.0024	0.1468	0.0019	0.1668	0.0035	0.1101	0.0009
		2.007	2.007	0.5608	0.0027	0.5769	0.0025	0.5584	0.0023	0.4961	0.0076
			3.083	0.4269	0.0028	0.4352	0.0032	0.4318	0.0026	0.3585	0.0057
75	37	1.001	2.007	0.2790	0.0031	0.2680	0.0031	0.2924	0.0035	0.2041	0.0040
			3.083	0.1535	0.0026	0.1471	0.0019	0.1699	0.0039	0.0981	0.0012
		2.007	2.007	0.5670	0.0024	0.5769	0.0025	0.5739	0.0020	0.4758	0.0103
			3.083	0.4330	0.0028	0.4353	0.0032	0.4416	0.0029	0.3422	0.0074
	56	1.001	2.007	0.2681	0.0021	0.2629	0.0021	0.2752	0.0021	0.2217	0.0025
			3.083	0.1420	0.0015	0.1373	0.0014	0.1504	0.0019	0.1062	0.0009
		2.007	2.007	0.5603	0.0019	0.5733	0.0017	0.5619	0.0018	0.5086	0.0052
			3.083	0.4224	0.0020	0.4295	0.0022	0.4266	0.0020	0.3676	0.0042
100	50	1.001	2.007	0.2723	0.0023	0.2648	0.0022	0.2825	0.0026	0.2082	0.0034
			3.083	0.1453	0.0018	0.1419	0.0015	0.1555	0.0024	0.1310	0.0011
		2.007	2.007	0.5658	0.0019	0.5749	0.0017	0.5676	0.0018	0.4873	0.0078
			3.083	0.4284	0.0022	0.4316	0.0023	0.4342	0.0022	0.3510	0.0059
	75	1.001	2.007	0.2636	0.0016	0.2578	0.0018	0.2675	0.0016	0.2241	0.0020
			3.083	0.1358	0.0011	0.1384	0.0010	0.1412	0.0012	0.1291	0.0012
		2.007	2.007	0.5625	0.0016	0.5688	0.0015	0.5607	0.0015	0.5191	0.0037
			3.083	0.4213	0.0016	0.4237	0.0019	0.4242	0.0015	0.3753	0.0031

Table (4): The estimates and MSEs values for the reliability, based on Bayes methods for $\alpha = 1,2$ and $\beta = 2,3$at $t = 0.7$ based on the progressive type-II censored samples at $(m = n/2, 3n/4)$.

