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New Modification of Robust Ridge Regression Estimator

Ehab A. Mahmood[1,] and E. Falih Fadel[2]*

[1] Department of Banking and Finance Sciences, College of Administration and Economics, University of Babylon, Babil, Iraq.
[2] Department of Business Administration, College of Administration and Economics, University of Babylon, Babil, Iraq.

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Abstract: Multicollinearity problem occurs in the multiple linear regression model when an independent variable is correlated with one or more of the other independent variables. This problem breaks the assumption of the ordinary least squares OLS method, so in this case we cannot use it to estimate the model coefficients. The ridge regression method is an alternative method used to deal with the multicollinearity problem to estimate parameters of the multiple linear regression model. However, this method causes misleading inferences when the data have leverage points. So, many robust methods have been proposed to deal with the multicollinearity problem and leverage points. Unfortunately, to date this problem still exists and researchers try to propose new estimators. This motivated us to propose a robust method to estimate the parameters of the multiple linear regression model to deal with leverage points and multicollinearity problems simultaneously. In this paper, we propose new modification of three formulas of ridge parameter (k) based on MM-estimator. The performance is evaluated with some other available estimators using the bias and the mean square error MSE criteria. Simulation results show that the robust proposed estimators outperform other considered estimators. Moreover, the ridgemed-MM is the best estimator at different percentages of leverage points and degree of correlation. Finally, the results of real-life examples show that the ridgemed-MM is the best estimator among the other considered estimators.

Keywords: Ridge regression, multicollinearity, robust estimation; leverage points, MM-estimator.

1. Introduction

The multiple linear regression model is one of famous models used for analyzing data in several fields of sciences. The multiple linear regression model is given by [1]

$$Y = X\beta + \varepsilon \tag{1}$$

where, Y is a vector (n*1) of dependent variable, X is a matrix (n*p) of explanatory variables, β is a vector (p*1) of model parameters, ε is a vector (n*1) of random error having multivariate normal distribution with mean vector equal to (0) and variance covariance matrix equal to ($\sigma^2 I_n$), I_n is (n*n) identity matrix. The ordinary least square OLS is usually used to estimate the model parameters, it is given as follows [2]

$$\hat{\beta}_{ols} = (X'X)^{-1}X'Y \tag{2}$$

The OLS estimator is the best linear unbiased estimator BLUE if all assumptions of the linear regression model are met. However, not all the assumptions are met in real data. One of them there is no correlation between two or more explanatory variables. The correlation leads to a square error of OLS estimator to be high; this is defined as muticollinearity problem. The multicollinearity is an important problem faced in many applications. To deal with this problem, Hoerl and Kennard [2] suggested the ridge regression estimator. The main idea is to add a small positive constant (k) to the diagonal of the ($X'X$) matrix. The ridge regression is defined as

$$\hat{\beta}_{ridge} = (X'X + kI_p)^{-1}X'Y \tag{3}$$

where, k is known as ridge or shrinkage parameter. In literature, many methods have been proposed to estimate it, interested readers are referred to [1], [3], [4], [5], [6], [7], [8], [9] among others. However, the ridge regression method is highly sensitive to the outliers and the leverage points. The existing outliers and leverage points in the data set cause to increase mean square error. To overcome this problem, several robust estimators have been proposed by different researchers. Göktas et al. [10] proposed some robust methods to estimate parameters of ridge regression model. Obadina et al. [11] compared ordinary least square, modified ridge regression and generalized Liu_Kejian to estimate parameters of linear regression with multicollinearity. Suhail et al. [12] suggested some quantile-based ridge M-estimators to parameters

*Corresponding author e-mail: a.awamrah@ju.edu.Jo

with muticollinearity problem and outliers. Yasin et al. [13] proposed modified robust ridge M- estimators for two parameters ridge regression model. Shih et al. [14] Proposed several ridge M-estimators by using the Jimichi-type ridge matrix to estimate regression coefficients. Jegede et al. [15] proposed robust Jackknife Kibria Lukman estimator based on M_estimator to handle outliers and multicollinearity together in the linear regression model. Wasim et al. [16] suggested some robust ridge M-estimators to circumstance the multicollinearity and outliers in the linear regression model. Unfortunately, to date the multicollinearity problem and leverage points are serious. In this paper, we handle the problem of multicollinearity and leverage points simultaneously. We propose new modification of the ridge estimator defined by Alkhamisi et al. [5]. We compare the proposed robust estimators with ridge regression and some other robust ridge estimators. The bias and the mean square error MSE are used as measures for comparison.

2. Materials and Methods

2.1 Ridge Regression Estimator

The ridge regression estimator is defined in equation (3). If If k = 0, the output is similar to the OLS estimator and if k is very large, the coefficients will become zero. The shrinkage parameter (k) is estimated as follows [3]

$$\hat{k}_{ols} = \frac{d * \hat{\sigma}^2_{OLS}}{\hat{\beta}'_{ols} * \hat{\beta}_{ols}} \tag{4}$$

$$\hat{\sigma}^2_{OLS} = \frac{(Y - X\hat{\beta}_{ols})'(Y - X\hat{\beta}_{ols})}{n - p} \tag{5}$$

where, d: No. of explanatory variables

p: No. of parameters.

By substituting equation (4) in in the ridge regression (equation 3), we can get the estimator. It is called as ridgeOLS.

2.2 Robust Ridge Regression Estimator

Some robust methods are combined with ridge regression to handle the problem of multicollinearity and leverage points simultaneously. The robust methods are used to estimate the shrinkage parameter (k) rather than OLS. The following methods are the well-known robust estimators.

2.2.1 M-Estimator

It is the robust method proposed to estimate parameters of linear regression model when the data have influential observations. It was proposed to minimize the objective function as follows [12]

$$\min \sum \rho\left(\frac{y_i - x'_{i\hat{\beta}}}{\hat{\sigma}}\right)$$

The estimators can be computed by differentiating the objective function with respect to β ̂ and it equals zero

$$\min \sum x_i \quad \psi\left(\frac{y_i - x'_{i\hat{\beta}}}{\hat{\sigma}}\right) = 0 \tag{6}$$

where, $\psi = \rho'$. The M-estimator is used rather than OLS in equation (4) to estimate the shrinkage parameter (k). By substituting this estimator in the ridge regression (equation 3), we can get the robust ridge estimator. It is called as ridgeM estimator.

2.2.2 Least Median of Squares LMS

It was proposed to consider minimizing the median of squared errors instead of the sum of squares error in the OLS method. IT is given as [17]

$$\min[\text{Med}(r_i^2)] = \min\left[\text{Med}\left(y_i - x'_{i\hat{\beta}}\right)^2\right] \tag{7}$$

The robust ridge estimator can be got by applying LMS rather than OLS to estimate (k). It is called as ridgeLMS estimator.

2.2.3 MM-Estimator

It is the most popular and commonly used in the robust regression field and it has high breakdown points. It is introduced by solving the following formula [17]:

$$\sum_{i=1}^{n} \omega_i \left(y_i - x'_{i\hat{\beta}} / \hat{\sigma}_r \right) x_i = 0 \tag{8}$$

where, ω_i can be chosen as Huber or bisquare weights.

The MM-estimator is used rather than OLS to get robust estimation of the shrinkage parameter (k). The robust ridge estimator is called as ridgeMM estimator.

