Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/190118

Averaging Principle of ABC Fractional Stochastic Differential Equations with Rosenblatt Process - Controllability Analysis

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Received: 8 Sep. 2024, Revised: 29 Oct. 2024, Accepted: 1 Nov. 2024
Published online: 1 Jan. 2025

Abstract: The controllability analysis of an averaging technique is investigated using the Rosenblatt process for the Atangana-Baleanu-Caputo fractional derivative (ABC derivative) system. The results of the distinctive averaged system can be used to find solutions to the underlying system's problems in terms of convergence in mean square and probability. Furthermore, by employing the Banach contraction principle, controllability results are proven. Also, numerical examples are given to demonstrate the theoretical findings.

Keywords: Stochastic Fractional Systems; Atangana-Baleanu Caputo Fractional Derivative; Averaging Principle; Controllability; Rosenblatt Process.

1 Introduction

In a real world problems, when we modeling a situations that more and more unpredictable and having fluctuations and noises we need the help of non-integer (random) differential equations([12]). To get the accuracy we ties up the random elements in non-integer order differential equations([5]). These are called the stochastic fractional differential equation, it helps the researchers for a several decades to get the accuracy of complex modeling solutions. We use multiple tactics for averting errors in the presence and distinctiveness of stochastic fractional differential equations (SFDE) like fixed point theorems, integral operator, successive approximation and averaging principle etc. These models are very useful to absorb the applications in several fields in sciences, life-sciences, and biology etc([6–9]).

We reinforced the stochastic component of our system, which is determined by the Rosenblatt process (RP), a straightforward non-Gaussian Hermite function. It also evolved as a constraint in the Non-Central Restrain theoretic. This process is identical in nature and exhibits regular disintegration. The SFDE driven by RP has been investigated by many researchers, (see [3])and references there in. There are many different derivatives and integrals for fractional calculus, quite a few which clash with one another in some areas of their definitional fields. Due to the presence of derivatives, it has become necessitate to examine the characteristics of fractional derivatives that make them ideal for modelling specific intricate structures from many fields of STEM fields. Here, using a series of Riemann-Liouville fractional integrals to represent the ABC fractional derivative with Mittag-Leffler core, we can see the non-locality of the fractional derivative more clearly. It can be implemented for a range of computational jobs and is simpler to utilise with than derivatives. These derivatives, which are easier to employ from a numerical perspective, are used to more accurately represent the hidden characteristics of non-local fluid dynamics. It was introduced in [1]. Some properties of ABC derivative is explained in [2], to more about the derivative (see [4]).

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The popularity of fractional calculus has driven many educators to develop a variety of analytical or numerical techniques to investigate the approximations of nonlinear differential equations using the fractional operator. On the other hand, a typical approach that is highly useful for researching the use of SFDEs in numerous fascinating disciplines is the averaging principle in SFDE. The averaging approach is a crucial tool for striking a balance between complex and straightforward models. A simplified system is used to approximate the original system as part of the averaging process. In other words, using related averaging equations to examine complicated equations allows us to easily and conveniently study its attributes. Some scholars attempted to establish averaging principles for stochastic dynamical systems involving fractional calculus, which is dealing with the fractional derivatives describe the system properties the other one is handling the FBM as the external exication of the system, thus providing reasonable ways for simplifying such kinds of equations. The averaging principle for a dynamical system is crucial in mechanics, control, and a variety of other fields. As in known to all, a lot of problems in theory of differential systems can be solved effectively by the averaging principle. In [6] Guangjun shen et al(2020), investigate averaging principle and stability of hybrid stochastic fractional differential equations driven by Lévy noise. In [7] Hamdy M. Ahmed et al (2021), investigate the averaging principle of Hilfer fractional stochastic delay differential equations with poisson jumps. In [8] Liu et al (2021), study averaging result for implusive fractional neutral stochastic differential equations. In [9] Luo Danfend et al (2020), discussed an averaging principle for stochastic fractional differential equations with time-delays. In [11] Pengju Duan et al(2018), investigate averaging principle for Caputo FSDE driven by FBM with delays. In [14] Wenjing et al (2020), study averaging principle for fractional stochastic differential equations with Lévy noise. In [15] Yong Xu et al (2014), study the averaging principle for SDDE with FBM.

Controllability is one of the most important concepts in mathematical control theory. Since the controllability of fractional stochastic differential equations is typically too powerful to comprehend the dynamical behaviour of such systems, the latter type of control system is more suited for study [10]. There are many deterministic and stochastic structures, and controllability properties play a crucial role in these systems (see [13]). There is no carry out pertinent to the solution of the averaging principle for the ABC fractional SDE with RP in the current corpus of research. Therefore, we have demonstrated in this work how to examine an averaging principle for Atangana–Baleanu Caputo fractional stochastic differential equations with Rosenblatt process - A controllability analysis.
The notable contribution of our work as follows:

– We establish sufficient conditions of an averaging principle for ABC fractional derivative equations with the Rosenblatt process.
– The resulting conclusion in this publication is brand-new in the sense that it generalises a lot of previously published findings, namely for the RP case of ABC fractional derivative stochastic situations.
– We represent the existence of original system and averaged system.
– By demonstrating that the solutions of the averaged equation approach the solutions of the original equation under certain assumptions, we are able to derive an averaging principle for the solution of the system under consideration.
– The controllability criteria of non-linear is proved by employing Banach contraction principle.
– Numerical illustrations were given.

The lineation of this manuscript is manifested here. In section 2, investigate indispensable definitions. In section 3, the representation of our consider system is shown. In Section 4, we established the solution for original and averaged systems. In section 5, we instigate the essential conditions of an averaging principle for our considered systems. In section 6, we instigate the essential conditions of the controllability basis. In section 7, we came up with two numerical examples For proving the value and relevance of theoretical findings. Finally conclusions are worn in Section 8.