Methods				Exact Bayes		Monte-Carlo Integration		Markov Chain Monte Carlo		Tierney-Kadane	
n	m	α	β	$\hat{R}_{EB}$	MSE	$\hat{R}_{MCI}$	MSE	$\hat{R}_{MCMC}$	MSE	$\hat{R}_{TK}$	MSE
25	12	1.001	2.007	0.1730	0.0087	0.1311	0.0048	0.2089	0.0162	0.0965	0.0010
			3.083	0.0884	0.0050	0.0496	0.0011	0.1053	0.0073	0.0408	4.5E-04
		2.007	2.007	0.3254	0.0072	0.3145	0.0086	0.3650	0.0141	0.2205	0.0043
			3.083	0.2156	0.0103	0.1795	0.0063	0.2372	0.0144	0.1222	0.0013
	18	1.001	2.007	0.1443	0.0045	0.1186	0.0030	0.1667	0.0075	0.0898	9.5E-04
			3.083	0.0667	0.0024	0.0440	0.0009	0.0803	0.0037	0.0358	3.1E-04
		2.007	2.007	0.3010	0.0046	0.2981	0.0059	0.3259	0.0072	0.2198	0.0044
			3.083	0.1866	0.0057	0.1642	0.0041	0.2027	0.0075	0.1153	0.0013
50	25	1.001	2.007	0.1344	0.0034	0.1088	0.0019	0.1562	0.0058	0.0713	9.9E-04
			3.083	0.0573	0.0015	0.0394	0.0005	0.0699	0.0025	0.0240	1.1E-04
		2.007	2.007	0.3000	0.0041	0.2848	0.0040	0.3228	0.0063	0.2072	0.0049
			3.083	0.1788	0.0044	0.1521	0.0026	0.1944	0.0059	0.1071	0.0013
	37	1.001	2.007	0.1184	0.0019	0.1038	0.0014	0.1323	0.0028	0.0739	9.0E-04
			3.083	0.0457	0.0008	0.0360	0.0004	0.0558	0.0013	0.0234	1.1E-04
		2.007	2.007	0.2847	0.0028	0.2782	0.0032	0.2950	0.0031	0.2189	0.0037
			3.083	0.1603	0.0025	0.1454	0.0020	0.1734	0.0034	0.1111	0.0011
75	37	1.001	2.007	0.1198	0.0020	0.1040	0.0014	0.1342	0.0031	0.0661	0.0011
			3.083	0.0462	0.0008	0.0361	0.0004	0.0567	0.0014	0.0195	1.1E-04
		2.007	2.007	0.2886	0.0030	0.2782	0.0032	0.3072	0.0040	0.2074	0.0046
			3.083	0.1633	0.0028	0.1454	0.0020	0.1784	0.0039	0.1047	0.0013
	56	1.001	2.007	0.1100	0.0013	0.0998	0.0009	0.1178	0.0015	0.0720	8.7E-04
			3.083	0.0393	0.0004	0.0319	0.0002	0.0447	0.0006	0.0204	1.0E-04
		2.007	2.007	0.2778	0.0020	0.2732	0.0021	0.2866	0.0022	0.2239	0.0028
			3.083	0.1509	0.0016	0.1400	0.0013	0.1599	0.0020	0.1116	0.0010
100	50	1.001	2.007	0.1125	0.0014	0.1011	0.0010	0.1227	0.0019	0.0657	0.0011
			3.083	0.0406	0.0005	0.0337	0.0003	0.0473	0.0008	0.0178	1.2E-04
		2.007	2.007	0.2823	0.0023	0.2750	0.0022	0.2939	0.0028	0.2123	0.0038
			3.083	0.1550	0.0019	0.1416	0.0014	0.1668	0.0026	0.1063	0.0011
	75	1.001	2.007	0.1043	0.0009	0.0964	0.0008	0.1096	0.0010	0.0710	7.9E-04
			3.083	0.0350	0.0003	0.0319	0.0002	0.0388	0.0003	0.0187	9.8E-05
		2.007	2.007	0.2736	0.0016	0.2681	0.0018	0.2795	0.0016	0.2261	0.0024
			3.083	0.1448	0.0012	0.1355	0.0011	0.1524	0.0014	0.1118	0.0008

Table (5): The estimates ($\hat{\theta}_o$)and MSEs values for the parameters α, β, and $R(t)$ at t=0.8, 0.82 based on the Bayes methods under progressive type-II censoring from the first Shasta Reservoir dataset.

The first Shasta Reservoir dataset									
Methods		Exact Bayes		Monte-Carlo Integration		Markov Chain Monte Carlo		Tierney-Kadane	
m	*Par.*	$\hat{\theta}_{EB}$	MSE	$\hat{\theta}_{MCI}$	MSE	$\hat{\theta}_{MCMC}$	MSE	$\hat{\theta}_{TK}$	MSE
n/2	α	4.4403	6.19E-01	4.1138	2.12E-01	4.1748	2.72E-01	4.3123	4.14E-01
	β	2.4314	5.89E-01	1.9990	1.12E-01	2.0910	1.82E-01	2.3860	5.21E-01
3n/4	α	4.4089	1.40E-01	4.2572	4.95E-02	4.2799	6.01E-02	4.3965	1.39E-02
	β	2.7971	1.52E-01	2.5602	2.36E-02	2.5859	3.21E-02	2.7395	1.11E-01
n/2	$R(t1)$	0.3382	1.60E-03	0.3110	4.52E-03	0.3322	2.12E-03	0.2180	2.65E-02
	$R(t2)$	0.2884	1.92E-03	0.2661	4.37E-03	0.2825	2.47E-03	0.1773	2.40E-02
3n/4	$R(t1)$	0.2828	4.96E-06	0.2630	4.81E-04	0.2834	2.50E-06	0.2052	6.36E-03
	$R(t2)$	0.2349	1.13E-05	0.2174	4.34E-04	0.2356	6.95E-06	0.1645	5.43E-03