2.3 Proposed Robust Ridge Regression Estimator

Hoerl and Kennard [3] proposed ridge regression estimator as alternative to the OLS estimator in the presence of muticollinearity problem. They proposed to add shrinkage parameter (k) to the diagonal elements of OLS estimator. However, this estimator is sensitive in the presence of outliers or leverage points. Many robust methods have proposed to handle the muticollinearity problem and leverage points. Unfortunately, to date researchers still focus on estimating the shrinkage parameter (k) to handle this problem. So, we concentrate on combining ridge regression and robust regression techniques to deal with multicollinearity problems and leverage points simultaneously. Alkhamisi et al. [5] proposed three modifications of Khalaf and Shukur [4] versions to estimate the shrinkage parameter (k) when multicollinearity problem exists in the regression model. The proposed formulas were given as follows

$$\hat{k}_{max} = \max\left[\frac{\lambda_{max}\hat{\sigma}^2_{OLS}}{(n-p)\hat{\sigma}^2_{OLS} + \lambda_{max}\hat{\beta}^2_{iOLS}}\right] \tag{9}$$

$$\hat{k}_{mean} = \mathrm{mean}\left[\frac{\lambda_{max}\hat{\sigma}^2_{OLS}}{(n-p)\hat{\sigma}^2_{OLS} + \lambda_{max}\hat{\beta}^2_{iOLS}}\right] \tag{10}$$

$$\hat{k}_{med} = \mathrm{median}\left[\frac{\lambda_{max}\hat{\sigma}^2_{OLS}}{(n-p)\hat{\sigma}^2_{OLS} + \lambda_{max}\hat{\beta}^2_{iOLS}}\right] \tag{11}$$

where, λ_{max} is the largest eigenvalue of the matrix $(X'X)$.

They explained that every one of the above estimators has good properties according to the power of correlation among explanatory variables and error distribution. However, they have not mentioned the existing leverage points in the explanatory variables. So, we propose to apply modification of their proposed estimators by relying on MM- estimator rather than OLS to estimate the shrinkage parameters (k) in the equations (9-11). The MM- estimator has high breakdown points and we expect it to be more useful to handle the multicollinearity problem and leverage points simultaneously. Our suggestion of robust ridge estimators is namely ridgemax-MM, ridgemean-MM, ridgemed-MM, respectively.

3. Results and Discussion

3.1 Simulation Study

In this section, a simulation study has been made to compare the performance of the estimators because a theoretical comparison among them is not possible. We suppose three explanatory variables, so the dependent variable (y) is generated by

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \varepsilon_i, \quad i = 1, 2, .., n \tag{12}$$

Following the articles of Kibria and Lukman [6], Göktas et al. [10], Suhail et al. [12], Yasin et al. [13], Suhail et al. [18], Suhail et al. [19], we choose the following cases: $\beta_0 = 0\ ,\ \beta_1 = \beta_2 = \beta_3 = 1$, the residuals are generated as $\varepsilon_i \sim N(0, \sigma^2)$ and the explanatory variables (x) are generated as

$$x_{ij} = z_{ij}\sqrt{(1 - r^2)} + r z_{ip}, \quad i = 1, 2, .., n;\ \ j = 1, 2,..,p \tag{13}$$

Where, $z_{ij} \sim N(0,1)$ and (r) represent the association between two predictor variables. In order to conduct a meaningful simulation, we should specify the effective properties of the estimators. Effective factors are data size (n), degree of association (r), residual variance (σ^2), percentage of leverage points. For this reason, we divided the values of the effective factors as:

- Three sample sizes are small, moderate and large (20, 60 and 100), respectively.
- Two degrees of association moderate and high (0.5, 0.9), respectively.
- The three values of residual variance are low, moderate and high (0.1, 1, and 10), respectively.
- Three percentages of leverage points clean, low and high (0%, 5%, 10%).

So, we evaluate our proposed robust ridge estimator by considering the following six scenarios:

Scenario I:

n = 20, r = 0.5, $\sigma^2 = 0.1$, 1, 10, percentage of leverage points (0%, 5%, 10%).

Scenario II:

n = 20, r = 0.9, $\sigma^2 = 0.1$, 1, 10, percentage of leverage points (0%, 5%, 10%).

Scenario III:

n = 60, r = 0.5, $\sigma^2 = 0.1$, 1, 10, percentage of leverage points (0%, 5%, 10%).

Scenario IV:

n = 60, r = 0.9, $\sigma^2 = 0.1$, 1, 10, percentage of leverage points (0%, 5%, 10%).

Scenario V:

n = 100, r = 0.5, $\sigma^2 = 0.1$, 1, 10, percentage of leverage points (0%, 5%, 10%).

Scenario VI:

n = 100, r = 0.9, $\sigma^2 = 0.1$, 1, 10, percentage of leverage points (0%, 5%, 10%).

The simulation experiments involved 3000 replications. The bias and the mean square error MSE of the estimators are chosen to be the performance criteria for the simulation study. The bias and MSE are estimated by [13], [18], [19], [20], [21].

$$\text{bias}(\hat{\beta}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\beta}_j - \beta) \tag{14}$$

$$\text{MSE}(\hat{\beta}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\beta} - \beta)^2 \tag{15}$$

The results of bias and MSE of the ridge estimators for the six scenarios are exhibited in Tables I:VI, respectively.

Scenario I: Table I gives the estimated bias and MSE for the considered estimators. Generally, the increase in error variance and percentage of leverage points adversely affects the performance of all the estimators but our robust proposed estimators are least affected. Moreover, it can be observed that the ridgemed-MM estimator outperforms other considered estimators for any percentage of contamination.

Scenario II: In this scenario, the degree of association is relatively larger than Scenario I. The results are given in Table II. As expected, the increase in the degree of multicollinearity results an increase in the bias and MSE for each of the considered estimators. However, the performance of our proposed estimators is better than other estimators, especially the ridgemed-MM estimator. It is the least estimator affected by leverage points and multicollinearity.

Scenario III: In this scenario, the sample size is relatively larger than Scenario-I while the degree of association is same. Generally, results in relatively higher bias and MSE but our suggested estimators are least affected by this situation (see Table III). Our suggested ridgemed-MM estimator still outperforming other estimators in all cases. Moreover, the ridgemed-MM estimator produces smaller MSE with respect to the Scenario-I.

Scenario IV: Although the general findings are the same, unlike Scenario II, there are cases where the MSE are increase and others are decrease by comparison with the Scenario-II (see Table IV). In this scenario, the sample size is relatively larger than Scenario II with the degree of association is same. In all the cases considered, our proposed estimators outperform the other estimators. Furthermore, the ridgemed-MM shows the best performance with regard to the bias and MSE criteria.

Scenario V: Table V gives the estimated bias and MSE for the considered estimators. In this scenario, the sample size is large (n=100) while the degree of association is same as Scenarios I and III. As the previous scenarios, the estimators are affected by increasing in error variance and percentage of leverage points, but our robust proposed estimators are least affected. Although, the MSE here is relatively smaller than MSE of scenarios I and III, we find that the ridgemed-MM estimator produces the least MSE in all cases.

Scenario VI: In this scenario, the degree of association is relatively larger than Scenario V while the sample size is same. The results are given in Table VI. the MSE of the estimators are relatively smaller than MSE of scenario V. However, the ridgemed-MM estimator is successfully estimated parameters with the least values of bias and MSE. Generally, it is noticed that the ridgemed-MM estimator is the best estimator among other considered estimators to handle the leverage points and multicollinearity problem.