2 Preludes:

Definition 1. *[12]: The Mittag-Leffler function as,*

$$A_{\rho,\mathscr{B}}(F)=\sum_{v=0}^{\infty}\frac{F^{v}}{\Gamma(v\rho+\mathscr{B})}.$$

for F is bounded linear operator,
If $\mathscr{B}=1$,

$$A_{\rho}(F)=A_{\rho}(F)=\sum_{v=0}^{\infty}\frac{F^{v}}{\Gamma(v\rho+1)}.$$

Definition 2. *[1]: The fractional integral associate to the ABC derivative is*

$$^{AB}I_{a}^{\rho}g(s)=\frac{1-\rho}{\Omega(\rho)}g(s)+\frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_{a}^{s}g(y)(s-y)^{\rho-1}dy$$

Definition 3. *[1]: The ABC fractional derivative is defined by*

$$^{ABC}D_{a^{+}}^{\rho}g(s)=\frac{\Omega(\rho)}{1-\rho}\int_{a}^{s}g'(y)E_{\rho}(\frac{-\rho}{1-\rho}(s-y)^{\rho})dy$$

for $0<\rho<1$, $a<s<b$, and g is differentiable on [a,b], the function E_{ρ} is the mittag-Leffler function. In general, the normalisation function $\Omega(\rho)$ can be $\Omega(0)=\Omega(1)=1$. Where assume that $\Omega(\rho)$ are real and strictly positive.

2.1 Rosenblatt process

Let $(\Omega, \mathscr{A}, \{\mathscr{A}_s\}_{s\geq 0}, P)$ be a filtered probability space. Suppose that $\{R(s), s \in [0,b]\}$ is the 1-dimensional RP with Hurst parameter $\mathscr{H} \in (\frac{1}{2}, 1)$.

$$E(R(s), R(v)) = \frac{1}{2}(|v|^{2\mathscr{H}} + |s|^{2\mathscr{H}} - |v-s|^{2\mathscr{H}}).$$

The Rp with Hurst parameter $\mathscr{H} > \frac{1}{2}$ is (see [3]):

$$R(s) = d(\mathscr{H}) \int_0^s \int_0^s \Big\{ \int_{X_1 \vee X_2}^s \frac{\partial K^{\mathscr{H}'}}{\partial t}(t, X_1) \frac{\partial K^{\mathscr{H}'}}{\partial t}(t, X_2) dt \Big\} dB(X_1) dB(X_2) \tag{1}$$

Where $\{B(s), s \in [0,b]\}$ is a BM, and $K^{\mathscr{H}}$ is the core

$$K^{\mathscr{H}}(s,v) = c_{\mathscr{H}} v^{\frac{1}{2}-\mathscr{H}} \int_v^s (t-v)^{\mathscr{H}-\frac{3}{2}} t^{\mathscr{H}-\frac{1}{2}} dt$$

Where $c_{\mathscr{H}} = \sqrt{\frac{\mathscr{H}(2\mathscr{H}-1)}{\Gamma(2-2\mathscr{H}, \mathscr{H}-\frac{1}{2})}}$.(for more see [10])
Let Y and X be a Separable Hilbert Space

$$\|\Psi\|_Y^2 = \sup_{t\in J} E\|\Psi\|^2,$$

Definition 4. *[10]: The SFDE* (7) *is said to be completely controllable on I if* $\forall v_1 \in X$, $\exists$ *a control* $w \in L^2(I,Y) \ni$: *the solution* $v(t)$ *is given in* (8) *satisfies* $v(b) = v_1$

Lemma 1. *[10]: **Banach Contraction Principle***
$\mathscr{T}$ has a singular fixed point if U is a Banach space and $\mathscr{T}: U \to U$ is a contraction mapping.

3 System Representation

Consider Atangana - Baleanu Caputo fractional stochastic differential equation with Rosenblatt process

$$\begin{aligned} &{}^{ABC}D^{\rho}(v(g)) = J(g, v(g)) + \Delta(g, v(g)) dz_{\mathscr{H}}(g) \quad g \in I := [0,b], b > 0, \\ &v(0) = v_0. \end{aligned} \tag{2}$$

- Where, ${}^{ABC}D^{\rho}$ is ABC derivative of the order $0 < \rho \leq 1$.
- $v(.) \in X$.
- $J: I \times X \to X$ is a bounded linear operator on X.
- $\Delta: I \times X \to L_2^0$ is a Hilbert-Schmidt operator for all $g \in I$, here $L_2^0 = L_2(Q^{\frac{1}{2}}K, Y)$.
- $z_{\mathscr{H}}(s)$ is a Rp with Hurst parameter $\mathscr{H} \in (\frac{1}{2}, 1)$.
- v_0 is the initial function.

Let consider the assumptions: $(\mathscr{A}_{11})$: For each $v_i \in X$, $U_i \in Y$, i=1,2.
$\exists$ a non-negative function $\mu(t)$, $\ni$:

$$\|J(g,v_1,U_1) - J(g,v_2,U_2)\|^2 + \|\Delta(g,v_1,U_1) - \Delta(g,v_2,U_2)\|^2 \leq \mu(g)(\|(v_1 - v_2)\|^2 + \|U_1 - U_2\|^2).$$

Where

$$\sup_{0\leq t\leq b} \|\mu(g)\|^2 < \infty$$

Remark:
In assumption $(\mathscr{A}_{11})$, if we let $\mu(t) = \mathbf{c}$ (where **c** is a constant), then it becomes Lipschitz condition.

$(\mathscr{A}_{12})$: For each $b_1 \in [0,b]$, $v \in X$ and $U \in Y$,
$\exists$ bounded and measurable functions $\mu_i(t) > 0$, i=1,2,3,4. $J^* : X \times \mathbf{R} \to X$, $\Delta^* : X \times \mathbf{R} \to L_2^0(Y,X)$, such that

$$\|J(s,v,U) - J^*(v,U)\|^2 \le \mu_1(b_1)(\|v\|^2 + \|U\|^2),$$
$$\frac{1}{b_1}\int_0^{b-1}(t-s)^{2\rho-2}\|A(s,v,U) - A^*(v,U)\|^2 ds \le \mu_2(b_1)(\|v\|^2 + \|U\|^2),$$
$$\|\Delta(s,v,U) - \Delta^*(v,U)\|^2 \le \mu_3(b_1)(\|v\|^2 + \|U\|^2),$$
$$\frac{1}{b_1}\int_0^{b-1}(t-s)^{2\rho-2}\|\Delta(s,v,U) - \Delta^*(v,U)\|^2 ds \le \mu_4(b_1)(\|v\|^2 + \|U\|^2),$$

4 Solution Representation

Theorem 1. *Under the $(\mathscr{A}_{11})$, $\exists$ a singular trivial solution v(g) to ABC fractional stochastic differential equation with Rp* (2).

