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Cost-Effective Management of Bulk Arrival Queuing Systems with Vacation and Setup Periods: A Case Study on Engine Piston Production

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Abstract: In this paper, we explore a single-server queuing system where arrivals follow a compound Poisson process, while service times have a general distribution. We introduce a concept of a threshold for idle times, assuming that when the server has been idle for a certain duration, it needs a random setup time before starting to serve again. We derive steady-state distributions for both system size and waiting time, and we also calculate the expected cycle time for each model. Additionally, we show that the system size and waiting time distributions can be broken down into three distinct, meaningful parts. For the threshold model, we present a method to determine the optimal threshold that minimizes the total expected operating cost. A numerical example, focusing on the cost model?s relevance to engine piston production, further demonstrates the model?s practical application.

Keywords: Queue models with batch arrivals, optimal threshold determination, closure period, setup time requirements

1 Introduction

Queuing models with vacationing servers are widely used in fields such as manufacturing, telecommunications, and various service industries. In these models, after serving a set number of customers or packets?either randomly or at fixed intervals?or after a specific busy period, the server may take a ?vacation.? These breaks can serve various purposes, such as handling personal matters or completing other work-related tasks. Examples of work tasks might include assisting colleagues or managing unresolved overhead left by previous clients. Many authors have extensively studied queuing models with server vacations [6,7,11,12,13,14,4,1]. Survey studies, such as [6], also offer practical examples of vacation queuing models, including cases where servers experience malfunctions and require repairs.

Server vacations are characterized by two phases: idle time and setup time. The idle period begins when the system is completely empty, assuming an exhaustive service system. In threshold model, the idle period ends when the number of customers in the queue first reaches or exceeds a specified threshold. At that point, the server resumes service. This study outlines and focuses on how this model impacts system efficiency and server utilization.

After an idle period, the server undergoes an independent, uniformly distributed setup phase. During this time, any customers who arrived while the server was idle, as well as those arriving during setup, are not served. Once setup is complete, the server starts serving customers immediately in a first-come, first-served (FCFS) order, initiating a busy period that continues until the queue is empty. To analyze these models, we use the supplementary variable approach to derive the steady-state distributions of the system size for each model. We then demonstrate that these distributions can be split into two components: one part representing the number of customers who arrived during the idle (vacation) period, and the other part corresponding to the system size distribution in a standard $M^X/G/1$ model without an idle phase.

Additionally, we calculate the waiting time distributions and provide each model with a unique decomposition property. We demonstrate that the expected cycle durations for each model are determined based on the time a customer spends

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waiting during the remaining vacation period. Finally, we propose a method to minimize the total expected operating cost per unit time, allowing us to determine the optimal threshold value for the threshold model.

The $M/G/1$ system with generalized vacation policies has been widely studied, with key insights documented in numerous research articles. Doshi?s survey [6] offers an in-depth overview of queuing systems featuring vacation policies. Chae et al. [1] introduced the arrival time approach, a powerful method for analyzing various $M/G/1$ systems with generalized vacations. Their findings show that the steady-state queue size distribution for $M^X/G/1$ systems with multiple vacations can be divided into two parts: the number of customers present during the vacation period and the distribution for a standard $M^X/G/1$ queue. Fuhrmann and Cooper [7] extended the vacation model to include threshold mechanisms, server setup times, single and multiple vacations, and their combinations. Lee and Srinivasan [11] studied a control policy for $M^X/G/1$ queues with multiple vacations, where M^X represents a compound Poisson process. Lee et al. further analyzed the $M^X/G/1$ model under configurations such as N-policy with multiple vacations [12], in addition to isolated studies on N-policy alone [13] and single vacation models with N-policy [14]. Choudhury [4] applied Fuhrmann?s decomposition property to the context of group arrivals, demonstrating that the queue size distribution at departure points can be represented as the convolution of three independent random variables.

The authors in [15] explored equilibrium methods for an $M/M/1$ queue experiencing working breakdowns. In [10], a queuing model was analyzed that accounts for operational failures and disasters. A limited capacity queue with Bernoulli feedback and working breakdowns was investigated by the authors in [9]. The study in [16] focused on a working-breakdown-reneging queuing system that has limited capacity and retains impatient customers. Ye and Liu [17] later examined a Markovian arrival process (MAP) that includes functional breakdowns and repair mechanisms. In [5], the steady-state characteristics of a breakdown queue during vacation periods were identified. Furthermore, recent queuing models have begun to incorporate elements such as crowdsourcing, optional services, phase-type (PH) distribution services during vacations, and MAP arrivals (see, e.g., [2,3]).

In this paper, we examine an exhaustive queuing system, where an idle period begins as soon as the system empties. We specifically look at a threshold model, which pauses operation once the queue reaches or exceeds a set number of customers. After an idle period, a setup phase begins, with its duration following a general distribution and remaining independent of other time intervals. During both the idle and setup phases, arriving customers are not served until the setup is complete. Once ready, the server resumes service in a FCFS manner, marking the start of a busy period that continues until the queue is again empty. We calculate the expected cycle lengths for this threshold model and propose a method for finding the optimal threshold value that minimizes the expected operating cost per unit time. To identify the best threshold level (or minimum batch size) for this system, we develop a cost model. A major contribution of our study is the analysis of a cost model that has practical relevance, illustrating how the results can support effective cost optimization in real-world applications.

The structure of the paper is as follows: Section 2 introduces the threshold model. In Section 3, we derive the steady-state system size distribution for the $M^X/G/1$ threshold model. Section 4 focuses on deriving the joint Laplace transformations of the waiting time for a randomly selected (tagged) customer, covering cases where the customer arrives during the idle, setup, or busy period. Here, we assume a FCFS principle and calculate the expected waiting time and the expected cycle length, which includes an idle period, setup period, and busy period. Section 5 examines a cost model using real data, discussing how the findings can be applied to optimize costs. The paper concludes with Section 6, which offers recommendations for future research directions.