Table I: Bias and MSE of ridge regression estimators (n=20, r =0.5)

			B0		B1		B2		B3	
	Leverage Points		**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**
$\sigma^2 = 0.1$	0% (clean data)	$ridge_{OLS}$	0.001	0.005	0.005	0.008	0.002	0.008	0.002	0.006
		$ridge_{M}$	0.001	0.006	0.005	0.008	0.002	0.008	0.002	0.006
		$ridge_{LMS}$	0.007	0.006	0.015	0.009	0.012	0.009	0.016	0.007
		$ridge_{MM}$	0.001	0.006	0.005	0.008	0.002	0.008	0.002	0.006
		$ridge_{max\text{-}MM}$	0.007	0.063	0.532	0.134	0.352	0.134	0.402	0.167
		$ridge_{mean\text{-}MM}$	0.003	0.015	0.124	0.023	0.124	0.023	0.167	0.032
		$ridge_{med\text{-}MM}$	0.001	0.006	0.017	0.008	0.016	0.008	0.026	0.005
	5%	$ridge_{OLS}$	0.029	0.062	0.956	0.917	0.079	0.059	0.067	0.056
		$ridge_{M}$	0.029	0.063	0.957	0.918	0.085	0.061	0.056	0.056
		$ridge_{LMS}$	0.023	0.338	0.996	0.993	0.954	0.926	0.955	0.931
		$ridge_{MM}$	0.021	0.346	0.997	0.995	0.974	0.951	0.976	0.957
		$ridge_{max\text{-}MM}$	0.016	0.268	0.987	0.974	0.794	0.641	0.813	0.668
		$ridge_{mean\text{-}MM}$	0.005	0.175	0.973	0.948	0.537	0.313	0.561	0.331
		$ridge_{med\text{-}MM}$	0.022	0.082	0.959	0.923	0.198	0.067	0.162	0.039
	10%	$ridge_{OLS}$	0.036	0.066	0.977	0.956	0.076	0.059	0.071	0.057
		$ridge_{M}$	0.036	0.068	0.977	0.956	0.084	0.062	0.056	0.059
		$ridge_{LMS}$	0.005	0.341	0.999	0.997	0.962	0.950	0.959	0.953
		$ridge_{MM}$	0.007	0.349	0.999	0.999	0.986	0.980	0.985	0.982
		$ridge_{max\text{-}MM}$	0.006	0.301	0.996	0.992	0.879	0.778	0.891	0.798
		$ridge_{mean\text{-}MM}$	0.001	0.222	0.990	0.981	0.675	0.474	0.699	0.501
		$ridge_{med\text{-}MM}$	0.030	0.086	0.980	0.961	0.205	0.069	0.172	0.043
$\sigma^2 = 1$	0% (clean data)	$ridge_{OLS}$	0.004	0.069	0.137	0.078	0.134	0.076	0.189	0.074
		$ridge_{M}$	0.004	0.070	0.144	0.080	0.141	0.078	0.197	0.077
		$ridge_{LMS}$	0.001	0.118	0.330	0.190	0.330	0.188	0.385	0.207
		$ridge_{MM}$	0.004	0.070	0.143	0.080	0.140	0.078	0.195	0.076
		$ridge_{max\text{-}MM}$	0.002	0.101	0.337	0.136	0.331	0.134	0.386	0.163
		$ridge_{mean\text{-}MM}$	0.004	0.060	0.136	0.057	0.111	0.058	0.203	0.067
		$ridge_{med\text{-}MM}$	0.004	0.059	0.069	0.060	0.068	0.058	0.122	0.043
	5%	$ridge_{OLS}$	0.024	0.153	0.967	0.938	0.276	0.163	0.247	0.141
		$ridge_{M}$	0.028	0.155	0.967	0.939	0.286	0.170	0.260	0.148
		$ridge_{LMS}$	0.001	0.359	0.990	0.982	0.813	0.752	0.820	0.758
		$ridge_{MM}$	0.004	0.342	0.989	0.979	0.770	0.731	0.761	0.728
		$ridge_{max\text{-}MM}$	0.009	0.272	0.979	0.961	0.643	0.451	0.660	0.468
		$ridge_{mean\text{-}MM}$	0.020	0.173	0.961	0.928	0.369	0.192	0.406	0.197
		$ridge_{med\text{-}MM}$	0.034	0.126	0.962	0.929	0.164	0.078	0.138	0.046
	10%	$ridge_{OLS}$	0.012	0.156	0.981	0.963	0.275	0.168	0.263	0.142
		$ridge_{M}$	0.015	0.159	0.981	0.964	0.288	0.176	0.280	0.151
		$ridge_{LMS}$	0.008	0.302	0.994	0.988	0.776	0.710	0.780	0.710
		$ridge_{MM}$	0.014	0.269	0.990	0.982	0.679	0.634	0.669	0.626
		$ridge_{max\text{-}MM}$	0.015	0.287	0.990	0.982	0.693	0.527	0.715	0.549

		$ridge_{mean\text{-}MM}$	0.017	0.203	0.979	0.960	0.450	0.263	0.461	0.265
		$ridge_{med\text{-}MM}$	0.024	0.131	0.978	0.959	0.169	0.084	0.139	0.047
$\sigma^2 = 10$	0% (clean data)	$ridge_{OLS}$	0.028	0.758	0.817	0.707	0.814	0.701	0.843	0.732
		$ridge_{M}$	0.028	0.759	0.820	0.711	0.817	0.705	0.846	0.735
		$ridge_{LMS}$	0.030	0.784	0.870	0.777	0.867	0.773	0.887	0.797
		$ridge_{MM}$	0.028	0.758	0.819	0.710	0.816	0.704	0.845	0.734
		$ridge_{max\text{-}MM}$	0.031	0.588	0.246	0.291	0.235	0.278	0.306	0.211
		$ridge_{mean\text{-}MM}$	0.030	0.579	0.119	0.403	0.104	0.390	0.182	0.230
		$ridge_{med\text{-}MM}$	0.030	0.578	0.078	0.483	0.064	0.461	0.151	0.259
	5%	$ridge_{OLS}$	0.027	0.797	0.993	0.987	0.849	0.751	0.868	0.772
		$ridge_{M}$	0.028	0.797	0.993	0.987	0.851	0.753	0.870	0.775
		$ridge_{LMS}$	0.034	0.836	0.997	0.994	0.938	0.894	0.947	0.905
		$ridge_{MM}$	0.030	0.811	0.994	0.989	0.886	0.811	0.900	0.827
		$ridge_{max\text{-}MM}$	0.025	0.717	0.977	0.961	0.573	0.424	0.595	0.421
		$ridge_{mean\text{-}MM}$	0.015	0.650	0.966	0.946	0.304	0.278	0.307	0.207
		$ridge_{med\text{-}MM}$	0.002	0.612	0.960	0.942	0.099	0.335	0.051	0.178
	10%	$ridge_{OLS}$	0.002	0.798	0.996	0.993	0.855	0.760	0.867	0.775
		$ridge_{M}$	0.002	0.798	0.996	0.993	0.857	0.763	0.870	0.778
		$ridge_{LMS}$	0.005	0.824	0.998	0.996	0.929	0.875	0.937	0.886
		$ridge_{MM}$	0.002	0.805	0.997	0.994	0.875	0.792	0.885	0.805
		$ridge_{max\text{-}MM}$	0.005	0.751	0.989	0.981	0.650	0.510	0.658	0.508
		$ridge_{mean\text{-}MM}$	0.015	0.687	0.983	0.972	0.396	0.323	0.387	0.271
		$ridge_{med\text{-}MM}$	0.031	0.631	0.978	0.967	0.133	0.343	0.058	0.187

Table II: Bias and MSE of ridge regression estimators (n=20, r =0.9)