The solution representation of the system (2) is

$$v(g) = v_0 + \frac{1-\rho}{\Omega(\rho)}J(g,v(g)) + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g J(s,v(s))(g-s)^{\rho-1}ds + \frac{1-\rho}{\Omega(\rho)}\Delta(g,v(g)) + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g \Delta(s,v(s))(g-s)^{\rho-1}dz_{\mathscr{H}}(s) \quad (3)$$

It follows from [6] and [11], exists a singular trivial solution v(g) to (2). We omit the proof.
Let us consider the canonical form of (3).

$$v^\varepsilon(g) = v_0 + \frac{\varepsilon(1-\rho)}{\Omega(\rho)}J(g,v^\varepsilon(g)) + \frac{\varepsilon\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g J(s,v^\varepsilon(s))(g-s)^{\rho-1}ds + \frac{\varepsilon^{\mathscr{H}}(1-\rho)}{\Omega(\rho)}\Delta(g,v^\varepsilon(t)) + \frac{\varepsilon^{\mathscr{H}}\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g \Delta(s,v^\varepsilon(s))(g-s)^{\rho-1}dz_{\mathscr{H}}(s) \quad (4)$$

Where $\varepsilon > 0 \in (0,\varepsilon_0]$ with a fixed number ε_0.

The original solution $v^\varepsilon(t)$ converges, as $\varepsilon \to 0$, to the solution $U^\varepsilon(t)$ of the averaged system:

$$U^\varepsilon(g) = v_0 + \frac{\varepsilon(1-\rho)}{\Omega(\rho)}J^*(g,U^\varepsilon(g)) + \frac{\varepsilon\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g J^*(s,U^\varepsilon(s))(g-s)^{\rho-1}ds + \frac{\varepsilon^{\mathscr{H}}(1-\rho)}{\Omega(\rho)}\Delta^*(t,U^\varepsilon(t)) + \frac{\varepsilon^{\mathscr{H}}\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g \Delta^*(s,U^\varepsilon(s))(g-s)^{\rho-1}dz_{\mathscr{H}}(s) \quad (5)$$

Where $J^* : X \times \mathbf{R} \to X$, $\Delta^* : X \times \mathbf{R} \to L_2^0(Y,X)$, are measurable functions.

5 An Averaging Principle

Theorem 2. *Suppose the assumptions $(\mathscr{A}_{11})$ and $(\mathscr{A}_{12})$ hold. Then, for a given arbitrary $\lambda > 0$, $\exists$ constants $p > 0$, $\varepsilon \in (0,\varepsilon_0]$ and $\gamma \in (0,1]$, $\ni$: $\forall$ $\Delta \in (0,\varepsilon_1]$,*

$$\sup_{t\in[0,p\varepsilon^{-\gamma}]} E(\|v^\varepsilon(t) - U^\varepsilon(t)\|^2) \le \lambda$$

Proof: Based on the canonical forms of (4) and (5),

$$v^\varepsilon(g) - U^\varepsilon(g) = \frac{\varepsilon(1-\rho)}{\Omega(\rho)}[J(g,v^\varepsilon(g)) - J^*(g,U^\varepsilon(g))] + \frac{\varepsilon\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g (g-s)^{\rho-1}[J(s,v^\varepsilon(s)) - J^*(s,U^\varepsilon(s))]ds + \frac{\varepsilon^{\mathscr{H}}(1-\rho)}{\Omega(\rho)}[\Delta(g,v^\varepsilon(t)) - \Delta^*(t,U^\varepsilon(t))] + \frac{\varepsilon^{\mathscr{H}}\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g (g-s)^{\rho-1}[\Delta(s,v^\varepsilon(s)) - \Delta^*(s,U^\varepsilon(s))]dz_{\mathscr{H}}(s) \quad (6)$$

$$
\begin{aligned}
E(\|v^{\varepsilon}(g)-U^{\varepsilon}(g)\|^2) \leq\ & E\|\frac{3\varepsilon(1-\rho)}{\Omega(\rho)}[J(g,v^{\varepsilon}(g))-J^*(g,U^{\varepsilon}(g))]\|^2+E\|\frac{3\varepsilon\rho}{\Omega(\rho)\Gamma(\rho)} \\
& \int_0^g (g-s)^{\rho-1}[J(s,v^{\varepsilon}(s))-J^*(s,U^{\varepsilon}(s))]ds\|^2+E\|\frac{3\varepsilon^{\mathscr{H}}(1-\rho)}{\Omega(\rho)} \\
& [\Delta(g,v^{\varepsilon}(g))-\Delta^*(g,U^{\varepsilon}(g))]\|^2+E\|\frac{3\varepsilon^{\mathscr{H}}\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g (g-s)^{\rho-1} \\
& [\Delta(s,v^{\varepsilon}(s))-\Delta^*(s,U^{\varepsilon}(s))]dz_{\mathscr{H}}(s)\|^2 \\
:=\ & I_1+I_2+I_3+I_4
\end{aligned}
$$