Assuming that the informative prior with hyper-parameter is a = 36, b = 8, c = 6, and d = 2, we compute the Bayes estimates in the first dataset. For the reliability characteristic, we use times t = 0.80 and 0.82. Additionally, in the second dataset, we assume that the hyper-parameter informative prior is a = 12, b = 4, c = 7, and d = 2, and we take t = 0.52 and 0.61 for the reliability characteristic. In the MCMC algorithm, we generate 10,000 samples from the posterior densities, and we consider the MLEs of the parameters as initial values of α, β. Also, we choose the gamma distribution as a proposal distribution. To have low autocorrelation between the generated samples.

We estimated the unknown parameters α, β, and $R(t)$ using the above methods. The results listed in Tables [5, 6] showed that, for both parameters α, β, the Bayes estimators based on Markov Chain Monte Carlo and Monte Carlo Integration techniques had the smallest MSE values when compared to the estimators based on the exact Bayes and Tierney-Kadane approximation methods. Also, the reliability estimators based on the Markov Chain Monte Carlo and exact Bayes had the smallest MSE values compared to the estimators based on the other approaches. Additionally, we observe that the reliability estimated by the estimating methods decreases with time t.
Plotting of the MCMC estimates for α, β, and $R(t)$ under a progressive type-II censoring scheme with uncensored levels m equal to [3n/4] based on fuzzy lifetime data. Both the histograms of estimates and the convergence of estimates graph are displayed in Figs. (2, 3).

In Fig. 2, the trace plot shows the iterations for posterior densities of α, β, and $R(t)$ with time t = 0.80 and 0.82, respectively. All the trace plots are scattered around the mean and converge very well. Furthermore, in the histogram plots of the marginal posterior density estimate for α, β, and $R(t)$, we observe that the data distribution for α is almost symmetrical. The data distributions for β, and $R(t)$ are positively quite skewed with a right tail.

In Fig. 3, the trace plot shows the iterations for posterior densities of α, β, and $R(t)$ with time t = 0.52 and 0.61, respectively. All the trace plots are scattered around the mean and converge very well. Also, the histogram plots of the marginal posterior density estimate for α, β, and $R(t)$ show that all the data distributions are almost symmetrical.

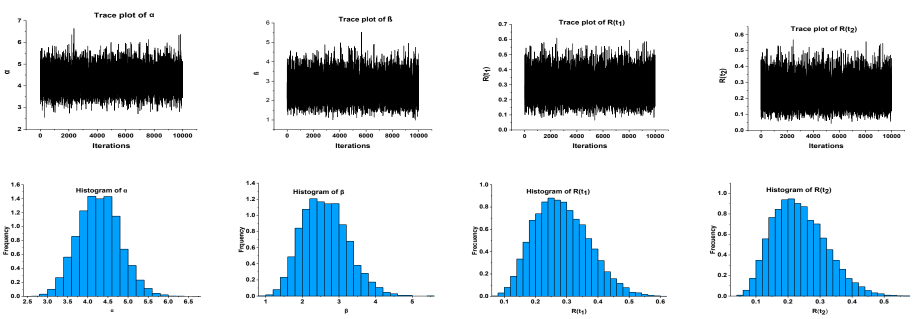

Fig. 2: MCMC plots for the first Shasta Reservoir dataset were established for the parameters α, β, and $R(t)$ with $t = (0.80, 0.82)$.

Table (6): The estimates $(\hat{\theta}_o)$ and MSEs values for the parameters α, β, and $R(t)$ at $t = 0.52, 0.61$ based on the Bayes methods under progressive type-II censoring from the first Shasta Reservoir dataset.