			B0		**B1**		**B2**		**B3**	
	Leverage Points		**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**
$\sigma^2 = 0.1$	0% (clean data)	$ridge_{OLS}$	0.002	0.006	0.007	0.032	0.006	0.032	0.011	0.030
		$ridge_{M}$	0.002	0.006	0.007	0.032	0.006	0.032	0.012	0.030
		$ridge_{LMS}$	0.002	0.006	0.012	0.027	0.012	0.027	0.031	0.025
		$ridge_{MM}$	0.002	0.006	0.007	0.032	0.006	0.032	0.012	0.030
		$ridge_{max\text{-}MM}$	0.003	0.057	0.226	0.058	0.227	0.058	0.384	0.151
		$ridge_{mean\text{-}MM}$	0.002	0.012	0.026	0.007	0.027	0.007	0.201	0.044
		$ridge_{med\text{-}MM}$	0.002	0.006	0.028	0.021	0.024	0.021	0.061	0.018
	5%	$ridge_{OLS}$	0.014	0.017	0.985	0.971	0.099	0.077	0.558	0.356
		$ridge_{M}$	0.014	0.017	0.984	0.971	0.102	0.077	0.554	0.352
		$ridge_{LMS}$	0.002	0.413	0.992	0.985	0.795	0.771	0.776	0.812
		$ridge_{MM}$	0.002	0.444	0.995	0.990	0.877	0.852	0.850	0.889
		$ridge_{max\text{-}MM}$	0.006	0.313	0.980	0.961	0.642	0.433	0.713	0.524
		$ridge_{mean\text{-}MM}$	0.001	0.157	0.967	0.936	0.262	0.092	0.396	0.176
		$ridge_{med\text{-}MM}$	0.028	0.049	0.966	0.934	0.128	0.028	0.019	0.006
	10%	$ridge_{OLS}$	0.012	0.018	0.992	0.985	0.103	0.082	0.563	0.361
		$ridge_{M}$	0.013	0.018	0.992	0.985	0.107	0.082	0.557	0.354
		$ridge_{LMS}$	0.020	0.405	0.997	0.994	0.748	0.775	0.692	0.820
		$ridge_{MM}$	0.025	0.429	0.998	0.996	0.835	0.847	0.762	0.898
		$ridge_{max\text{-}MM}$	0.030	0.376	0.992	0.985	0.756	0.597	0.807	0.668
		$ridge_{mean\text{-}MM}$	0.023	0.228	0.985	0.971	0.431	0.224	0.542	0.322
		$ridge_{med\text{-}MM}$	0.021	0.056	0.982	0.965	0.125	0.027	0.031	0.007
$\sigma^2 = 1$	0% (clean data)	$ridge_{OLS}$	0.001	0.059	0.017	0.078	0.014	0.076	0.189	0.084
		$ridge_{M}$	0.000	0.060	0.014	0.074	0.011	0.078	0.197	0.087
		$ridge_{LMS}$	0.002	0.078	0.090	0.070	0.095	0.066	0.265	0.107
		$ridge_{MM}$	0.000	0.059	0.013	0.075	0.014	0.073	0.195	0.086
		$ridge_{max\text{-}MM}$	0.002	0.101	0.227	0.066	0.221	0.064	0.381	0.153
		$ridge_{mean\text{-}MM}$	0.001	0.060	0.036	0.037	0.031	0.038	0.213	0.067

		ridge$_{\text{med-MM}}$	0.000	0.059	0.069	0.101	0.028	0.098	0.162	0.073
	5%	ridge$_{\text{OLS}}$	0.024	0.083	0.967	0.948	0.196	0.113	0.157	0.091
		ridge$_{\text{M}}$	0.028	0.085	0.967	0.939	0.186	0.109	0.148	0.098
		ridge$_{\text{LMS}}$	0.020	0.349	0.984	0.972	0.513	0.562	0.563	0.571
		ridge$_{\text{MM}}$	0.014	0.262	0.989	0.969	0.267	0.441	0.291	0.438
		ridge$_{\text{max-MM}}$	0.012	0.272	0.979	0.941	0.443	0.251	0.550	0.341
		ridge$_{\text{mean-MM}}$	0.018	0.153	0.961	0.932	0.089	0.052	0.246	0.091
		ridge$_{\text{med-MM}}$	0.034	0.096	0.962	0.934	0.144	0.058	0.008	0.016
	10%	ridge$_{\text{OLS}}$	0.022	0.086	0.981	0.973	0.200	0.118	0.153	0.092
		ridge$_{\text{M}}$	0.025	0.089	0.981	0.974	0.198	0.116	0.141	0.091
		ridge$_{\text{LMS}}$	0.018	0.292	0.994	0.981	0.376	0.463	0.441	0.465
		ridge$_{\text{MM}}$	0.011	0.209	0.988	0.972	0.119	0.344	0.149	0.336
		ridge$_{\text{max-MM}}$	0.004	0.307	0.988	0.972	0.533	0.357	0.625	0.439
		ridge$_{\text{mean-MM}}$	0.007	0.173	0.983	0.960	0.182	0.093	0.321	0.155
		ridge$_{\text{med-MM}}$	0.024	0.098	0.988	0.969	0.149	0.054	0.007	0.017
$\sigma^2 = 10$	0% (clean data)	ridge$_{\text{OLS}}$	0.012	0.728	0.594	0.453	0.594	0.452	0.687	0.521
		ridge$_{\text{M}}$	0.018	0.726	0.592	0.449	0.592	0.447	0.686	0.518
		ridge$_{\text{LMS}}$	0.018	0.737	0.623	0.462	0.623	0.460	0.705	0.537
		ridge$_{\text{MM}}$	0.018	0.727	0.592	0.450	0.593	0.448	0.686	0.519
		ridge$_{\text{max-MM}}$	0.016	0.566	0.153	0.170	0.152	0.180	0.320	0.169
		ridge$_{\text{mean-MM}}$	0.013	0.555	0.004	0.439	0.018	0.443	0.195	0.233
		ridge$_{\text{med-MM}}$	0.012	0.559	0.095	0.999	0.021	0.899	0.184	0.558
	5%	ridge$_{\text{OLS}}$	0.031	0.819	0.992	0.984	0.625	0.495	0.697	0.555
		ridge$_{\text{M}}$	0.031	0.818	0.992	0.984	0.625	0.496	0.698	0.555
		ridge$_{\text{LMS}}$	0.028	0.891	0.995	0.990	0.779	0.679	0.825	0.725
		ridge$_{\text{MM}}$	0.031	0.839	0.992	0.985	0.660	0.539	0.726	0.595
		ridge$_{\text{max-MM}}$	0.026	0.765	0.987	0.976	0.451	0.326	0.555	0.387
		ridge$_{\text{mean-MM}}$	0.028	0.662	0.983	0.972	0.099	0.182	0.245	0.169
		ridge$_{\text{med-MM}}$	0.036	0.591	0.985	0.981	0.198	0.324	0.061	0.152
	10%	ridge$_{\text{OLS}}$	0.006	0.857	0.991	0.982	0.632	0.497	0.707	0.559
		ridge$_{\text{M}}$	0.005	0.855	0.990	0.982	0.631	0.495	0.706	0.558
		ridge$_{\text{LMS}}$	0.002	0.925	0.993	0.988	0.773	0.670	0.819	0.717
		ridge$_{\text{MM}}$	0.004	0.878	0.991	0.984	0.662	0.538	0.732	0.596
		ridge$_{\text{max-MM}}$	0.000	0.793	0.986	0.975	0.448	0.316	0.554	0.385
		ridge$_{\text{mean-MM}}$	0.009	0.677	0.981	0.969	0.093	0.171	0.244	0.169
		ridge$_{\text{med-MM}}$	0.030	0.602	0.981	0.973	0.202	0.305	0.056	0.152

Table III: Bias and MSE of ridge regression estimators (n=60, r =0.5)

			B0		**B1**		**B2**		**B3**	
	Leverage Points		**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**
$\sigma^2 = 0.1$	0% (clean data)	ridge$_{\text{OLS}}$	0.001	0.002	0.002	0.002	0.002	0.002	0.003	0.002
		ridge$_{\text{M}}$	0.001	0.002	0.002	0.002	0.002	0.002	0.003	0.002
		ridge$_{\text{LMS}}$	0.001	0.002	0.004	0.002	0.003	0.002	0.005	0.002
		ridge$_{\text{MM}}$	0.001	0.002	0.002	0.002	0.002	0.002	0.003	0.002
		ridge$_{\text{max-MM}}$	0.000	0.018	0.330	0.111	0.330	0.111	0.380	0.146
		ridge$_{\text{mean-MM}}$	0.001	0.004	0.113	0.015	0.112	0.014	0.153	0.024
		ridge$_{\text{med-MM}}$	0.001	0.002	0.015	0.002	0.015	0.002	0.024	0.002
	5%	ridge$_{\text{OLS}}$	0.030	0.018	0.972	0.945	0.070	0.020	0.083	0.022
		ridge$_{\text{M}}$	0.030	0.019	0.972	0.945	0.076	0.024	0.075	0.025
		ridge$_{\text{LMS}}$	0.001	0.121	0.999	0.999	0.990	0.980	0.991	0.982
		ridge$_{\text{MM}}$	0.001	0.121	0.999	0.999	0.991	0.982	0.992	0.984
		ridge$_{\text{max-MM}}$	0.004	0.097	0.994	0.987	0.840	0.708	0.854	0.732
		ridge$_{\text{mean-MM}}$	0.012	0.065	0.985	0.970	0.592	0.359	0.613	0.383
		ridge$_{\text{med-MM}}$	0.028	0.025	0.974	0.949	0.173	0.038	0.114	0.018