By $(\mathscr{A}_{11})$,$(\mathscr{A}_{12})$ and the elementary, Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
\boldsymbol{I_1} &= \frac{6\varepsilon^2(1-\rho)^2}{\Omega(\rho)^2}E\|J(g,v^{\varepsilon}(g))-J^*(g,U^{\varepsilon}(g))\|^2 \\
&\leq \frac{6\varepsilon^2(1-\rho)^2}{\Omega(\rho)^2}E\|J(g,v^{\varepsilon}(g))-J(g,U^{\varepsilon}(g))\|^2+\frac{6\varepsilon^2(1-\rho)^2}{\Omega(\rho)^2}E\|J(g,U^{\varepsilon}(g))-J^*(g,U^{\varepsilon}(g))\|^2 \\
&\leq \frac{6\varepsilon^2(1-\rho)^2}{\Omega(\rho)^2}\sup_{0\leq g\leq b}\mu(g)(E\|v^{\varepsilon}(g)-U^{\varepsilon}(g)\|^2+E\|v^{\varepsilon}(s)-U^{\varepsilon}(s)\|^2) \\
&\quad +\frac{6\varepsilon^2(1-\rho)^2}{\Omega(\rho)^2}\sup_{0\leq s\leq g}\mu_1(s)(\sup_{0\leq s\leq g}E\|v^{\varepsilon}(s)\|^2+\sup_{0\leq s\leq g}E\|U^{\varepsilon}(s)\|^2) \\
\boldsymbol{I_2} &= \frac{6\varepsilon^2(\rho)^2}{\Omega(\rho)^2(\Gamma(\rho))^2}E\|\int_0^g (g-s)^{\rho-1}[J(s,v^{\varepsilon}(s))-J^*(s,U^{\varepsilon}(s)ds\|^2 \\
&\leq \frac{6\varepsilon^2(\rho)^2}{\Omega(\rho)^2(\Gamma(\rho))^2}E\|\int_0^g (g-s)^{\rho-1}[J(s,v^{\varepsilon}(s))-J(s,U^{\varepsilon}(s)ds\|^2 \\
&\quad +\frac{6\varepsilon^2(\rho)^2}{\Omega(\rho)^2(\Gamma(\rho))^2}E\|\int_0^g (g-s)^{\rho-1}[J(s,U^{\varepsilon}(s))-J^*(s,U^{\varepsilon}(s)ds\|^2 \\
&\leq \frac{6\varepsilon^2(\rho)^2}{\Omega(\rho)^2(\Gamma(\rho))^2}\int_0^g (g-s)^{2\rho-2}E\|[J(s,v^{\varepsilon}(s))-J(s,U^{\varepsilon}(s)\|^2 ds \\
&\quad +\frac{6\varepsilon^2(\rho)^2}{\Omega(\rho)^2(\Gamma(\rho))^2}\int_0^g (g-s)^{2\rho-2}E\|[J(s,U^{\varepsilon}(s))-J^*(s,U^{\varepsilon}(s)\|^2 ds \\
&\leq \frac{6\varepsilon^2(\rho)^2}{\Omega(\rho)^2(\Gamma(\rho))^2}\sup_{0\leq g\leq b}\mu(g)\int_0^g (g-s)^{2\rho-2}E\|[(v^{\varepsilon}(r))-(U^{\varepsilon}(r)\|^2 \\
&\quad +E\|[(v^{\varepsilon}(s))-(U^{\varepsilon}(s)\|^2 ds+\frac{6\varepsilon^2(\rho)^2}{\Omega(\rho)^2(\Gamma(\rho))^2}\sup_{0\leq s\leq g}\mu_2(s) \\
&\quad \times(\sup_{0\leq s\leq g}E\|v^{\varepsilon}(s)\|^2+\sup_{0\leq s\leq g}E\|U^{\varepsilon}(s)\|^2) \\
\boldsymbol{I_3} &= \frac{6\varepsilon^{2\mathscr{H}}(1-\rho)^2}{\Omega(\rho)^2}E\|\Delta(g,v^{\varepsilon}(g))-\Delta^*(g,U^{\varepsilon}(g))\|^2 \\
&\leq \frac{6\varepsilon^{2\mathscr{H}}(1-\rho)^2}{\Omega(\rho)^2}E\|\Delta(g,v^{\varepsilon}(g))-\Delta(g,U^{\varepsilon}(g))\|^2+\frac{6\varepsilon^{2\mathscr{H}}(1-\rho)^2}{\Omega(\rho)^2} \\
&\quad \times E\|\Delta(g,U^{\varepsilon}(g))-\Delta^*(g,U^{\varepsilon}(g))\|^2 \\
&\leq \frac{6\varepsilon^{2\mathscr{H}}(1-\rho)^2}{\Omega(\rho)^2}\sup_{0\leq g\leq b}\mu(g)(E\|v^{\varepsilon}(g)-U^{\varepsilon}(g)\|^2+E\|v^{\varepsilon}(s)-U^{\varepsilon}(s)\|^2) \\
&\quad +\frac{6\varepsilon^{2\mathscr{H}}(1-\rho)^2}{\Omega(\rho)^2}\sup_{0\leq s\leq g}\mu_3(s)(\sup_{0\leq s\leq g}E\|v^{\varepsilon}(s)\|^2+\sup_{0\leq s\leq g}E\|U^{\varepsilon}(s)\|^2)
\end{aligned}
$$

$$\begin{aligned}
\boldsymbol{I_4} &= \frac{6\varepsilon^{2\mathscr{H}}(\rho)^2}{\Omega(\rho)^2(\Gamma(\rho))^2}E\|\int_0^g (g-s)^{\rho-1}[\Delta(s,v^{\varepsilon}(s)) - \Delta^*(s,U^{\varepsilon}(s)dz_{\mathscr{H}}(s)\|^2 \\
&\leq \frac{6\mathscr{C}_{\mathscr{H}}\varepsilon^{2\mathscr{H}}(\rho)^2 g^{2\mathscr{H}-1}}{\Omega(\rho)^2(\Gamma(\rho))^2}\int_0^g (g-s)^{2\rho-2}E\|[\Delta(s,v^{\varepsilon}(s)) - \Delta^*(s,U^{\varepsilon}(s)\|^2_{L_2^0}dz_{\mathscr{H}}(s) \quad (where\ \mathscr{C}_{\mathscr{H}} > 0\ is\ a\ constant). \\
&\leq \frac{6\mathscr{C}_{\mathscr{H}}\varepsilon^{2\mathscr{H}}(\rho)^2 g^{2\mathscr{H}-1}}{\Omega(\rho)^2(\Gamma(\rho))^2}\sup_{0\leq g\leq b}\mu(g)\int_0^g (g-s)^{2\rho-2}E\|[(v^{\varepsilon}(r)) - (U^{\varepsilon}(r)\|^2 + E\|[(v^{\varepsilon}(s)) - (U^{\varepsilon}(s)\|^2 ds \\
&\quad + \frac{6\mathscr{C}_{\mathscr{H}}\varepsilon^{2\mathscr{H}}(\rho)^2 g^{2\mathscr{H}-1}}{\Omega(\rho)^2(\Gamma(\rho))^2}\sup_{0\leq s\leq g}\mu_4(s)\times(\sup_{0\leq s\leq g}E\|v^{\varepsilon}(s)\|^2 + \sup_{0\leq s\leq g}E\|U^{\varepsilon}(s)\|^2)
\end{aligned}$$