The second Shasta Reservoir dataset									
Methods		Exact Bayes		Monte-Carlo Integration		Markov Chain Monte Carlo		Tierney-Kadane	
m	*Par.*	θ_{EB}	MSE	θ_{MCI}	MSE	θ_{MCMC}	MSE	θ_{TK}	MSE
n/2	α	2.8524	3.50E-02	2.7202	3.02E-03	2.7311	4.34E-03	2.8268	2.61E-02
	β	3.0747	1.83E-01	2.8260	3.22E-02	2.8675	4.88E-02	3.0516	1.64E-01
3n/4	α	3.0277	7.81E-03	2.9553	2.55E-04	2.9651	6.63E-04	3.0025	4.0E-03
	β	3.0646	5.03E-02	2.9498	1.20E-02	2.9780	1.89E-02	3.0291	3.56E-02
n/2	$R(t1)$	0.6014	2.18E-07	0.5806	4.16E-04	0.6013	1.05E-07	0.5044	9.33E-03
	$R(t2)$	0.4323	3.46E-05	0.4144	5.68E-04	0.4318	4.14E-05	0.3324	1.12E-02
3n/4	$R(t1)$	0.6374	4.32E-07	0.6271	1.20E-04	0.6375	3.29E-07	0.5835	2.98E-03
	$R(t2)$	0.4667	5.99E-06	0.4557	1.82E-04	0.4666	6.52E-06	0.4051	4.10E-03

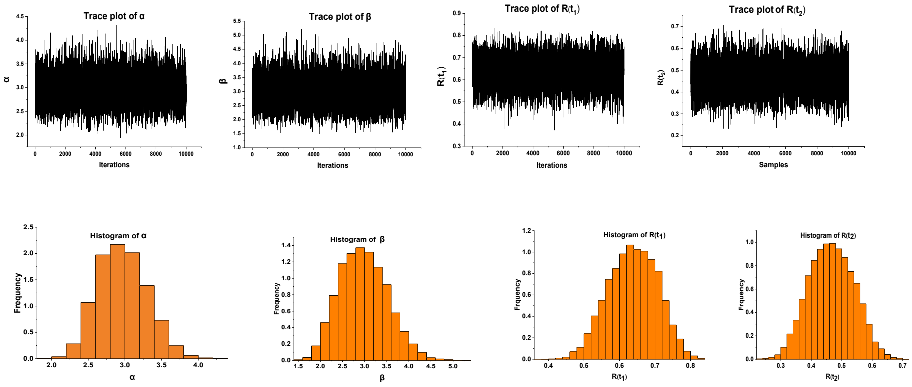

Fig. 3: MCMC plots for the first Shasta Reservoir dataset were established for the parameters α, β, and $R(t)$ with $t = (0.52, 0.61)$.

5 Conclusions

This paper provides the Bayesian inference for the Kumaraswamy distribution based on the progressive type-II censoring fuzzy data. We obtained the MLE for the unknown parameters of the Kumaraswamy distribution. Under the symmetric (squared error) loss function, the MSE values based on the Bayes method for the parameters α, β, and $R(t)$ have been compared to the exact Bayes estimators using the Monte Carlo Integration, Markov Chain Monte Carlo, and Tierney-Kadane approximation methods. In comparison to the estimators of α, β based on the exact Bayes and Tierney-Kadane approximation methods, the simulation results showed that the Bayes estimators for the parameters α, β based on Markov Chain Monte Carlo and Monte Carlo Integration methods have the minimum MSE values.
We also found that the reliability estimators based on the Monte Carlo Integration and Tierney-Kadane approximation methods have the smallest MSE values compared to the estimators based on the other methods. Numerical examples were conducted to examine and compare the performance of the proposed methods. Also, we recommend using Bayes estimators obtained by approximation Bayes methods under the squared loss function to provide the best estimate of the parameters and the reliability function of the Kumaraswamy distribution.