	10%	$ridge_{OLS}$	0.032	0.020	0.985	0.971	0.074	0.020	0.081	0.021
		$ridge_{M}$	0.032	0.020	0.985	0.971	0.075	0.020	0.079	0.021
		$ridge_{LMS}$	0.001	0.117	0.999	0.999	0.997	0.995	0.998	0.995
		$ridge_{MM}$	0.001	0.117	0.999	0.999	0.998	0.996	0.998	0.996
		$ridge_{max\text{-}MM}$	0.003	0.104	0.998	0.996	0.912	0.832	0.920	0.847
		$ridge_{mean\text{-}MM}$	0.008	0.080	0.995	0.989	0.732	0.542	0.752	0.570
		$ridge_{med\text{-}MM}$	0.030	0.026	0.986	0.973	0.177	0.039	0.117	0.018
$\sigma^2 = 1$	0% (clean data)	$ridge_{OLS}$	0.001	0.020	0.142	0.036	0.143	0.036	0.190	0.045
		$ridge_{M}$	0.001	0.020	0.143	0.036	0.144	0.036	0.190	0.045
		$ridge_{LMS}$	0.001	0.025	0.215	0.067	0.217	0.067	0.267	0.085
		$ridge_{MM}$	0.002	0.020	0.143	0.036	0.143	0.036	0.190	0.045
		$ridge_{max\text{-}MM}$	0.002	0.032	0.317	0.107	0.318	0.107	0.370	0.140
		$ridge_{mean\text{-}MM}$	0.001	0.018	0.146	0.023	0.149	0.034	0.188	0.046
		$ridge_{med\text{-}MM}$	0.002	0.017	0.069	0.022	0.073	0.021	0.107	0.019
	5%	$ridge_{OLS}$	0.017	0.053	0.976	0.954	0.307	0.120	0.290	0.110
		$ridge_{M}$	0.018	0.053	0.976	0.954	0.310	0.121	0.297	0.112
		$ridge_{LMS}$	0.006	0.129	0.997	0.995	0.941	0.910	0.943	0.911
		$ridge_{MM}$	0.005	0.130	0.998	0.996	0.952	0.936	0.953	0.938
		$ridge_{max\text{-}MM}$	0.002	0.104	0.991	0.982	0.768	0.600	0.786	0.627
		$ridge_{mean\text{-}MM}$	0.009	0.063	0.973	0.948	0.490	0.262	0.501	0.279
		$ridge_{med\text{-}MM}$	0.028	0.040	0.973	0.947	0.154	0.039	0.085	0.016
	10%	$ridge_{OLS}$	0.028	0.050	0.989	0.978	0.314	0.125	0.298	0.115
		$ridge_{M}$	0.028	0.050	0.989	0.978	0.316	0.126	0.300	0.117
		$ridge_{LMS}$	0.010	0.108	0.998	0.995	0.860	0.812	0.861	0.813
		$ridge_{MM}$	0.009	0.108	0.997	0.995	0.846	0.809	0.844	0.808
		$ridge_{max\text{-}MM}$	0.006	0.103	0.997	0.993	0.834	0.706	0.848	0.727
		$ridge_{mean\text{-}MM}$	0.010	0.072	0.987	0.974	0.594	0.363	0.610	0.398
		$ridge_{med\text{-}MM}$	0.031	0.039	0.986	0.973	0.154	0.041	0.080	0.016
$\sigma^2 = 10$	0% (clean data)	$ridge_{OLS}$	0.010	0.265	0.898	0.810	0.896	0.808	0.910	0.830
		$ridge_{M}$	0.011	0.265	0.898	0.810	0.896	0.807	0.910	0.830
		$ridge_{LMS}$	0.009	0.264	0.892	0.801	0.890	0.800	0.903	0.821
		$ridge_{MM}$	0.011	0.265	0.898	0.810	0.896	0.807	0.909	0.830
		$ridge_{max\text{-}MM}$	0.006	0.179	0.239	0.122	0.232	0.123	0.297	0.124
		$ridge_{mean\text{-}MM}$	0.005	0.173	0.118	0.126	0.110	0.130	0.175	0.092
		$ridge_{med\text{-}MM}$	0.004	0.172	0.081	0.1434	0.073	0.147	0.138	0.098
	5%	$ridge_{OLS}$	0.014	0.275	0.998	0.994	0.916	0.841	0.922	0.858
		$ridge_{M}$	0.014	0.274	0.997	0.994	0.916	0.841	0.925	0.858
		$ridge_{LMS}$	0.013	0.278	0.998	0.996	0.942	0.892	0.949	0.903
		$ridge_{MM}$	0.014	0.277	0.998	0.995	0.932	0.873	0.940	0.888
		$ridge_{max\text{-}MM}$	0.020	0.247	0.989	0.980	0.709	0.522	0.731	0.549
		$ridge_{mean\text{-}MM}$	0.030	0.214	0.980	0.963	0.409	0.212	0.421	0.209
		$ridge_{med\text{-}MM}$	0.039	0.191	0.973	0.952	0.100	0.112	0.011	0.057
	10%	$ridge_{OLS}$	0.001	0.267	0.998	0.995	0.912	0.836	0.922	0.851
		$ridge_{M}$	0.001	0.267	0.998	0.996	0.912	0.835	0.921	0.851
		$ridge_{LMS}$	0.001	0.267	0.998	0.997	0.921	0.854	0.929	0.867
		$ridge_{MM}$	0.001	0.268	0.998	0.997	0.918	0.846	0.926	0.860
		$ridge_{max\text{-}MM}$	0.007	0.254	0.995	0.991	0.775	0.620	0.791	0.643
		$ridge_{mean\text{-}MM}$	0.019	0.227	0.991	0.982	0.500	0.297	0.512	0.304
		$ridge_{med\text{-}MM}$	0.035	0.198	0.985	0.973	0.098	0.114	0.006	0.061

Table IV: Bias and MSE of ridge regression estimators (n=60, r =0.9)

			B0		**B1**		**B2**		**B3**	
	Leverage Points		**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**
$\sigma^2 = 0.1$	0%	$ridge_{OLS}$	0.001	0.002	0.003	0.009	0.006	0.009	0.009	0.009