$$\begin{aligned}
E(\|v^{\varepsilon} - U^{\varepsilon}\|^2) &\leq \frac{6\varepsilon^2(1-\rho)^2}{\Omega(\rho)^2} + \frac{6\varepsilon^{2\mathscr{H}}(1-\rho)^2}{\Omega(\rho)^2}\sup_{0\leq g\leq b}\mu(g)(E\|v^{\varepsilon}(g) - U^{\varepsilon}(g)\|^2 \\
&\quad + E\|v^{\varepsilon}(s) - U^{\varepsilon}(s)\|^2) + \frac{6\varepsilon^2(1-\rho)^2 + 6\varepsilon^{2\mathscr{H}}(1-\rho)^2}{\Omega(\rho)^2}\times\sup_{0\leq s\leq g}\mu_1(s)\sup_{0\leq s\leq g}\mu_3(s)(\sup_{0\leq s\leq g}E\|v^{\varepsilon}(s)\|^2 \\
&\quad + \sup_{0\leq s\leq g}E\|U^{\varepsilon}(s)\|^2) + \frac{6\varepsilon^2\rho^2 g + 6C_{\mathscr{H}}\varepsilon^{2\mathscr{H}}(\rho)^2 g^{2\mathscr{H}-1}}{\Omega(\rho)^2(\Gamma(\rho))^2} \\
&\quad \times\sup_{0\leq g\leq b}\mu(g)\int_0^g (g-s)^{2\rho-2}(E\|v^{\varepsilon}(g) - U^{\varepsilon}(g)\|^2 + E\|v^{\varepsilon}(s) - U^{\varepsilon}(s)\|^2)ds \\
&\quad + \frac{6\varepsilon^2\rho^2 t^2 + 6C_{\mathscr{H}}\varepsilon^{2\mathscr{H}}(\rho)^2 t^{2\mathscr{H}}}{\Omega(\rho)^2(\Gamma(\rho))^2}\times\sup_{0\leq s\leq g}\mu_2(s)\sup_{0\leq s\leq g}\mu_4(s)(g-s)^{2\rho-2} \\
&\quad \times(\sup_{0\leq s\leq g}E\|v^{\varepsilon}(s)\|^2 + \sup_{0\leq s\leq g}E\|U^{\varepsilon}(s)\|^2). \\
&\leq \frac{6\varepsilon^2(1-\rho)^2 + 6\varepsilon^{2\mathscr{H}}(1-\rho)^2}{\Omega(\rho)^2}\sup_{0\leq g\leq b}\mu(g)(E\|v^{\varepsilon}(g) - U^{\varepsilon}(g)\|^2 \\
&\quad + E\|v^{\varepsilon}(s) - U^{\varepsilon}(s)\|^2) + \frac{6\varepsilon^2\rho^2 g + 6C_{\mathscr{H}}\varepsilon^{2\mathscr{H}}(\rho)^2 g^{2-1}}{\Omega(\rho)^2(\Gamma(\rho))^2} \\
&\quad \times\sup_{0\leq g\leq b}\mu(g)\int_0^g (g-s)^{2\rho-2}(E\|v^{\varepsilon}(g) - U^{\varepsilon}(g)\|^2 + E\|v^{\varepsilon}(s) - U^{\varepsilon}(s)\|^2)ds + \Lambda\sum_{i=1}^{4}\sup_{0\leq s\leq g}\mu_i(s) \\
&\quad \times(\sup_{0\leq s\leq g}E\|v^{\varepsilon}(s)\|^2 + \sup_{0\leq s\leq g}E\|U^{\varepsilon}(s)\|^2).
\end{aligned}$$

Where $\Lambda = \frac{(\Gamma(\rho))^2(6\varepsilon^2(1-\rho)^2 + 6\varepsilon^{2\mathscr{H}}(1-\rho)^2) + (\Omega(\rho))^2(6\varepsilon^2\rho^2 g^2 + 6C_{\mathscr{H}}\varepsilon^{2\mathscr{H}}(\rho)^2 g^{2\mathscr{H}})}{\Omega(\rho)^2(\Gamma(\rho))^2}$

When $g \in (0,b) \Rightarrow E(\|v^{\varepsilon}(g) - U^{\varepsilon}(g)\|^2) = 0$

$$\begin{aligned}
E(\|v^{\varepsilon} - U^{\varepsilon}\|^2) &\leq \frac{12\varepsilon^2(1-\rho)^2 + 12\varepsilon^{2\mathscr{H}}(1-\rho)^2}{\Omega(\rho)^2}\sup_{0\leq g\leq b}\mu(g)(E\|v^{\varepsilon}(g) - U^{\varepsilon}(g)\|^2) \\
&\quad + \frac{12\varepsilon^2\rho^2 g + 12C_{\mathscr{H}}\varepsilon^{\mathscr{H}}(\rho)^2 g^{\mathscr{H}} - 1}{\Omega(\rho)^2(\Gamma(\rho))^2}\sup_{0\leq g\leq b}\mu(g)\int_0^g (g-s)^{2\rho-2} \\
&\quad \times(E\|v^{\varepsilon}(s) - U^{\varepsilon}(s)\|^2)ds + \Lambda\sum_{i=1}^{4}\sup_{0\leq s\leq g}\mu_i(s)\times(\sup_{0\leq s\leq t}E\|v^{\varepsilon}(s)\|^2 + \sup_{0\leq s\leq g}E\|U^{\varepsilon}(s)\|^2).
\end{aligned}$$

The Gronwall-Bellman inequality provides us with,

$$\begin{aligned}
E(\|v^{\varepsilon} - U^{\varepsilon}\|^2) &\leq \frac{12\varepsilon^2(1-\rho)^2 + 12\varepsilon^{2\mathscr{H}}(1-\rho)^2}{\Omega(\rho)^2}\sup_{0\leq g\leq b}\mu(g)(E\|v^{\varepsilon}(g) - U^{\varepsilon}(g)\|^2) \\
&\quad + \sum_{v=0}^{\infty}\frac{(12\varepsilon^2\rho^2 g^{\rho+1} + 12C_{\mathscr{H}}\varepsilon^{2\mathscr{H}}(\rho)^2 g^{2\mathscr{H}+\rho-1})^v}{(\Omega(\rho))^v(\Gamma(\rho))^v\Gamma(v\rho+1)}(\sup_{0\leq g\leq b}\mu(g))^v \\
&\quad + \Lambda\sum_{i=1}^{4}\sup_{0\leq s\leq g}\mu_i(s)(\sup_{0\leq s\leq g}E\|v^{\varepsilon}(s)\|^2 + \sup_{0\leq s\leq g}E\|U^{\varepsilon}(s)\|^2).
\end{aligned}$$