2 Threshold model description

A cycle in this system starts when the queue is empty, causing the server to go idle. The server remains idle until the number of waiting customers reaches N (where $N \geq 1$). During this idle phase, customers arrive in groups according to a compound Poisson process with a rate of μ. Let χ represent the random variable for group size, with its probability generating function (PGF) denoted as $\kappa(z)$. The probability that a batch of k customers arrives is represented by $g_k = \Pr(\chi = k)$, for $k = 1, 2, \ldots, q$. For service times, we denote the probability density function (pdf) as $S(x)$ and its Laplace transform (LT) as $\widetilde{S}(\theta)$. Similarly, for setup times, let U, $u(x)$, and $\tilde{U}(\theta)$ represent the setup time variable, its pdf, and LT, respectively. The remaining service time of a customer in service at time t is given by $S^+(t)$, and the remaining setup time at time t by $U^+(t)$. The variable $N(t)$ denotes the system size at time t. Additionally, we define $\varphi(t)$ as the indicator of the server?s state at any time t, with the following values:

$$\varphi(t) = \begin{cases} 0, & \text{if the server is idle at time } t, \\ 1, & \text{if the server is in the setup phase at time } t, \\ 2, & \text{if the server is actively serving (busy) at time } t. \end{cases}$$

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For steady-state probabilities, we use the following variables:
1. Idle State Probability:

$$\vartheta_n = \lim_{t\to\infty} \vartheta_n(t), \quad \vartheta_n(t) = \Pr[N(t) = n, \varphi(t) = 0], \quad n = 0,1,2,\ldots,N-1.$$

2. Setup State Probability:

$$U_n(x)\,dx = \lim_{t\to\infty} U_n(x,t)\,dx, \quad U_n(x,t)\,dx = \Pr[N(t) = n, \varphi(t) = 1, x < U^+(t) \le x+dx],$$

$$\tilde{U}_n(\theta) = \int_0^\infty e^{-\theta x} U_n(x)\,dx, \quad n = N, N+1, \ldots.$$

3. Busy State Probability:

$$\nu_n(x)\,dx = \lim_{t\to\infty} \nu_n(x,t)\,dx, \quad \nu_n(x,t)\,dx = \Pr[N(t) = n, \varphi(t) = 2, x < S^+(t) \le x+dx],$$

$$\tilde{\nu}_n(\theta) = \int_0^\infty e^{-\theta x} \nu_n(x)\,dx, \quad n = 1,2,\ldots.$$

These definitions allow us to model the system's behavior across idle, setup, and busy states, providing the basis for deriving steady-state probabilities and other performance metrics.
By observing the state changes over the interval $(t, t+\Delta t)$ for any given t, we derive the following system of temporary state equations:

$$\vartheta_0(t+\Delta t) = \vartheta_0(t)(1-\lambda\Delta t) + \nu_1(0,t)\Delta t, \tag{2.1}$$

$$\vartheta_n(t+\Delta t) = \vartheta_n(t)(1-\lambda\Delta t) + \sum_{k=1}^{n} \vartheta_{n-k}(t)\lambda g_k \Delta t, \quad 1 \le n \le N-1, \tag{2.2}$$

$$U_N(x-\Delta t, t+\Delta t) = U_N(x,t)(1-\lambda\Delta t) + \sum_{k=0}^{N-1} \vartheta_k(t)\lambda g_{N-k} u(x)\Delta t, \tag{2.3}$$

$$U_n(x-\Delta t, t+\Delta t) = U_n(x,t)(1-\lambda\Delta t) + \sum_{k=0}^{N-1} \vartheta_k(t)\lambda g_{n-k} u(x)\Delta t + \sum_{k=1}^{n-N} U_{n-k}(x,t)\lambda g_k \Delta t, \quad n \ge N+1, \tag{2.4}$$

$$\nu_1(x-\Delta t, t+\Delta t) = \nu_1(x,t)(1-\lambda\Delta t) + \nu_2(0,t)S(x)\Delta t, \tag{2.5}$$

$$\nu_n(x-\Delta t, t+\Delta t) = \nu_n(x,t)(1-\lambda\Delta t) + \nu_{n+1}(0,t)S(x)\Delta t + \sum_{k=1}^{n-1} \nu_{n-k}(x,t)\lambda g_k \Delta t, \quad 2 \le n \le N-1, \tag{2.6}$$

$$\begin{aligned} \nu_n(x-\Delta t, t+\Delta t) =& \nu_n(x,t)(1-\lambda\Delta t) + \nu_{n+1}(0,t)S(x)\Delta t \\ &+ U_n(0,t)S(x)\Delta t + \sum_{k=1}^{n-1} \nu_{n-k}(x,t)\lambda g_k \Delta t, \quad n \ge N. \end{aligned} \tag{2.7}$$

To derive the steady-state balance equations, we divide both sides of equations (2.1) through (2.7) by Δt, then take the limits as $\Delta t \to 0$ and $t \to \infty$:

$$0 = -\lambda\vartheta_0 + \nu_1(0), \tag{2.8}$$

$$0 = -\lambda\vartheta_n + \lambda \sum_{k=1}^{n} \vartheta_{n-k} g_k, \quad 1 \le n \le N-1, \tag{2.9}$$

$$-\frac{d}{dx}U_N(x) = -\lambda U_N(x) + \lambda \sum_{k=0}^{N-1} \vartheta_k g_{N-k} u(x), \tag{2.10}$$

$$-\frac{d}{dx}U_n(x) = -\lambda U_n(x) + \lambda \sum_{k=1}^{n-N} U_{n-k}(x) g_k + \lambda \sum_{k=0}^{N-1} \vartheta_k g_{n-k} u(x), \quad n \ge N+1, \tag{2.11}$$

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$$-\frac{d}{dx}v_1(x) = -\lambda v_1(x) + v_2(0)S(x), \tag{2.12}$$

$$-\frac{d}{dx}v_n(x) = -\lambda v_n(x) + v_{n+1}(0)S(x) + \lambda \sum_{k=1}^{n-1} v_{n-k}(x)g_k, \quad 2 \le n \le N-1, \tag{2.13}$$

$$-\frac{d}{dx}v_n(x) = -\lambda v_n(x) + v_{n+1}(0)S(x) + U_n(0)S(x) + \lambda \sum_{k=1}^{n-1} v_{n-k}(x)g_k, \quad n \ge N. \tag{2.14}$$

The solution to this system of equations holds under the condition $\rho = \lambda E(\chi)E(q) < 1$. To proceed, we apply the Laplace Transform (LT) to equations (2.10)-(2.14) to derive the following difference equations. If we multiply both sides of Eq. (2.10) by $e^{-\theta x}$ and integrate from 0 to ∞, we obtain

$$\int_0^\infty e^{-\theta x}\frac{d}{dx}U_N(x)\,dx = \lambda \int_0^\infty e^{-\theta x}U_N(x)\,dx - \lambda \sum_{k=0}^{N-1} \vartheta_k g_{N-k} \int_0^\infty e^{-\theta x}u(x)\,dx,$$

where we define:

$$\tilde{U}_N(\theta) = \int_0^\infty e^{-\theta x}U_N(x)\,dx,$$

$$\tilde{U}(\theta) = \int_0^\infty e^{-\theta x}u(x)\,dx.$$

So,

$$\theta \tilde{U}_N(\theta) - U_N(0) = \lambda \tilde{U}_N(\theta) - \lambda \sum_{k=0}^{N-1} \vartheta_k g_{N-k}\tilde{U}(\theta), \tag{2.15}$$