	(clean data)	$ridge_{M}$	0.001	0.002	0.003	0.009	0.006	0.009	0.009	0.009
		$ridge_{LMS}$	0.001	0.002	0.005	0.009	0.008	0.008	0.014	0.008
		$ridge_{MM}$	0.001	0.002	0.003	0.009	0.006	0.009	0.009	0.009
		$ridge_{max\text{-}MM}$	0.001	0.016	0.220	0.050	0.216	0.049	0.383	0.147
		$ridge_{mean\text{-}MM}$	0.001	0.003	0.024	0.002	0.024	0.002	0.200	0.041
		$ridge_{med\text{-}MM}$	0.001	0.002	0.018	0.006	0.021	0.006	0.050	0.007
	5%	$ridge_{OLS}$	0.010	0.005	0.991	0.983	0.091	0.028	0.574	0.342
		$ridge_{M}$	0.010	0.005	0.991	0.983	0.091	0.028	0.573	0.341
		$ridge_{LMS}$	0.002	0.164	0.998	0.997	0.961	0.936	0.966	0.949
		$ridge_{MM}$	0.001	0.168	0.999	0.998	0.979	0.963	0.982	0.971
		$ridge_{max\text{-}MM}$	0.006	0.119	0.990	0.980	0.723	0.529	0.782	0.615
		$ridge_{mean\text{-}MM}$	0.014	0.062	0.980	0.961	0.343	0.129	0.473	0.232
		$ridge_{med\text{-}MM}$	0.023	0.014	0.978	0.958	0.155	0.027	0.007	0.002
	10%	$ridge_{OLS}$	0.007	0.006	0.995	0.991	0.094	0.029	0.574	0.341
		$ridge_{M}$	0.007	0.006	0.995	0.991	0.094	0.030	0.573	0.340
		$ridge_{LMS}$	0.001	0.162	0.999	0.998	0.975	0.970	0.968	0.978
		$ridge_{MM}$	0.007	0.164	0.999	0.999	0.989	0.983	0.985	0.984
		$ridge_{max\text{-}MM}$	0.002	0.136	0.997	0.994	0.844	0.713	0.872	0.771
		$ridge_{mean\text{-}MM}$	0.007	0.088	0.992	0.985	0.552	0.315	0.642	0.421
		$ridge_{med\text{-}MM}$	0.021	0.015	0.989	0.978	0.151	0.023	0.002	0.001
$\sigma^2 = 1$	0% (clean data)	$ridge_{OLS}$	0.001	0.019	0.020	0.016	0.023	0.016	0.199	0.047
		$ridge_{M}$	0.001	0.018	0.023	0.016	0.024	0.016	0.199	0.045
		$ridge_{LMS}$	0.001	0.025	0.045	0.017	0.057	0.017	0.227	0.065
		$ridge_{MM}$	0.002	0.019	0.023	0.016	0.023	0.016	0.190	0.045
		$ridge_{max\text{-}MM}$	0.002	0.032	0.217	0.047	0.218	0.047	0.370	0.140
		$ridge_{mean\text{-}MM}$	0.001	0.018	0.026	0.013	0.039	0.014	0.218	0.046
		$ridge_{med\text{-}MM}$	0.002	0.017	0.039	0.032	0.023	0.028	0.127	0.029
	5%	$ridge_{OLS}$	0.017	0.023	0.976	0.954	0.217	0.060	0.136	0.037
		$ridge_{M}$	0.018	0.023	0.976	0.954	0.210	0.061	0.137	0.032
		$ridge_{LMS}$	0.016	0.119	0.997	0.985	0.481	0.560	0.543	0.561
		$ridge_{MM}$	0.015	0.101	0.988	0.976	0.362	0.506	0.403	0.498
		$ridge_{max\text{-}MM}$	0.008	0.114	0.981	0.972	0.618	0.396	0.696	0.497
		$ridge_{mean\text{-}MM}$	0.019	0.063	0.973	0.948	0.210	0.062	0.361	0.149
		$ridge_{med\text{-}MM}$	0.021	0.023	0.973	0.957	0.174	0.039	0.035	0.006
	10%	$ridge_{OLS}$	0.018	0.027	0.989	0.979	0.224	0.065	0.146	0.035
		$ridge_{M}$	0.018	0.027	0.989	0.979	0.216	0.066	0.140	0.037
		$ridge_{LMS}$	0.010	0.078	0.998	0.985	0.175	0.332	0.251	0.313
		$ridge_{MM}$	0.014	0.068	0.997	0.985	0.031	0.263	0.094	0.248
		$ridge_{max\text{-}MM}$	0.001	0.130	0.997	0.983	0.714	0.536	0.778	0.612
		$ridge_{mean\text{-}MM}$	0.010	0.072	0.990	0.984	0.354	0.153	0.470	0.252
		$ridge_{med\text{-}MM}$	0.021	0.030	0.986	0.973	0.174	0.041	0.030	0.006
$\sigma^2 = 10$	0% (clean data)	$ridge_{OLS}$	0.008	0.293	0.776	0.618	0.775	0.617	0.826	0.691
		$ridge_{M}$	0.008	0.292	0.773	0.614	0.772	0.613	0.823	0.687
		$ridge_{LMS}$	0.007	0.273	0.676	0.498	0.675	0.497	0.750	0.581
		$ridge_{MM}$	0.008	0.292	0.772	0.613	0.772	0.612	0.823	0.686
		$ridge_{max\text{-}MM}$	0.007	0.187	0.156	0.063	0.153	0.064	0.326	0.122
		$ridge_{mean\text{-}MM}$	0.006	0.177	0.021	0.124	0.018	0.127	0.189	0.090
		$ridge_{med\text{-}MM}$	0.006	0.177	0.028	0.272	0.015	0.257	0.140	0.156
	5%	$ridge_{OLS}$	0.017	0.312	0.992	0.985	0.779	0.622	0.826	0.691
		$ridge_{M}$	0.017	0.312	0.992	0.985	0.778	0.620	0.825	0.690
		$ridge_{LMS}$	0.016	0.317	0.993	0.987	0.798	0.682	0.842	0.734
		$ridge_{MM}$	0.016	0.317	0.993	0.987	0.800	0.659	0.842	0.721
		$ridge_{max\text{-}MM}$	0.019	0.270	0.984	0.969	0.530	0.312	0.626	0.412
		$ridge_{mean\text{-}MM}$	0.026	0.220	0.977	0.956	0.119	0.060	0.275	0.107
		$ridge_{med\text{-}MM}$	0.030	0.195	0.979	0.962	0.206	0.124	0.080	0.050

	10%	$ridge_{OLS}$	0.035	0.296	0.996	0.993	0.773	0.615	0.822	0.686
		$ridge_{M}$	0.035	0.296	0.996	0.993	0.773	0.614	0.822	0.686
		$ridge_{LMS}$	0.036	0.294	0.996	0.992	0.747	0.595	0.800	0.666
		$ridge_{MM}$	0.035	0.297	0.996	0.993	0.777	0.621	0.824	0.691
		$ridge_{max\text{-}MM}$	0.035	0.276	0.994	0.988	0.636	0.442	0.712	0.532
		$ridge_{mean\text{-}MM}$	0.038	0.228	0.990	0.981	0.243	0.115	0.386	0.190
		$ridge_{med\text{-}MM}$	0.036	0.188	0.991	0.983	0.219	0.129	0.082	0.055

Table V: Bias and MSE of ridge regression estimators (n=100, r =0.5)