So,

$$\sup_{0\le s\le g} E(\|v^{\varepsilon}-U^{\varepsilon}\|^2) \le \frac{12\varepsilon^2(1-\rho)^2+12\varepsilon^{2\mathscr{H}}(1-\rho)^2}{\Omega(\rho)^2}\sup_{0\le g\le b}\mu(g)(E\|v^{\varepsilon}(g)-U^{\varepsilon}(g)\|^2)$$
$$+\sum_{v=0}^{\infty}\frac{(12\varepsilon^2\rho^2 g^{\rho+1}+12C_{\mathscr{H}}\varepsilon^{2\mathscr{H}}(\rho)^2 g^{2\mathscr{H}+\rho-1})^v}{(\Omega(\rho))^v(\Gamma(\rho))^v\Gamma(v\rho+1)}(\sup_{0\le g\le b}\mu(g))^v$$
$$+\Lambda\sum_{i=1}^{4}\sup_{0\le s\le g}\mu_i(s)(\sup_{0\le s\le g}E\|v^{\varepsilon}(s)\|^2+\sup_{0\le s\le g}E\|U^{\varepsilon}(s)\|^2).$$

Let $\beta\in(0,1)$, $p>0$, $\ni$: $\forall\, g\in[0,p\varepsilon^{-\beta}]\subseteq[0,G]$,

$$\sup_{0\le s\le g} E(\|v^{\varepsilon}-U^{\varepsilon}\|^2)\le L\varepsilon^{1-\beta},$$

Here,

$$L=\frac{12\varepsilon^{1-\beta}(1-\rho)^2+12\varepsilon^{(2\mathscr{H}-1)(1-\beta)}}{\Omega(\rho)^2}\sup_{0\le g\le b}\mu(g)(E\|v^{\varepsilon}(g)-U^{\varepsilon}(g)\|^2)$$
$$+\sum_{v=0}^{\infty}\frac{(12\varepsilon^2\rho^2 g^{\rho+1}+12C_{\mathscr{H}}\varepsilon^{2\mathscr{H}}(\rho)^2 g^{2\mathscr{H}+\rho-1})^v}{(\Omega(\rho))^v(\Gamma(\rho))^v\Gamma(v\rho+1)}(\sup_{0\le g\le b}\mu(g))^v$$
$$+\Lambda\sum_{i=1}^{4}\sup_{0\le s\le g}\mu_i(s)(\sup_{0\le s\le g}E\|v^{\varepsilon}(s)\|^2+\sup_{0\le s\le g}E\|U^{\varepsilon}(s)\|^2).$$

Therefore, for any $\lambda>0$, there exists $\varepsilon_1\in(0,\varepsilon_0)$, $\in$: for any $\varepsilon\in(0,\varepsilon_1)$ and $g\in[0,p\varepsilon^{-\beta}]$,

$$\sup_{0\le g\le p\varepsilon^{-\beta}} E(\|v^{\varepsilon}-U^{\varepsilon}\|^2)\le\lambda.$$

Hence the Proof.

Theorem 3. *Under the assumptions* $(\mathscr{A}_{11})$ *and* $(\mathscr{A}_{12})$, *for a arbitrary* $\lambda_1>0$, $\exists$ *constants* $p>0$, $\varepsilon\in(0,\varepsilon_0]$ *and* $\beta\in(0,1]$, $\in$: $\forall\,\Delta\in(0,\varepsilon_1]$,

$$\lim_{\varepsilon\to 0}\mathbb{P}(\sup_{t\in[0,p\varepsilon^{-\beta}]}E(\|v^{\varepsilon}(t)-U^{\varepsilon}(t)\|^2)>\lambda_1)=0.$$

Proof: The Chebyshev- Markov inequality and theorem (2) provides us with, for any $\lambda_1>0$,

$$\mathbb{P}(\sup_{g\in[0,p\varepsilon^{-\beta}]}E(\|v^{\varepsilon}(g)-U^{\varepsilon}(g)\|^2)>\lambda_1)\le\frac{1}{\lambda_1^2}E(\sup_{g\in[0,p\varepsilon^{-\beta}]}\|v^{\varepsilon}(g)-U^{\varepsilon}(g)\|^2)$$
$$\le\frac{L}{\lambda_1^2}\varepsilon^{1-\beta}$$

If $\varepsilon\to 0$, the necessary outcomes are as follows.

Remark:

The probability of the initial solution $v^{\varepsilon}(t)$ and the averaged solution $U^{\varepsilon}(t)$ converges, according to the theorem (3).

6 Controllability Criteria

6.1 System Representation with Control

Consider Atangana - Baleanu Caputo fractional stochastic differential equation with Rosenblatt process

$$\begin{aligned}{}^{ABC}D^{\rho}(v(g))&=A(g,v(g))+\Delta(g,v(g))dz_{\mathscr{H}}(g)+Hw(g)\quad g\in I:=[0,b],b>0,\\ v(0)&=v_0.\end{aligned}\tag{7}$$

–$H : Y \to X$ is a bounded linear operator on X.
–$w(.) \in L^2(I,Y)$,

System (7) has the solution as,

$$\begin{aligned} v(g) = v_0 &+ \frac{1-\rho}{\Omega(\rho)}A(g,v(g)) + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g A(s,v(s))(g-s)^{\rho-1}ds \\ &+ \frac{1-\rho}{\Omega(\rho)}\Delta(g,v(g)) + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g \Delta(s,v(s))(g-s)^{\rho-1}dz_{\mathscr{H}}(s) \\ &+ \frac{1-\rho}{\Omega(\rho)}Hw(g) + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g Hw(s)(g-s)^{\rho-1}ds \end{aligned} \tag{8}$$

we assume the Assumptions,
$(\mathscr{C}_{11})$: Assume that there exists constants $h_1\ h_2 > 0$ for all $v_1, v_2 \in X$ and $g \in I$.

$$\|A(g,v_1) - A(g,v_2)\|^2 \leq h_1\|v_1 - v_2\|^2$$
$$\|\Delta(g,v_1) - \Delta(g,v_2)\|^2 \leq h_2\|v_1 - v_2\|^2$$

$(\mathscr{C}_{12})$: $\Theta : L^2(J,Y) \to X$, the linear operator defined by

$$\Theta = \frac{1-\rho}{\Omega(\rho)}Hw(g) + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g Hw(s)(g-s)^{\rho-1}E\{v/\mathscr{F}_s\}ds$$