similarly,

$$\theta \tilde{U}_n(\theta) - U_n(0) = \lambda \tilde{U}_n(\theta) - \lambda \sum_{k=1}^{n-N} g_k \tilde{U}_{n-k}(\theta) - \lambda \sum_{k=0}^{N-1} \vartheta_k g_{n-k}\tilde{U}(\theta), \quad n \ge N+1, \tag{2.16}$$

$$\theta \tilde{v}_1(\theta) - v_1(0) = \lambda \tilde{v}_1(\theta) - v_2(0)\tilde{S}(\theta), \tag{2.17}$$

$$\theta \tilde{v}_n(\theta) - v_n(0) = \lambda \tilde{v}_n(\theta) - v_{n+1}(0)\tilde{S}(\theta) - \lambda \sum_{k=1}^{n-1} g_k \tilde{v}_{n-k}(\theta), \quad 2 \le n \le N-1, \tag{2.18}$$

$$\theta \tilde{v}_n(\theta) - v_n(0) = \lambda \tilde{v}_n(\theta) - v_{n+1}(0)\tilde{S}(\theta) - U_n(0)\tilde{S}(\theta) - \lambda \sum_{k=1}^{n-1} g_k \tilde{v}_{n-k}(\theta), \qquad n \ge N. \tag{2.19}$$

We define the following generating functions (GF) for $|Z| \le 1$:

– $\vartheta(Z) = \sum_{n=0}^{N-1} \vartheta_n Z^n$: the generating function for the number of customers in the system when the server is idle.

– $\tilde{U}(Z,\theta) = \sum_{n=N}^{\infty} \tilde{U}_n(\theta)Z^n$: the joint transform of the number of customers in the system during the setup period and the remaining setup time.

– $U(Z,0) = \sum_{n=N}^{\infty} U_n(0)Z^n$: the generating function for the number of customers in the system at the end of the setup period.

– $\tilde{v}(Z,\theta) = \sum_{n=1}^{\infty} \tilde{v}_n(\theta)Z^n$: the joint transform of the number of customers in the system during the busy period and the remaining service time of the customer currently being served.

– $v(Z,0) = \sum_{n=1}^{\infty} v_n(0)Z^n$: the generating function for the number of customers in the system at the time of departure.

With these definitions, we proceed to derive expressions for $\tilde{U}(Z,\theta)$ and $\tilde{v}(Z,\theta)$ as stated in the following theorem.

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Theorem 2.1. *Theorem 2.1 establishes the following results for $\tilde{U}(Z,\theta)$ and $\tilde{v}(Z,\theta)$, given that*

$$\lambda\sum_{n=N}^{\infty}\left(\sum_{k=0}^{N-1}\vartheta_k g_{n-k}\right)Z^n=\lambda[\vartheta(Z)\chi(Z)-\vartheta(Z)+\vartheta_0].$$

Then, $\tilde{U}(Z,\theta)$ and $\tilde{v}(Z,\theta)$ are obtained as follows:
1. For $\tilde{U}(Z,\theta)$:

$$\tilde{U}(Z,\theta)=\frac{\lambda[\tilde{U}(\lambda-\lambda\chi(Z))-\tilde{U}(\theta)][\vartheta_0-\vartheta(Z)(1-\chi(Z))]}{\theta-\lambda+\lambda\chi(Z)},$$

2. For $\tilde{v}(Z,\theta)$:

$$\tilde{v}(Z,\theta)=\frac{Z\left[\tilde{S}(\lambda-\lambda\chi(Z))-\tilde{S}(\theta)\right]}{\tilde{S}(\lambda-\lambda\chi(Z))-Z}\times\frac{\lambda\vartheta_0\left[1-\tilde{U}(\lambda-\lambda\chi(Z))\right]-\lambda\tilde{U}(\lambda-\lambda\chi(Z))\left[\chi(Z)\vartheta(Z)-\vartheta(Z)\right]}{\theta-\lambda+\lambda\chi(Z)}.$$

Proof. Let $\Xi_n(Z)=\lambda\sum_{n=N}^{\infty}\left(\sum_{k=0}^{N-1}\vartheta_k g_{n-k}\right)Z^n$. Using (2.9), then we have

$$\begin{aligned}\Xi_n(Z)&=\lambda\left\{\left(\sum_{k=0}^{N-1}\vartheta_k g_{N-k}\right)Z^N+\left(\sum_{k=0}^{N-1}\vartheta_k g_{N+1-k}\right)Z^{N+1}+\left(\sum_{k=0}^{N-1}\vartheta_k g_{N+2-k}\right)Z^{N+2}+\cdots\right\}\\&=\lambda\left\{\sum_{k=0}^{N-1}\vartheta_k Z^k\left[g_{N-k}Z^{N-k}+g_{N+1-k}Z^{N+1-k}+g_{N+2-k}Z^{N+2-k}+\cdots\right]\right\},\end{aligned}$$

we can write

$$\sum_{j=N-k}^{\infty}g_jZ^j=\sum_{j=1}^{\infty}g_jZ^j-\sum_{j=1}^{N-k-1}g_jZ^j,$$

so,

$$\begin{aligned}\Xi_n(Z)&=\lambda\left\{\sum_{k=0}^{N-1}\vartheta_k Z^k\left[\sum_{j=1}^{\infty}g_jZ^j\right]-\sum_{k=0}^{N-1}\vartheta_k Z^k\left[\sum_{j=1}^{N-k-1}g_jZ^j\right]\right\}\\&=\lambda\left\{\vartheta(Z)\chi(Z)-\vartheta_0\left[g_1Z+g_2Z^2+\cdots+g_{N-1}Z^{N-1}\right]-\vartheta_1Z\left[g_1Z+g_2Z^2+\cdots+g_{N-2}Z^{N-2}\right]\right.\\&\quad\left.-\cdots-\vartheta_{N-1}Z^{N-1}[g_1Z]\right\}.\end{aligned}\qquad(*)$$

Also from (2.9), we have

$$\lambda\sum_{n=1}^{N-1}\left(\sum_{k=1}^{n}\vartheta_{n-k}g_k\right)Z^n=\lambda\sum_{n=1}^{N-1}\vartheta_nZ^n=\lambda[\vartheta(Z)-\vartheta_0],$$

so, we can get

$$\begin{aligned}\lambda\Big\{&\vartheta_0g_1Z+[\vartheta_1g_1+\vartheta_0g_2]Z^2+[\vartheta_2g_1+\vartheta_1g_2+\vartheta_0g_3]Z^3\\&+\cdots+[\vartheta_{N-2}g_1+\vartheta_{N-3}g_2+\cdots+\vartheta_0g_{N-1}]Z^{N-1}\Big\}=\lambda[\vartheta(Z)-\vartheta_0],\end{aligned}$$

hence,

$$\begin{aligned}\lambda\Big\{&\vartheta_0[g_1Z+g_2Z^2+\cdots+g_{N-1}Z^{N-1}]+\vartheta_1Z[g_1Z+g_2Z^2+\cdots+g_{N-2}Z^{N-2}]\\&+\cdots+\vartheta_{N-2}Z^{N-2}[g_1Z]\Big\}=\lambda[\vartheta(Z)-\vartheta_0].\end{aligned}$$