Appendix A:

$$H(\alpha,\beta) = \frac{1}{n}[(m+a-1)log\alpha + (m+c-1)log\beta - \alpha b - \beta[d - \sum_{i=1}^{m} R_i log(1-c_{(i)}^{\alpha})] + \sum_{i=1}^{m} log\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx]$$

Therefore, at $\hat{\alpha}$ and $\hat{\beta}$ can be obtained by solving the following two equations:

$$H_1 = \frac{\partial H(\alpha,\beta)}{\partial\alpha} = \frac{1}{n}\left[\frac{m+a-1}{\alpha} - b - \beta\sum_{i=1}^{m} R_i \frac{c_{(i)}^{\alpha} logc_{(i)}}{1-c_{(i)}^{\alpha}} + \sum_{i=1}^{m}\frac{\int[1-(\beta-1)x^{\alpha}(1-x^{\alpha})^{-1}]x^{\alpha-1}logx(1-x^{\alpha})^{(\beta-1)}\mu_{\tilde{x}}(x)dx}{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx}\right],$$

$$H_2 = \frac{\partial H(\alpha,\beta)}{\partial\beta} = \frac{1}{n}\left[\frac{m+c-1}{\beta} - d + \sum_{i=1}^{m} R_i log(1-c_{(i)}^{\alpha}) + \sum_{i=1}^{m}\frac{\int x^{\alpha-1}log(1-x^{\alpha})(1-x^{\alpha})^{(\beta-1)}\mu_{\tilde{x}_i}(x)dx}{\int x^{\alpha-1}(1-x^{\alpha})^{(\beta-1)}\mu_{\tilde{x}}(x)dx}\right]$$

The second derivative for H and H^* will be obtained as follows:

In the following, we will derive the second derivatives for H and H^*

$$H_{11}=\frac{\partial^2 H(\alpha,\beta)}{\partial\alpha^2}=\frac{1}{n}\left[-\frac{m+a-1}{\alpha^2}-\beta\sum_{i=1}^{m}R_i[c_{(i)}^{\alpha}(logc_{(i)})^2(1-c_{(i)}^{\alpha})^{-1}+(c_{(i)}^{\alpha}logc_{(i)})^2(1-c_{(i)}^{\alpha})^{-2}]\right.$$
$$+\sum_{i=1}^{m}\left[\frac{\int x^{\alpha-1}(logx)^2(1-x^{\alpha})^{(\beta-1)}I_{11}\mu_{\tilde{x}}(x)dx}{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx}-\frac{\int(\beta-1)x^{2\alpha-1}(logx)^2(1-x^{\alpha})^{(\beta-2)}I_{11}\mu_{\tilde{x}}(x)dx}{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx}\right]$$
$$\left.-\sum_{i=1}^{m}\frac{\int(\beta-1)x^{2\alpha-1}(logx)^2(1-x^{\alpha})^{(\beta-2)}I_{22}\mu_{\tilde{x}}(x)dx}{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx}-\sum_{i=1}^{m}\left[\frac{\int[1-(\beta-1)x^{\alpha}(1-x^{\alpha})^{-1}]x^{\alpha-1}logx(1-x^{\alpha})^{(\beta-1)}\mu_{\tilde{x}}(x)dx}{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx}\right]^2\right]$$

where $I_{11}=1-(\beta-1)x^{\alpha}(1-x^{\alpha})^{-1}$
$I_{22}=1+x^{\alpha}(1-x^{\alpha})^{-1}$.

$$H_{22}=\frac{\partial^2 H(\alpha,\beta)}{\partial\beta^2}=\frac{1}{n}\left[-\frac{m+c-1}{\beta^2}+\sum_{i=1}^{n}\frac{\int x^{\alpha-1}(1-x^{\alpha})^{(\beta-1)}(log(1-x^{\alpha}))^2\mu_{\tilde{x}}(x)dx}{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx}\right.$$
$$\left.-\sum_{i=1}^{n}\left[\frac{\int x^{\alpha-1}(1-x^{\alpha})^{(\beta-1)}logx(1-x^{\alpha})\mu_{\tilde{x}}(x)dx}{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx}\right]^2\right],$$