			B0		**B1**		**B2**		**B3**	
	Leverage Points		**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**
$\sigma^2 = 0.1$	0% (clean data)	$ridge_{OLS}$	0.001	0.001	0.001	0.001	0.001	0.001	0.003	0.001
		$ridge_{M}$	0.001	0.001	0.001	0.001	0.001	0.001	0.003	0.001
		$ridge_{LMS}$	0.001	0.001	0.002	0.001	0.002	0.001	0.004	0.001
		$ridge_{MM}$	0.001	0.001	0.001	0.001	0.001	0.001	0.003	0.001
		$ridge_{max\text{-}MM}$	0.002	0.010	0.325	0.107	0.325	0.107	0.376	0.142
		$ridge_{mean\text{-}MM}$	0.001	0.002	0.110	0.013	0.110	0.013	0.150	0.023
		$ridge_{med\text{-}MM}$	0.001	0.001	0.014	0.001	0.014	0.001	0.023	0.001
	5%	$ridge_{OLS}$	0.031	0.012	0.976	0.951	0.071	0.012	0.083	0.015
		$ridge_{M}$	0.031	0.012	0.976	0.951	0.072	0.014	0.081	0.016
		$ridge_{LMS}$	0.008	0.072	0.999	0.999	0.993	0.986	0.994	0.988
		$ridge_{MM}$	0.008	0.072	0.999	0.999	0.994	0.987	0.994	0.989
		$ridge_{max\text{-}MM}$	0.012	0.060	0.995	0.990	0.856	0.735	0.869	0.757
		$ridge_{mean\text{-}MM}$	0.018	0.041	0.988	0.976	0.616	0.386	0.638	0.412
		$ridge_{med\text{-}MM}$	0.030	0.015	0.977	0.955	0.169	0.033	0.105	0.014
	10%	$ridge_{OLS}$	0.033	0.012	0.988	0.976	0.069	0.013	0.083	0.015
		$ridge_{M}$	0.033	0.012	0.988	0.976	0.069	0.013	0.082	0.015
		$ridge_{LMS}$	0.011	0.071	0.999	0.999	0.998	0.997	0.998	0.997
		$ridge_{MM}$	0.011	0.072	0.999	0.999	0.998	0.997	0.999	0.997
		$ridge_{max\text{-}MM}$	0.013	0.064	0.999	0.997	0.921	0.850	0.929	0.864
		$ridge_{mean\text{-}MM}$	0.017	0.050	0.996	0.992	0.753	0.572	0.773	0.600
		$ridge_{med\text{-}MM}$	0.032	0.015	0.989	0.978	0.169	0.033	0.108	0.014
$\sigma^2 = 1$	0% (clean data)	$ridge_{OLS}$	0.001	0.012	0.145	0.031	0.141	0.030	0.183	0.039
		$ridge_{M}$	0.001	0.011	0.145	0.031	0.141	0.030	0.184	0.039
		$ridge_{LMS}$	0.001	0.013	0.193	0.049	0.190	0.048	0.235	0.064
		$ridge_{MM}$	0.001	0.011	0.145	0.031	0.141	0.030	0.184	0.040
		$ridge_{max\text{-}MM}$	0.003	0.017	0.315	0.103	0.312	0.102	0.362	0.133
		$ridge_{mean\text{-}MM}$	0.002	0.010	0.121	0.021	0.126	0.022	0.172	0.032
		$ridge_{med\text{-}MM}$	0.001	0.010	0.073	0.015	0.069	0.015	0.100	0.015
	5%	$ridge_{OLS}$	0.016	0.023	0.981	0.963	0.310	0.111	0.293	0.150
		$ridge_{M}$	0.016	0.022	0.981	0.963	0.312	0.112	0.295	0.102
		$ridge_{LMS}$	0.008	0.077	0.999	0.998	0.968	0.950	0.968	0.945
		$ridge_{MM}$	0.008	0.078	0.999	0.999	0.980	0.971	0.981	0.972
		$ridge_{max\text{-}MM}$	0.001	0.063	0.994	0.987	0.809	0.659	0.823	0.682
		$ridge_{mean\text{-}MM}$	0.008	0.044	0.978	0.958	0.501	0.380	0.597	0.327
		$ridge_{med\text{-}MM}$	0.021	0.022	0.977	0.956	0.151	0.033	0.071	0.017
	10%	$ridge_{OLS}$	0.020	0.030	0.990	0.981	0.312	0.113	0.295	0.103
		$ridge_{M}$	0.021	0.029	0.990	0.981	0.312	0.103	0.295	0.103
		$ridge_{LMS}$	0.006	0.068	0.999	0.998	0.913	0.885	0.913	0.885
		$ridge_{MM}$	0.007	0.069	0.999	0.998	0.913	0.887	0.900	0.886
		$ridge_{max\text{-}MM}$	0.004	0.067	0.998	0.996	0.870	0.762	0.881	0.781
		$ridge_{mean\text{-}MM}$	0.006	0.047	0.949	0.929	0.635	0.421	0.684	0.440
		$ridge_{med\text{-}MM}$	0.029	0.024	0.989	0.977	0.150	0.033	0.069	0.010
$\sigma^2 = 10$	0%	$ridge_{OLS}$	0.003	0.158	0.909	0.828	0.910	0.830	0.919	0.847

	(clean data)	ridge$_{M}$	0.003	0.158	0.909	0.828	0.910	0.829	0.919	0.846
		ridge$_{LMS}$	0.003	0.157	0.900	0.813	0.901	0.814	0.910	0.831
		ridge$_{MM}$	0.003	0.157	0.909	0.828	0.909	0.830	0.919	0.846
		ridge$_{max\text{-}MM}$	0.003	0.106	0.236	0.094	0.247	0.100	0.283	0.107
		ridge$_{mean\text{-}MM}$	0.003	0.103	0.119	0.081	0.134	0.085	0.156	0.061
		ridge$_{med\text{-}MM}$	0.003	0.103	0.083	0.088	0.100	0.092	0.115	0.058
	5%	ridge$_{OLS}$	0.001	0.163	0.998	0.995	0.925	0.858	0.932	0.870
		ridge$_{M}$	0.001	0.163	0.998	0.995	0.925	0.858	0.932	0.870
		ridge$_{LMS}$	0.001	0.165	0.998	0.997	0.945	0.896	0.950	0.904
		ridge$_{MM}$	0.001	0.164	0.998	0.996	0.939	0.884	0.945	0.894
		ridge$_{max\text{-}MM}$	0.005	0.149	0.992	0.984	0.751	0.574	0.767	0.596
		ridge$_{mean\text{-}MM}$	0.015	0.130	0.983	0.968	0.455	0.233	0.459	0.231
		ridge$_{med\text{-}MM}$	0.025	0.116	0.976	0.955	0.120	0.077	0.005	0.034
	10%	ridge$_{OLS}$	0.009	0.162	0.999	0.998	0.924	0.855	0.932	0.870
		ridge$_{M}$	0.009	0.162	0.999	0.998	0.924	0.855	0.932	0.870
		ridge$_{LMS}$	0.009	0.162	0.999	0.998	0.926	0.861	0.934	0.874
		ridge$_{MM}$	0.009	0.162	0.999	0.998	0.925	0.859	0.933	0.873
		ridge$_{max\text{-}MM}$	0.013	0.156	0.997	0.994	0.817	0.678	0.833	0.702
		ridge$_{mean\text{-}MM}$	0.020	0.139	0.994	0.988	0.553	0.333	0.572	0.351
		ridge$_{med\text{-}MM}$	0.031	0.120	0.990	0.981	0.099	0.076	0.007	0.035

Table VI: Bias and MSE of ridge regression estimators (n=100, r =0.9)

			B0		B1		B2		B3	
	Leverage Points		**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**	**bias**	**MSE**
$\sigma^2 = 0.1$	0% (clean data)	ridge$_{OLS}$	0.001	0.001	0.001	0.005	0.006	0.005	0.008	0.006
		ridge$_{M}$	0.001	0.001	0.001	0.005	0.006	0.005	0.008	0.006
		ridge$_{LMS}$	0.001	0.001	0.003	0.005	0.007	0.005	0.010	0.005
		ridge$_{MM}$	0.001	0.001	0.001	0.005	0.006	0.005	0.008	0.006
		ridge$_{max\text{-}MM}$	0.001	0.009	0.219	0.050	0.216	0.045	0.381	0.146
		ridge$_{mean\text{-}MM}$	0.004	0.002	0.024	0.002	0.023	0.002	0.198	0.040
		ridge$_{med\text{-}MM}$	0.001	0.001	0.016	0.004	0.016	0.005	0.048	0.005
	5%	ridge$_{OLS}$	0.009	0.003	0.991	0.985	0.086	0.020	0.577	0.340
		ridge$_{M}$	0.009	0.003	0.992	0.985	0.087	0.020	0.577	0.340
		ridge$_{LMS}$	0.007	0.095	0.999	0.995	0.986	0.966	0.985	0.973
		ridge$_{MM}$	0.006	0.095	0.999	0.995	0.986	0.973	0.991	0.981
		ridge$_{max\text{-}MM}$	0.012	0.069	0.992	0.985	0.751	0.568	0.805	0.649
		ridge$_{mean\text{-}MM}$	0.018	0.037	0.982	0.965	0.381	0.158	0.505	0.262
		ridge$_{med\text{-}MM}$	0.023	0.009	0.982	0.963	0.158	0.028	0.011	0.001
	10%	ridge$_{OLS}$	0.009	0.003	0.996	0.992	0.094	0.020	0.576	0.339
		ridge$_{M}$	0.009	0.003	0.996	0.992	0.094	0.020	0.575	0.338
		ridge$_{LMS}$	0.010	0.106	0.999	0.999	0.990	0.985	0.988	0.989
		ridge$_{MM}$	0.010	0.106	0.999	0.999	0.996	0.992	0.995	0.995
		ridge$_{max\text{-}MM}$	0.012	0.089	0.998	0.995	0.860	0.741	0.890	0.793
		ridge$_{mean\text{-}MM}$	0.017	0.060	0.994	0.988	0.584	0.349	0.671	0.455
		ridge$_{med\text{-}MM}$	0.024	0.010	0.991	0.982	0.156	0.025	0.007	0.001
$\sigma^2 = 1$	0% (clean data)	ridge$_{OLS}$	0.001	0.010	0.025	0.009	0.021	0.009	0.200	0.044
		ridge$_{M}$	0.001	0.011	0.022	0.011	0.023	0.010	0.201	0.054
		ridge$_{LMS}$	0.001	0.011	0.043	0.011	0.042	0.011	0.215	0.054
		ridge$_{MM}$	0.001	0.010	0.023	0.009	0.024	0.009	0.201	0.044
		ridge$_{max\text{-}MM}$	0.001	0.017	0.215	0.043	0.212	0.042	0.362	0.143
		ridge$_{mean\text{-}MM}$	0.001	0.010	0.031	0.007	0.036	0.007	0.210	0.042
		ridge$_{med\text{-}MM}$	0.001	0.010	0.029	0.019	0.020	0.015	0.120	0.025
	5%	ridge$_{OLS}$	0.016	0.015	0.981	0.963	0.220	0.058	0.133	0.030
		ridge$_{M}$	0.018	0.015	0.981	0.963	0.222	0.052	0.135	0.032