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Substituting in (*), then

$$\lambda \sum_{n=N}^{\infty} \left(\sum_{k=0}^{N-1} \vartheta_k g_{n-k} \right) Z^n = \lambda [\vartheta(Z)\chi(Z) - \vartheta(Z) + \vartheta_0]. \tag{2.20}$$

By multiplying Eqs. (2.15) and (2.16) by the appropriate powers of Z and then summing each term, we obtain:

$$\theta \tilde{U}(Z,\theta) - U(Z,0) = \lambda \tilde{U}(Z,\theta) - \lambda \sum_{k=1}^{n-N} g_k Z^k \sum_{n-k=N-k+1}^{\infty} \tilde{U}_{n-k}(\theta) Z^{n-k} - \lambda \tilde{U}(\theta) \sum_{n=N}^{\infty} \left(\sum_{k=0}^{N-1} \vartheta_k g_{n-k} \right) Z^n,$$

which is equivalent to

$$\theta \tilde{U}(Z,\theta) - U(Z,0) = \lambda \tilde{U}(Z,\theta) - \lambda \chi_1(Z) \tilde{U}(Z,\theta) - \lambda \tilde{U}(\theta)[\vartheta(Z)\chi(Z) - \vartheta(Z) + \vartheta_0],$$

where $\chi_1(Z) = \sum_{k=1}^{n-N} g_k Z^k$, then,

$$[\theta - \lambda + \lambda \chi(Z)] \tilde{U}(Z,\theta) = U(Z,0) - \lambda \tilde{U}(\theta)[\vartheta(Z)\chi(Z) - \vartheta(Z) + \vartheta_0]. \tag{2.21}$$

Letting $\theta = \lambda - \lambda \chi(Z)$ in Eq. (2.21), we obtain

$$U(Z,0) = \lambda \tilde{U}[\lambda - \lambda \chi(Z)][\chi(Z)\vartheta(Z) - \vartheta(Z) + \vartheta_0], \tag{**}$$

Substitute from Eq. (**) in Eqn. (2.21), then

$$\tilde{U}(Z,0) = \frac{\lambda [\tilde{U}(\lambda - \lambda \chi(Z)) - \tilde{U}(\theta)][\vartheta_0 - \vartheta(Z)(1 - \chi(Z))]}{\theta - \lambda + \lambda \chi(Z)}. \tag{2.22}$$

Continuing with Eqs. (2.17)-(2.19), we can proceed as follows: Multiply Eq. (2.17) by Z, Eq. (2.18) by Z^n for $2 \le n \le N-1$, and Eq. (2.19) by Z^n for $n \ge N$. After performing these multiplications and adding all the terms together, we arrive at the resulting expression:

$$[\theta - \lambda + \lambda \chi_2(Z)] \tilde{v}(Z,\theta) = v(Z,0) - \frac{\tilde{S}(\theta)}{Z}[v(Z,0) - v_1(0)Z] - \tilde{S}(\theta) U(Z,0), \tag{2.23}$$

where $\chi_2(Z) = \sum_{k=1}^{n-1} g_k Z^k$.
We can derive $v(Z,0)$ from Eq. (2.23) as follows:

$$v(Z,0) = \tilde{S}(\lambda - \lambda \chi(Z))[U(Z,0) - v_1(0)] \left[\frac{Z}{Z - \tilde{S}(\lambda - \lambda \chi(Z))} \right].$$

Substituting this expression into Eq. (2.23), we obtain:

$$[\theta - \lambda + \lambda \chi(Z)] \tilde{v}(Z,\theta) = [U(Z,0) - v_1(0)] \left[\frac{Z \left[\tilde{S}(\lambda - \lambda \chi(Z)) - \tilde{S}(\theta) \right]}{Z - \tilde{S}(\lambda - \lambda \chi(Z))} \right],$$

hence,

$$\tilde{v}(Z,\theta) = \frac{Z \left[\tilde{S}(\lambda - \lambda \chi(Z)) - \tilde{S}(\theta) \right]}{Z - \tilde{S}(\lambda - \lambda \chi(Z))} \frac{[U(Z,0) - v_1(0)]}{\theta - \lambda + \lambda \chi(Z)}.$$

Using (**), we get

$$\tilde{v}(Z,\theta) = \frac{Z \left[\tilde{S}(\lambda - \lambda \chi(Z)) - \tilde{S}(\theta) \right]}{Z - \tilde{S}(\lambda - \lambda \chi(Z))} \frac{[\lambda \tilde{U}[\lambda - \lambda \chi(Z)][\chi(Z)\vartheta(Z) - \vartheta(Z) + \vartheta_0] - v_1(0)]}{\theta - \lambda + \lambda \chi(Z)},$$

since $v_1(0) = \lambda \vartheta_0$, then we can write $\tilde{v}(Z,\theta)$ as

$$\tilde{v}(Z,\theta) = \frac{Z \left[\tilde{S}(\lambda - \lambda \chi(Z)) - \tilde{S}(\theta) \right]}{\tilde{S}(\lambda - \lambda \chi(Z)) - Z} \frac{\lambda \vartheta_0 \left[1 - \tilde{U}[\lambda - \lambda \chi(Z)] \right] - \lambda \tilde{U} \left[\lambda - \lambda \chi(Z) \right] \left[\chi(Z)\vartheta(Z) - \vartheta(Z) \right]}{\theta - \lambda + \lambda \chi(Z)}. \tag{2.24}$$

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Remark 1 *It has been shown that* $\vartheta(Z) = \vartheta_0 \sum_{n=0}^{N-1} \xi_n Z^n$, *where* ξ_n *represents the probability that the system reaches state n (the number of customers) during the idle period. This can be calculated using the relation* $\xi_n = \sum_{k=1}^{n} g_k \xi_{n-k}$. *Consequently, the expression* $\frac{\xi_i}{\sum_{n=0}^{N-1} \xi_n}$ *represents the probability that there are i customers waiting in the system at any given time, assuming the server is in an idle state.*

3 Queue size distribution

In Equations (2.22) and (2.24), $\tilde{U}(Z,\theta)$ and $\tilde{v}(Z,\theta)$ represent the joint transforms during the setup and busy periods. The PGF for the steady-state number of customers in the threshold model system is given by

$$\nu_N(Z) = \vartheta(Z) + \tilde{U}(Z,\theta)\Big|_{\theta=0} + \tilde{v}(Z,\theta)\Big|_{\theta=0}, \tag{3.1}$$

where $\vartheta(Z) = \vartheta_0 \sum_{n=0}^{N-1} \xi_n Z^n$. The unknown variable ϑ_0 in Equation (3.1) is determined by the boundary condition $\nu_N(Z)\Big|_{Z=1} = 1$. This gives us $\vartheta_0 = \frac{(1-\rho)}{(\sum_{n=0}^{N-1} \xi_n + \lambda E(U))}$, with $\rho = \lambda E(\chi)E(S)$. Finally, we present the PGF for the system size, as stated in the following theorem.