$$H_{12}=\frac{\partial^2 H(\alpha,\beta)}{\partial\alpha\partial\beta}=\frac{1}{n}\left[-\sum_{i=1}^{m}R_i\frac{c_{(i)}^{\alpha}logc_{(i)}}{1-c_{(i)}^{\alpha}}+\sum_{i=1}^{n}\frac{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}log(1-x^{\alpha})logx\mu_{\tilde{x}}(x)dx}{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx}\right.$$
$$-\sum_{i=1}^{n}\frac{\int x^{2\alpha-1}logx(1-x^{\alpha})^{\beta-2}\mu_{\tilde{x}}(x)dx}{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx}-\sum_{i=1}^{n}\frac{\int(\beta-1)x^{2\alpha-1}(1-x^{\alpha})^{\beta-2}log(1-x^{\alpha})logx\mu_{\tilde{x}}(x)dx}{\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx}$$
$$\left.-\sum_{i=1}^{m}\left[\frac{\int logxI_{33}\mu_{\tilde{x}}(x)dx\int x^{\alpha-1}(1-x^{\alpha})^{(\beta-1)}log(1-x^{\alpha})\mu_{\tilde{x}}(x)dx}{\left(\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx\right)^2}\right]\right],$$

$$I_{33}=[1-(\beta-1)x^{\alpha}(1-x^{\alpha})^{-1}]x^{\alpha-1}(1-x^{\alpha})^{(\beta-1)}.$$

$$H^*(\alpha,\beta)=\frac{1}{n}[\log(g(\alpha,\beta))+Q(\alpha,\beta)]$$
$$\text{for } g(\alpha,\beta)=\alpha\ \ then,$$
$$H^*(\alpha,\beta)=\frac{1}{n}[log\alpha+Q(\alpha,\beta)]$$

Similarly, for

$$H^*(\alpha,\beta)=\frac{1}{n}[(m+a)log\alpha+(m+c-1)log\beta-\alpha b-\beta[d-\sum_{i=1}^{m}R_ilog\left(1-c_{(i)}^{\alpha}\right)]$$
$$+\sum_{i=1}^{m}log\int x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\mu_{\tilde{x}}(x)dx]$$

Similarly, the derivatives of $H^*(\alpha,\beta)$ respectively. Finally, we compute the Bayes estimator of the reliability function R(t) under the square loss function. In this case, $g(\alpha,\beta)=(1-t^{\alpha})^{\beta}$ then,

$$H_{R(t)}^*(\alpha,\beta)=H(\alpha,\beta)+\frac{\beta log(1-t^{\alpha})}{n}.$$

Now for the reliability estimator, we compute $(\hat{\alpha}^*,\hat{\beta}^*)$ by solving the following two nonlinear equations:

$$\frac{\partial H_{R(t)}^*}{\partial\alpha}=\frac{\partial H}{\partial\alpha}-\frac{\beta t^{\alpha}logt}{n(1-t^{\alpha})},$$
$$\frac{\partial H_{R(t)}^*}{\partial\beta}=\frac{\partial H}{\partial\beta}+\frac{log(1-t^{\alpha})}{n}$$

and, from the second derivative of $H_{R(t)}^*(\alpha,\beta)$, the determinant of the negative of the inverse Hessian of $H_{R(t)}^*(\alpha,\beta)$ at $(\hat{\alpha}^*,\hat{\beta}^*)$ is given by

$$\det\Sigma_{R(t)}^*=(H_{R(t)11}^* * H_{R(t)22}^*-{H_{R(t)12}^*}^2)^{-1},$$

where

$$H_{R(t)11}^*=\frac{\partial^2 H_{R(t)}}{\partial\alpha^2}-\frac{\beta t^{\alpha}(logt)^2}{n(1-t^{\alpha})}[1+t^{\alpha}(1-t^{\alpha})^{-1}]$$
$$H_{R(t)12}^*=H_{R(t)21}^*=\frac{\partial^2 H_{R(t)}}{\partial\alpha\partial\beta}-\frac{t^{\alpha}logt}{n(1-t^{\alpha})}$$
$$H_{R(t)22}^*=0.$$

Conflicts of Interest

The author declares that there are no conflicts of interest.

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