Let $\Theta^{-1} : X \to L^2(J,Y)/ker(\Theta)$,inverse operator Θ^{-1} is bounded and $\exists$ constants $\tilde{M}1 > 0$ and $\tilde{M}2 > 0 \in$: $\|H\|^2 \leq \tilde{M}1$ and $\|\Theta^{-1}\|^2 \leq \tilde{M}2$

$(\mathscr{C}_{13})$: Let $\Sigma := D1 + D2 + D1D3\tilde{M}1\tilde{M}2 + D3D2 + D4D1\tilde{M}1\tilde{M}2 + D4D2$,
Where, $D1 = \frac{1-\rho}{\Omega(\rho)}(h1+h2)$, $D2 = \frac{\rho}{\Omega(\rho)\Gamma(\rho)}(h1+h2)$, $D3 = \frac{1-\rho}{\Omega(\rho)}$, $D4 = \frac{\rho}{\Omega(\rho)\Gamma(\rho)}$, be such that $0 \leq \Sigma < 1$

Theorem 4.*Suppose* $(\mathscr{C}_{11})$-$(\mathscr{C}_{13})$ *hold, moreover let* $\Sigma < 1$ *then* (7) *is completely controllable on I.*

Proof: Consider the operator F defined by

$$\begin{aligned} Fv(t) = v_0 &+ \frac{1-\rho}{\Omega(\rho)}A(g,v(g)) + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g A(s,v(s))(g-s)^{\rho-1}ds \\ &+ \frac{1-\rho}{\Omega(\rho)}\Delta(g,v(g)) + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g \Delta(s,v(s))(t-s)^{\rho-1}dz_{\mathscr{H}}(s) \\ &+ \frac{1-\rho}{\Omega(\rho)}Hw(g) + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g Hw(s)(g-s)^{\rho-1}ds \end{aligned}$$

Using assumption $(\mathscr{C}_{12})$, let v_1 be an arbitrary point in X. We have Θ^{-1} is bounded and the control variable w as

$$\begin{aligned} w(g) = E\{\Theta^{-1}[v_1 - v_0 &- \frac{1-\rho}{\Omega(\rho)}A(g,v(g)) - \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^b A(s,v(s))(b-s)^{\rho-1}ds \\ &- \frac{1-\rho}{\Omega(\rho)}\Delta(g,v(g)) - \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^b \Delta(s,v(s))(b-s)^{\rho-1}dz_{\mathscr{H}}(s)]/\mathscr{F}_s\} \end{aligned} \tag{9}$$

Clearly $F(v(b)) = v_1$, To show F has a fixed point.
Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence $\ni$: $v_n \to v$ in X.

$$
\begin{aligned}
\|F(v_n)(g) - F(v)(g)\|^2 \leq\ & \frac{1-\rho}{\Omega(\rho)}\|A(g,v_n(g)) - A(t,v(g))\|^2 + \frac{\rho}{\Omega(\rho)\Gamma(\rho)} \\
& \times \int_0^g \|A(s,v_n(s)) - A(s,v(s))\|^2 (g-s)^{\rho-1} ds \\
& + \frac{1-\rho}{\Omega(\rho)}\|\Delta(g,v_n(g)) - \Delta(g,v(g))\|^2 + \frac{\rho}{\Omega(\rho)\Gamma(\rho)} \\
& \times \int_0^g \|\Delta(s,v_n(s)) - \Delta(s,v(s))\|^2 (g-s)^{\rho-1} dz_{\mathscr{H}}(s) \\
& + \frac{1-\rho}{\Omega(\rho)}\|H\|^2\|w_{v_n}(g) - w_v(g)\|^2 + \frac{\rho}{\Omega(\rho)\Gamma(\rho)} \\
& \times \int_0^g \|H\|^2\|w_{v_n}(s) - w_v(s)\|^2 (g-s)^{\rho-1} ds \\
\leq\ & \frac{1-\rho}{\Omega(\rho)}h_1\|v_n(g) - v(g)\|^2 + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g \|A(s,v_n(s)) - A(s,v(s))\|^2 \\
& \times (g-s)^{\rho-1} ds + \frac{1-\rho}{\Omega(\rho)}h_2\|v_n(g) - v(g)\|^2 + \frac{\rho}{\Omega(\rho)\Gamma(\rho)} \\
& \times \int_0^g \|\Delta(s,v_n(s)) - \Delta(s,v(s))\|^2 (g-s)^{\rho-1} dz_{\mathscr{H}}(s) + \frac{1-\rho}{\Omega(\rho)}\tilde{M}1\tilde{M}2[\frac{1-\rho}{\Omega(\rho)}h_1 \\
& \times E\|v_n(r) - v(r)\|^2 + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^b E\|A(q,v_n(q)) - A(q,v(q))\|^2 \\
& \times (b-q)^{\rho-1} dq + \frac{1-\rho}{\Omega(\rho)}h_2 E\|v_n(r) - v(r)\|^2 + \frac{\rho}{\Omega(\rho)\Gamma(\rho)} \\
& \times \int_0^b E\|\Delta(q,v_n(q)) - \Delta(q,v(q))\|^2 (b-q)^{\rho-1} dz_{\mathscr{H}}(q)] \\
& + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g (g-s)^{\rho-1}\tilde{M}1\tilde{M}2[\frac{1-\rho}{\Omega(\rho)}h_1 E\|v_n(r) - v(r)\|^2 \\
& + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^b E\|A(q,v_n(q)) - A(q,v(q))\|^2 (b-q)^{\rho-1} dq \\
& + \frac{1-\rho}{\Omega(\rho)}h_2 E\|v_n(r) - v(r)\|^2 + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^b E\|\Delta(q,v_n(q)) - \Delta(q,v(q))\|^2 \\
& \times (b-q)^{\rho-1} dz_{\mathscr{H}}(q)] ds
\end{aligned}
$$

We obtain $\lim_{n\to\infty} F(v_n) = F(v)$ in X, because the linear operators A, Δ are continuous on X.