Theorem 3.1. *Let* $\nu_N(Z)$ *denote the PGF of the steady-state system size for the* $M^X|G|1$ *threshold model that includes setup time. Then, we have:*

$$\nu_N(Z) = \frac{(1-\rho)(1-Z)\tilde{S}(\lambda-\lambda\chi(Z))(1-\tilde{U}(\lambda-\lambda\chi(Z)))[1-(1-\chi(Z))\sum_{n=0}^{N-1}\xi_n Z^n]}{[\tilde{S}(\lambda-\lambda\chi(Z))-Z][1-\chi(Z)][\sum_{n=0}^{N-1}\xi_n + \lambda E(U)]}. \tag{3.2}$$

Proof. Using equations (2.22) and (2.24), we can derive:

$$\tilde{U}(Z,\theta)\Big|_{\theta=0} = \frac{\lambda[\tilde{U}(\lambda-\lambda\chi(Z)) - \tilde{U}(0)][\vartheta_0 - \vartheta(Z)(1-\chi(Z))]}{-\lambda+\lambda\chi(Z)}. \tag{3**}$$

$$\tilde{v}(Z,\theta)\Big|_{\theta=0} = [\frac{Z[\tilde{S}(\lambda-\lambda\chi(Z)) - \tilde{S}(0)]}{\tilde{S}(\lambda-\lambda\chi(Z)) - Z}][\frac{\lambda\vartheta_0[1-\tilde{U}[\lambda-\lambda\chi(Z)]] - \lambda\tilde{U}[\lambda-\lambda\chi(Z)][\chi(Z)\vartheta(Z) - \vartheta(Z)]}{-\lambda+\lambda\chi(Z)}]. \tag{3***}$$

Additionally, from (3**), and (3***) in (3.1), we have:

$$\nu_N(Z) = \frac{(1-\rho)\sum_{n=0}^{N-1}\xi_n Z^n}{\sum_{n=0}^{N-1}\xi_n + \lambda E(U)} - \frac{[\tilde{U}(\lambda-\lambda\chi(Z)) - 1][\vartheta_0 - \vartheta(Z)(1-\chi(Z))]}{[1-\chi(Z)]} - [\frac{Z[\tilde{S}(\lambda-\lambda\chi(Z)) - 1]}{\tilde{S}(\lambda-\lambda\chi(Z)) - Z}][\frac{\vartheta_0[1-\tilde{U}(\lambda-\lambda\chi(Z))] - \tilde{U}(\lambda-\lambda\chi(Z))\vartheta(Z)(\chi(Z)-1)}{1-\chi(Z)}].$$

Then,

$$\begin{aligned}\nu_N(Z) = \Bigg\{ &\sum_{n=0}^{N-1}\xi_n Z^n (1-\rho)[\tilde{S}(\lambda-\lambda\chi(Z)) - Z][1-\chi(Z)] \\ &- [\tilde{U}(\lambda-\lambda\chi(Z)) - 1][1-\vartheta(Z)(1-\chi(Z))][\tilde{S}(\lambda-\lambda\chi(Z)) - Z][\sum_{n=0}^{N-1}\xi_n + \lambda E(U)] \\ &- [Z[\tilde{S}(\lambda-\lambda\chi(Z)) - 1]][(1-\tilde{U}(\lambda-\lambda\chi(Z))) + \tilde{U}(\lambda-\lambda\chi(Z))\vartheta(Z)(1-\chi(Z))] \times \\ &[\sum_{n=0}^{N-1}\xi_n + \lambda E(U)]\Bigg\}\Bigg\{[\tilde{S}(\lambda-\lambda\chi(Z)) - Z][1-\chi(Z)][\sum_{n=0}^{N-1}\xi_n + \lambda E(U)]\Bigg\}^{-1},\end{aligned}$$

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after simple simplification, we can get

$$\nu_N(Z)=\frac{(1-\rho)(1-Z)\tilde{S}(\lambda-\lambda\chi(Z))(1-\tilde{U}(\lambda-\lambda\chi(Z)))[1-(1-\chi(Z))\sum_{n=0}^{N-1}\xi_n Z^n]}{[\tilde{S}(\lambda-\lambda\chi(Z))-Z][1-\chi(Z)][\sum_{n=0}^{N-1}\xi_n+\lambda E(U)]}. \tag{3.3}$$

Remark 2*We define the probabilities of the system being in the idle state, denoted as $\nu(I)$, and in the setup state, denoted as $\nu(S)$. By setting $Z=1$ in $\vartheta(Z)$, we obtain:*

$$\vartheta(Z)\Big|_{Z=1}=\nu(I)=\frac{(1-\rho)\sum_{n=0}^{N-1}\xi_n}{\sum_{n=0}^{N-1}\xi_n+\lambda E(U)}.$$

Additionally, we have:

$$\tilde{U}(Z,\theta)\Big|_{Z=1,\theta=0}=\nu(S)=\frac{(1-\rho)\lambda E(U)}{\sum_{n=0}^{N-1}\xi_n+\lambda E(U)}.$$

Here, $\vartheta(Z)\Big|_{Z=1}$ and $\tilde{U}(Z,\theta)\Big|_{Z=1,\theta=0}$ represent the probabilities that the system is in the idle and setup states, respectively. Consequently, the term $1-\rho$ indicates the probability that the system is in a vacation period, while $\nu(B)=\rho$ reflects the probability that the server is busy. Next, we examine the PGF of the system size, $\nu_N(Z)$, to gain further insight.

$$\nu_N(Z)=\frac{(1-Z)(1-\rho)\tilde{S}(\lambda-\lambda\chi(Z))}{\tilde{S}(\lambda-\lambda\chi(Z))-Z}\delta_N(Z), \tag{3.4}$$

where

$$\delta_N(Z)=\frac{1-\tilde{U}(\lambda-\lambda\chi(Z))\left[1-(1-\chi(Z))\sum_{n=0}^{N-1}\xi_n Z^n\right]}{[1-\chi(Z)]\left[\sum_{n=0}^{N-1}\xi_n+\lambda E(U)\right]}.$$

This formulation allows us to interpret the dynamics of the system size in relation to various factors, including the idle and busy states.