		ridge$_{LMS}$	0.010	0.067	0.992	0.988	0.508	0.580	0.558	0.585
		ridge$_{MM}$	0.010	0.065	0.999	0.989	0.432	0.559	0.471	0.542
		ridge$_{max\text{-}MM}$	0.011	0.073	0.994	0.987	0.679	0.469	0.743	0.562
		ridge$_{mean\text{-}MM}$	0.018	0.044	0.988	0.968	0.271	0.090	0.417	0.187
		ridge$_{med\text{-}MM}$	0.021	0.012	0.987	0.966	0.171	0.033	0.031	0.007
	10%	ridge$_{OLS}$	0.017	0.016	0.990	0.981	0.222	0.053	0.135	0.030
		ridge$_{M}$	0.018	0.015	0.990	0.981	0.222	0.053	0.135	0.031
		ridge$_{LMS}$	0.011	0.048	0.994	0.988	0.153	0.325	0.233	0.305
		ridge$_{MM}$	0.012	0.039	0.994	0.988	0.009	0.237	0.077	0.216
		ridge$_{max\text{-}MM}$	0.004	0.087	0.996	0.993	0.777	0.612	0.821	0.681
		ridge$_{mean\text{-}MM}$	0.012	0.057	0.992	0.985	0.435	0.217	0.550	0.320
		ridge$_{med\text{-}MM}$	0.020	0.014	0.991	0.982	0.178	0.033	0.039	0.004
$\sigma^2 = 10$	0% (clean data)	ridge$_{OLS}$	0.012	0.176	0.815	0.672	0.815	0.672	0.857	0.738
		ridge$_{M}$	0.012	0.176	0.814	0.669	0.813	0.669	0.856	0.735
		ridge$_{LMS}$	0.011	0.162	0.715	0.535	0.715	0.534	0.779	0.618
		ridge$_{MM}$	0.012	0.176	0.812	0.667	0.812	0.667	0.855	0.734
		ridge$_{max\text{-}MM}$	0.001	0.106	0.153	0.045	0.151	0.045	0.326	0.116
		ridge$_{mean\text{-}MM}$	0.001	0.104	0.015	0.072	0.015	0.062	0.196	0.068
		ridge$_{med\text{-}MM}$	0.002	0.103	0.030	0.154	0.023	0.141	0.147	0.100
	5%	ridge$_{OLS}$	0.006	0.184	0.994	0.989	0.810	0.662	0.851	0.728
		ridge$_{M}$	0.006	0.184	0.994	0.989	0.810	0.662	0.851	0.727
		ridge$_{LMS}$	0.008	0.185	0.994	0.989	0.798	0.666	0.841	0.725
		ridge$_{MM}$	0.007	0.186	0.995	0.990	0.820	0.681	0.858	0.742
		ridge$_{max\text{-}MM}$	0.000	0.163	0.988	0.977	0.596	0.373	0.681	0.475
		ridge$_{mean\text{-}MM}$	0.012	0.131	0.982	0.964	0.175	0.060	0.329	0.128
		ridge$_{med\text{-}MM}$	0.018	0.111	0.982	0.970	0.213	0.090	0.084	0.033
	10%	ridge$_{OLS}$	0.020	0.181	0.997	0.994	0.808	0.660	0.849	0.726
		ridge$_{M}$	0.021	0.181	0.997	0.994	0.808	0.660	0.849	0.726
		ridge$_{LMS}$	0.023	0.178	0.997	0.993	0.767	0.611	0.817	0.681
		ridge$_{MM}$	0.021	0.182	0.997	0.994	0.808	0.661	0.850	0.727
		ridge$_{max\text{-}MM}$	0.020	0.171	0.995	0.991	0.707	0.519	0.769	0.604
		ridge$_{mean\text{-}MM}$	0.025	0.140	0.991	0.983	0.322	0.140	0.454	0.232
		ridge$_{med\text{-}MM}$	0.025	0.113	0.991	0.984	0.217	0.092	0.090	0.032

3.2 Numerical Example

A real-life example is considered in this section to illustrate the usefulness of proposed estimators. The Stack-Loss data is used [22], it is result of 21 days of operations of a plant oxidizing ammonia to nitric acid. The data set includes three explanatory variables and one dependent variable with multicollinearity problem. The observation numbered 2 was identified as influential observation. Table VII shows the results of the parameter estimates and the mean square error of the methods: ridgeOLS, ridgeM, ridgeLMS, ridgeMM, ridgemax-MM, ridgemean-MM and ridgemed-MM. The results show that both ridgemax-MM and ridgemed-MM give less value of MSE than other estimators. Furthermore, the ridgemed-MM estimator produces the smallest MSE by comparing with the other considered estimators.

Table VII: Parameters estimation and MSE of ridge regression method

	b_0	b_1	b_2	b_3	MSE
ridge$_{OLS}$	-40.562	0.689	1.310	-0.130	11.22
ridge$_{M}$	-40.656	0.685	1.312	-0.126	10.72
ridge$_{LMS}$	-41.615	0.636	1.321	-0.083	10.97
ridge$_{MM}$	-40.711	0.682	1.313	-0.124	10.58
ridge$_{max\text{-}MM}$	-40.444	0.694	1.308	-0.134	10.46
ridge$_{mean\text{-}MM}$	-38.345	0.301	0.806	0.124	23.02
ridge$_{med\text{-}MM}$	-42.061	0.602	1.313	-0.052	10.42

4. Conclusions

In this paper, we propose new modification of three formulas that suggested by Alkhamisi et al. [6] to estimate the ridge parameter (k) based on MM-estimator. The main objective of this paper is to propose robust estimator to handle leverage points and multicollinearity problems simultaneously . The study has been conducted by means of Monte Carlo simulations of six scenarios where the strength of sample size, multicollinearity, residual variance, and the percentage of leverage points have been varied. For each combination, we have used 3,000 replications. The evaluation has mainly been done by using the bias and MSE criteria to compare between the modified versions and some other available estimators. The results indicate that the proposed ridge parameters have good properties in terms of MSE criteria. Moreover, the MSE for ridgemed-MM estimator is minimum among all the considered estimators.

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