Remark 3*To find the expected value L_N of the number of customers in the system, we can use the derivative $\frac{d}{dZ}\nu_N(Z)\Big|_{Z=1}$ along with L'Hopital's rule. Thus, the expected value of the steady-state system size for the $M^X|G|1$ threshold model, which includes setup time, is given by:*

$$L_N=\frac{\lambda E(S)E(\chi(\chi-1))+\lambda^2E^2(\chi)E(S^2)}{2(1-\rho)}+\rho+\frac{2\left[\lambda E(U)E(\chi)\sum_{n=0}^{N-1}\xi_n+\sum_{n=0}^{N-1}n\xi_n\right]+\lambda^2E(\chi)E(U^2)}{2\left(\sum_{n=0}^{N-1}\xi_n+\lambda E(U)\right)}. \tag{3.5}$$

4 Waiting-time distribution

Let?s define the following LT for the waiting time of an arbitrarily chosen customer, depending on the state of the system upon arrival:

– $\tilde{W}_I(\theta)$ represents the LT of the waiting time for a customer who arrives when the system is idle.
– $\tilde{W}_U(\theta)$ is the LT of the waiting time for a customer who arrives while the system is in its setup phase.
– $\tilde{W}_B(\theta)$ stands for the LT of the waiting time when the customer arrives during a busy period.

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Following the FCFS approach, our goal is to calculate the overall LT of the waiting time, which we'll call $\tilde{W}_{q,N}(\theta)$, for any customer arriving in the system. So, putting it all together, the LT of the waiting time for this customer, regardless of the system?s current state, is:

$$\tilde{W}_{q,N}(\theta) = \tilde{W}_I(\theta) + \tilde{W}_U(\theta) + \tilde{W}_B(\theta).$$

Next, we?ll look at each of these transforms individually, breaking down how the system's idle, setup, and busy states impact a customer?s wait.

Lee et al. [14] provide the LT for the waiting time distribution of a customer arriving during an idle period, without considering a setup phase. In our model, the customer must also wait for an additional setup time, independent of the idle period. Therefore, the LT of the waiting time $\tilde{W}_I(\theta)$ for a customer who arrives during an idle period is expressed as follows:

$$\tilde{W}_I(\theta) = \frac{\tilde{U}(\theta)(1-\rho)\sum_{n=0}^{N-1}\xi_n[\tilde{S}(\theta)]^n}{\sum_{n=0}^{N-1}\xi_n}\left(\sum_{r=1}^{N-n-1}\frac{g_r\left(1-[\tilde{S}(\theta)]^r\right)}{E(\chi)\left(1-\tilde{S}(\theta)\right)}[\tilde{I}_{N-n-1}(\theta)-1] + \frac{1-\chi[\tilde{S}(\theta)]}{E(\chi)\left(1-\tilde{S}(\theta)\right)}\right),$$

where the LT of the idle period, $\tilde{I}_N(\theta)$, with threshold N and conditioned on the size of the first arrival group, is defined by:

$$\tilde{I}_N(\theta) = \frac{\lambda}{\lambda+\theta}\left(1+\sum_{k=1}^{N-1} g_k\left(I_N^{\tilde{k}}(\theta)-1\right)\right).$$

The LT of the waiting time for a customer who arrives while the system is in its setup phase, see Hur and Ahn [8]

$$\begin{aligned}\tilde{W}_U(\theta) =& \frac{(1-\rho)\lambda E(U)}{\sum_{n=0}^{N-1}\xi_n + \lambda E(U)}\left[1-\left(1-\chi(\tilde{S}(\theta))\right)\sum_{n=0}^{N-1}\xi_n(\tilde{S}(\theta))^n\right] \\ & \left[\frac{\tilde{U}(\lambda-\lambda\chi[\tilde{S}(\theta)])-\tilde{U}(\theta)}{E(U)[\theta-\lambda+\lambda\chi[\tilde{S}(\theta)]]}\right]\left[\frac{1-\chi[\tilde{S}(\theta)]}{E(\chi)[1-\tilde{S}(\theta)]}\right].\end{aligned} \tag{4.1}$$

The waiting time for a customer arriving during a busy period depends on the number of customers already present in the system at the moment of arrival, including the customer currently being served. We have derived the joint transform $\tilde{v}(Z,\theta)$, which incorporates the system size (denoted by Z) and the remaining service time of the customer currently in service (denoted by θ). This leads to the following expression for calculating the waiting time during the busy period:

$$\begin{aligned}\tilde{W}_B(\theta) &= \frac{\tilde{v}(\tilde{S}(\theta),\theta)}{\tilde{S}(\theta)}\cdot\frac{1-\chi[\tilde{S}(\theta)]}{E(\chi)[1-\tilde{S}(\theta)]} \\ &= \frac{1-\rho}{\sum_{n=0}^{N-1}\xi_n+\lambda E(U)}\cdot\frac{1-\chi[\tilde{S}(\theta)]}{E(\chi)[1-\tilde{S}(\theta)]} \\ &\cdot\frac{\lambda[1-\tilde{U}(\lambda-\lambda\chi[\tilde{S}(\theta)])]+(\lambda-\lambda\chi[\tilde{S}(\theta)])\tilde{U}(\lambda-\lambda\chi[\tilde{S}(\theta)])\sum_{n=0}^{N-1}\xi_n\tilde{S}(\theta)^n}{\theta-\lambda+\lambda\chi[\tilde{S}(\theta)]}.\end{aligned} \tag{4.2}$$

4.1 Expected waiting time

To find the expected value of the waiting time $W_{q,N}$ from the LT given by

$$\tilde{W}_{q,N}(\theta) = \tilde{W}_I(\theta) + \tilde{W}_U(\theta) + \tilde{W}_B(\theta),$$

we differentiate $\tilde{W}_{q,N}(\theta)$ with respect to θ, evaluate the result at $\theta = 0$, and use LHopital's rule. This approach allows us to extract the expected waiting time. After carrying out these steps, we obtain the expected waiting time expressed as

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follows:

$$W_{q,N} = \frac{\lambda E^2(\chi)E(S^2) + E(S)E(\chi(\chi-1))}{2E(\chi)(1-\rho)} + \frac{2\left[\lambda E(U)E(\chi)\sum_{n=0}^{N-1}\xi_n + \sum_{n=0}^{N-1}n\xi_n\right] + \lambda^2 E(\chi)E(U^2)}{2\lambda E(\chi)\left[\sum_{n=0}^{N-1}\xi_n + \lambda E(U)\right]}. \tag{4.3}$$

This expression encompasses the contributions to the expected waiting time from various factors, such as service times, the number of customers, and the efficiency of the system as represented by the parameter ρ.

4.2 Expected length of a cycle

A complete cycle consists of three phases: an idle period, a setup period, and a busy period. We already have the LT for the distribution of the idle period. The expected length of the idle period is given by:

$$E(I_N) = -\tilde{I}'_N(0) = \frac{1}{\lambda}\sum_{k=1}^{N-1}\xi_k. \tag{4.4}$$

Next, let's denote N_B as the number of customers in the system at the start of the busy period and $N_B(Z)$ as its PGF. Since N_B represents the total number of customers that arrive during both the idle and setup periods?two independent components?we can express it as follows:

$$N_B(Z) = \left(1 - [1-\chi(Z)]\sum_{n=0}^{N-1}\xi_n Z^n\right)\tilde{U}(\lambda - \lambda\chi(Z)).$$

This equation highlights how the number of customers in the system at the beginning of the busy period is derived from the contributions of customers arriving during both the idle and setup phases. Next, let's define $B_N(\theta)$ as the LT of the length of the busy period. Since each customer experiences their own busy period, we can express this as follows:

$$\tilde{B}_N(\theta) = N_B(Z)\Big|_{Z=\tilde{B}(\theta)} = \left(1 - [1-\chi(\tilde{B}(\theta))]\sum_{n=0}^{N-1}\xi_n(\tilde{B}(\theta))^n\right)\tilde{U}(\lambda - \lambda\chi(\tilde{B}(\theta))), \tag{4.5}$$

where $\tilde{B}(\theta)$ represents the LT of the busy period that starts with one customer in a standard $M^X|G|1$ system. Consequently, we have:

$$\tilde{B}(\theta) = \tilde{S}(\theta + \lambda - \lambda\chi[\tilde{B}(\theta)]).$$

As a result, the expected length of the busy period is given by:

$$E(B_N) = E(N_B)E(B) = \frac{E(\chi)E(S)}{1-\rho}\left(\sum_{n=0}^{N-1}\xi_n + \lambda E(U)\right). \tag{4.6}$$

From the expected values above, we also find that:

$$E(N_B) = \lambda E(\chi)(E(I_N) + E(U)), \tag{4.7}$$

which follows from Wald's equation. Now, the expected length of a complete cycle, denoted $E(T_N)$, can be expressed as:

$$\begin{aligned} E(T_N) &= E(I_N) + E(U) + E(B_N) \\ &= \frac{1}{\lambda}\sum_{n=0}^{N-1}\xi_n + E(U) + \frac{E(\chi)E(S)}{1-\rho}\left(\sum_{n=0}^{N-1}\xi_n + \lambda E(U)\right) \\ &= \frac{1}{\lambda(1-\rho)}\left(\sum_{n=0}^{N-1}\xi_n + \lambda E(U)\right). \end{aligned} \tag{4.8}$$

This final expression encapsulates the expected duration of a full cycle, combining the contributions from idle, setup, and busy periods.

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5 Numerical results

A numerical study is conducted to determine the optimal threshold value N that minimizes the total operational cost in the threshold model, we define two main cost components:

1. Setup Cost (C_s): This cost occurs once per cycle whenever the server initiates the setup. Thus, its contribution to the cost per unit time depends on the cycle length $E(T_N)$, making the per-unit time setup cost $\frac{C_s}{E(T_N)}$.
2. Holding Cost (C_h): This cost applies to each customer present in the system per unit time. With an increase in N, while the per-unit time setup cost decreases, the holding cost increases due to a higher steady-state system size L_N.

Our objective is to balance these costs by choosing an optimal N that minimizes the long-run expected total cost per unit time. We introduce the cost criterion $T_c(N)$ as a function of N:

$$T_c(N) = \frac{C_s}{E(T_N)} + C_h L_N.$$

In this expression:

– The term $\frac{C_s}{E(T_N)}$ captures the setup cost per unit time, which decreases as N increases since a larger threshold reduces the frequency of setups.
– The term $C_h L_N$ represents the total holding cost, which generally increases with N due to the larger average number of customers in the system.

By optimizing $T_c(N)$ with respect to N, we can find the threshold level that minimizes the total cost, achieving an efficient balance between setup and holding costs in the system, see Hur and Ahn [8]. Hence, the optimal threshold N^* is given by $N^* = min\{a \geq m : T_c(a+1) - T_c(a) > 0\}$, where m be the first a such that $T_c(a+1) - T_c(a) > 0$ and $T_c(a)$ is monotone increasing for $a \geq m$.

5.1 Real data analysis

In this subsection, we provide a case study to demonstrate the methodology proposed in previous sections, using a practical scenario in a car engine pistons manufacturing industry. Specifically, this case focuses on a factory where pistons arrive in bulk from a turning center to a CNC copy turning center, and the process of handling these arrivals is modeled as an $M^X/G/1$ queuing system.

Case background

In this setup:

– Pistons arrive in bulk from the turning center to the CNC machine, following a Poisson process with arrival rate λ.
– The operator enters an idle periods whenever the available number of pistons is below a certain minimum threshold after finishing a machine operation. During these breaks, the operator engages in other tasks until the required threshold quantity, N, is reached.
– Upon return from vacation, if the piston count meets or exceeds N, a setup time is required for the operator to prepare the machine for operation.

To balance the workload efficiently between the operator and the CNC machine, management aims to determine the optimal threshold value N that minimizes the overall average cost.

Modeling the System

The queuing system is modeled as an $M^X/G/1$ queue, with the following specific parameter choices based on practical relevance:

1. Service Time Distribution: Follows a 2-Erlang distribution with parameters $k = 2$ and $\mu = 7$.
2. Batch Arrival Process: Modeled as a Poisson process.
3. Vacation Time and Setup Time: Both follow exponential distributions, with arrival rate $\lambda = 10$ and setup rate $\gamma = 7$.
4. Cost Parameters:

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–Threshold values considered: $N = 1-10, 15, 20, 25$
–Setup cost, $C_s = 4$ USD
–Holding cost per customer, $C_h = 6$ USD

Using this setup, we analyze the system?s behavior and determine the optimal threshold level N that minimizes the total cost per unit time, balancing setup and holding costs efficiently. This approach aids management in optimizing operational efficiency while minimizing associated costs.
Table 1 provides the numerical results across various threshold levels, detailing the probabilities of the system being in an idle state $P(I)$, setup state $P(S)$, and busy state $P(B)$. Additionally, we calculate the expected number of pistons in the system at steady state, L_N, along with the waiting time for a randomly selected piston who arrives during the idle ($\tilde{W}_I$), setup ($\tilde{W}_U$), and busy ($\tilde{W}_B$) periods, assuming a FCFS protocol. Following these calculations, we determine the overall waiting time in the queue for the selected piston, $\tilde{W}_{q,N}$, which accounts for all potential arrival periods. The analysis also includes the expected cycle length $E(T_N)$, composed of an idle, setup, and busy period. Finally, we determine the optimal threshold level $N^\star$ that minimizes the total operational cost T_c of the threshold model as outlined in Section 2. This optimal threshold balances the costs associated with setup and holding, providing guidance for efficient resource allocation. Figure 1 illustrates the relationship between the threshold value and the total average cost, as well as the optimal threshold value that minimizes the total average cost. This visual comparison highlights how adjusting the threshold affects overall operational expenses and identifies the threshold point at which costs are minimized.

Table 1: Optimal threshold values and performance measures

$N[N^\star]$	$P(I)$	$P(S)$	$P(B)$	L_N	$\tilde{W}_I$	$\tilde{W}_U$	$\tilde{W}_B$	$\tilde{W}_{q,N}$	$E(T_N)$	$T_c(N)$	$T_c(N^\star)$
1[2]	0.8666	0.0619	0.0714	1.5757	0.9285	0.0619	0.0714	1.0619	0.0616	9.7010	8.3128
2[2]	0.8803	0.0482	0.0714	2.5491	1.4918	0.0482	0.0714	1.6114	0.6798	18.0138	9.7010
3[2]	0.8855	0.0430	0.0714	3.4031	2.1150	0.0430	0.0714	2.2294	1.0218	24.5059	18.0138
4[5]	0.8879	0.0407	0.0714	4.0090	2.5970	0.0407	0.0714	2.7091	1.1982	28.8468	28.8468
5[6]	0.8889	0.0396	0.0714	4.4124	2.9266	0.0396	0.0714	3.0376	1.2886	31.6291	31.6291
6[7]	0.8895	0.0391	0.0714	4.6705	3.1380	0.0391	0.0714	3.2485	1.3350	33.3633	33.3633
7[8]	0.8897	0.0388	0.0714	4.8305	3.2683	0.0388	0.0714	3.3785	1.3589	34.4187	34.4187
8[8]	0.8899	0.0387	0.0714	4.9273	3.3463	0.0387	0.0714	3.4564	1.3711	35.0480	35.0480
9[9]	0.8899	0.0386	0.0714	4.9846	3.3920	0.0386	0.0714	3.5021	1.3774	35.4170	35.4170
10[10]	0.8900	0.0386	0.0714	5.0179	3.4184	0.0386	0.0714	3.5285	1.3806	35.6302	35.4170
15[15]	0.8900	0.0386	0.0714	5.0595	3.4509	0.0386	0.0714	3.5609	1.3839	35.8930	35.8811
20[18]	0.8900	0.0386	0.0714	5.0618	3.4526	0.0386	0.0714	3.5626	1.3840	35.9068	35.9032
25[21]	0.8900	0.0386	0.0714	5.0619	3.4527	0.0386	0.0714	3.5627	1.3842	35.9074	35.9071

Discussion

The numerical findings presented in Table 1 reveal the following insights:

1. System State Probabilities: The probability of the system being in the setup state decreases as the threshold value increases, whereas the probability of it being in the idle state rises with higher thresholds.
2. Total Probability Consistency: Across all threshold values examined, the sum of the probabilities for the system being in idle, setup, or busy states consistently equals one.
3. Waiting Time Trends: As threshold values increase, the waiting times for a randomly chosen piston arriving during idle, $\tilde{W}_I$, as well as the overall queue waiting time $\tilde{W}_{q,N}$, also rise. However, the waiting time for a piston arriving during the setup period, $\tilde{W}_U$, decreases with higher threshold values.
4. Cycle Lengths and Operational Costs: Both the expected cycle lengths and the total operational costs increase as the threshold values grow.
5. Optimal Threshold Level: The threshold level $N^\star$ minimizes the total operational cost T_c of the threshold model, providing a cost-efficient balance between setup and holding costs.
6. Threshold Value Recommendation: For a copy turning center with a maximum threshold of 25 pistons, setting the threshold to 2 achieves the minimum total average cost, according to the model?s findings.
7. Cost Efficiency of Optimal Threshold: The total operational costs T_c calculated for the optimal threshold values are lower than those calculated for the initial threshold values, confirming the cost-saving advantage of this model's threshold optimization approach.

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Figure 1 shows the side-by-side plots based on the table data:

– Left Plot: This shows the relationship between the threshold value N and the total average cost $T_c(N)$. We can see that as N increases, the cost rises sharply and then stabilizes at higher threshold values.
– Right Plot: This depicts the optimal threshold values $N^\star$ and their corresponding total average costs $T_c(N^\star)$. The cost $T_c(N^\star)$ increases with higher optimal threshold levels, highlighting that the minimal cost is achieved at a lower $N^\star$ value.

These plots provide a visual comparison of total costs across different threshold levels and emphasize the impact of selecting the optimal threshold on operational costs.

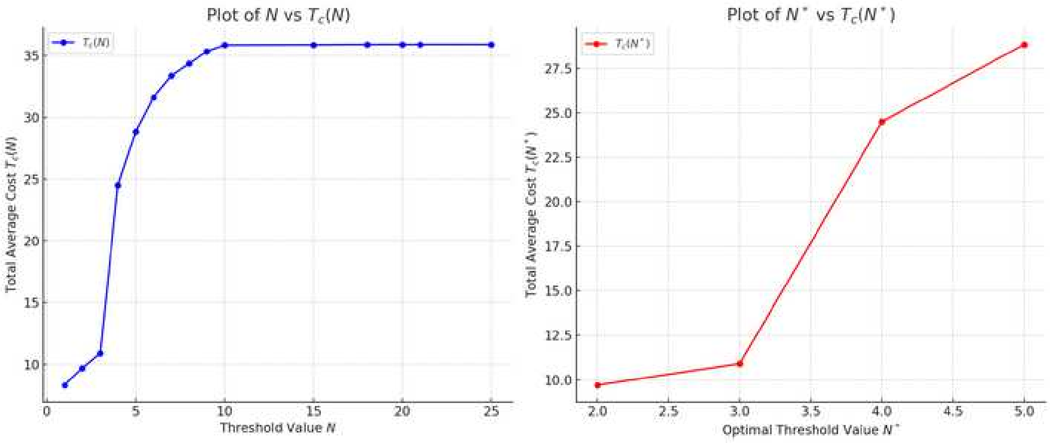

Fig. 1: Left: Threshold N vs. Total Average Cost $T_c(N)$. Right: Optimal Threshold $N^\star$ vs. Total Average Cost $T_c(N^\star)$.

6 Conclusion

We analyzed an $M^X/G/1$ queuing model with a setup period, where the system size can be decomposed into the queue size in a standard $M^X/G/1$ system and the count of customers waiting when the server is unavailable. The probability generating function of the queue size at any point in time was determined, along with key performance metrics. To optimize costs, a cost model was proposed and quantitatively assessed. A practical example was included to illustrate how this cost model can assist managers in manufacturing with informed decision-making. Our findings on system size distribution leveraged multiple analytical approaches. We observed that the system size, as viewed by an incoming group arriving during server downtime, is easier to derive due to the arrival occurring during a defined vacation period. Additionally, under the First-Come-First- Served protocol, the waiting time distribution reflects a combination of two distinct variables, accounting for both multiple and single vacation models within the system size.

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