Our claim is F maps X into itself.

$$
\begin{aligned}
\sup_{g\in I} E\|w(g)\|^2 \leq\ & \|\Theta^{-1}\|^2[E\|v_1\|^2 + E\|v_0\|^2 + \frac{1-\rho}{\Omega(\rho)}E\|A(g,v(g))\|^2 + \frac{\rho}{\Omega(\rho)\Gamma(\rho)} \\
& \times \int_0^b E\|A(s,v(s))\|^2 (b-s)^{\rho-1} ds - \frac{1-\rho}{\Omega(\rho)}E\|\Delta(g,v(g))\|^2 \\
& + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^b E\|\Delta(s,v(s))\|^2 (b-s)^{\rho-1} dz_{\mathscr{H}}(s)] \\
\leq\ & 2\tilde{M}2(h_1 + h_2)[\frac{1-\rho}{\Omega(\rho)} + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}] \\
<\ & \infty
\end{aligned}
$$

$$\sup_{g\in I}\|Fv(g)\|^2 \leq 4E\|v_0\|^2 + \frac{1-\rho}{\Omega(\rho)}(h_1+h_2) + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}(h_1+h_2)$$
$$+\tilde{M}1\{2\tilde{M}2[\frac{1-\rho}{\Omega(\rho)}(h_1+h_2) + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}(h_1+h_2)]\}$$
$$(\frac{1-\rho}{\Omega(\rho)} + \frac{\rho}{\Omega(\rho)\Gamma(\rho)})$$
$$< \infty$$

Now, for $v_1, v2 \in X$ we have

$$\sup_{g\in I}\|Fv_1(g) - Fv_2(g)\|^2 \leq \sup_{g\in I} E\|\frac{1-\rho}{\Omega(\rho)}[A(g,v_1(g)) - A(g,v_2(g))] + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}$$
$$\times \int_0^g [A(s,v_1(s)) - A(s,v_2(s))](g-s)^{\rho-1}ds$$
$$+\frac{1-\rho}{\Omega(\rho)}[\Delta(g,v_1(g)) - \Delta(g,v_2(g))] + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}$$
$$\times \int_0^g [\Delta(s,v_1(s)) - \Delta(s,v_2(s))](g-s)^{\rho-1}dz_{\mathscr{H}}(s)$$
$$+\frac{1-\rho}{\Omega(\rho)}H\Theta^{-1}\{\frac{1-\rho}{\Omega(\rho)}[A(r,v_1(r)) - A(r,v_2(r))]$$
$$+\frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^b [A(q,v_1(q)) - A(q,v_2(q))]$$
$$\times(b-q)^{\rho-1}dq + \frac{1-\rho}{\Omega(\rho)}[\Delta(r,v_1(r)) - \Delta(r,v_2(r))]$$
$$+\frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^b [\Delta(q,v_1(q)) - \Delta(q,v_2(q))](b-q)^{\rho-1}dz_{\mathscr{H}}(q)\}$$
$$+\frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^g (g-s)^{\rho-1}H\Theta^{-1}\{\frac{1-\rho}{\Omega(\rho)}[A(r,v_1(r)) - A(r,v_2(r))]$$
$$+\frac{\rho}{\Omega(\rho)\Gamma(\rho)}\int_0^b [A(q,v_1(q)) - A(q,v_2(q))](b-q)^{\rho-1}dq$$
$$+\frac{1-\rho}{\Omega(\rho)}[\Delta(r,v_1(r)) - \Delta(r,v_2(r))] + \frac{\rho}{\Omega(\rho)\Gamma(\rho)}$$
$$\times \int_0^b [\Delta(q,v_1(q)) - \Delta(q,v_2(q))](b-q)^{\rho-1}dz_{\mathscr{H}}(q)\}\|^2$$
$$\leq 8\{D1 + D2 + D3D1\tilde{M}1\tilde{M}2 + D3D2 + D4D1\tilde{M}1\tilde{M}2 + D4D2\}\|v_1 - v_2\|^2$$
$$\leq 8\Sigma\|v_1 - v_2\|^2$$

The assumptions of theorem (4) are satisfied, therefore F is a contraction mapping and $\exists$ a singular point $v_1 \in X$ for F satisfied $Fv(b) = v_1$. Hence from theorem(4) we conclude that the non-linear fractional stochastic system (7) is completely controllable on I.

7 Examples

*Example 1.*Let us consider the following fractional stochastic differential equation

$$^{ABC}D^{\rho}(v(g)) = (v^{\varepsilon}(g)sin^2g - gv^{\varepsilon}(g)cosg) + cdz_{\mathscr{H}}(g), \quad g \in I := [0,\pi], \tag{10}$$

Where $g \in I := [0,\pi]$, $A(g,v(g)) = (v^{\varepsilon}(g)sin^2g - gv^{\varepsilon}(g)cosg)$ and $\Delta(g,v(g)) = c$(c is a constant), $\frac{1}{2} < \rho < 1$.
Here, A, Δ satisfies $(\mathscr{A}_{11})$ and $(\mathscr{A}_{12})$,

Define $A^*(g,v(g))$ as follows

$$\begin{aligned}\int_0^\pi A^*(g,v(g))dt &= \frac{1}{\pi}\int_0^\pi A(g,v(g))dg\\ &= \frac{v}{\pi}\int_0^\pi (sin^2 g - gcosg)dg\\ &= \frac{v}{\pi}\{\int_0^\pi (sin^2 g)dg - \int_0^\pi (gcosg)dg\}\\ &= \frac{v}{\pi}\{\int_0^\pi \frac{(1-cos2g)}{2}dg - [gsing]_0^\pi - \int_0^\pi singdg]\}\end{aligned}$$

we can derive that

$$A^*(g,v(g)) = v(\frac{\pi+4}{2\pi}).$$

$$\Delta^*(g,v(g)) = c$$

The averaging from of (10) can be defined as

$$^{ABC}D^\rho(U(g)) = U(g)(\frac{\pi+4}{2\pi}) + cdz_{\mathscr{H}}(g), \quad g\in I := [0,\pi], \tag{11}$$

Clearly, for $\lambda = 2g^2 > 0$ and $g\in[0,\pi]$,

$$\sup_{g\in[0,\pi]} E(\|v^\varepsilon(g) - U^\varepsilon(g)\|^2) \le \lambda$$

According to theorems (2) and (3), the results can be checked here that the solution of averaged system (11) will converge to that the standard stochastic system (10) in the sense of mean square.

*Example 2.*Evaluate the non-linear SFDE with RP,

$$\begin{aligned} &^{ABC}D^\rho(v(g)) = A(g,v(g)) + \Delta(g,v(g))dz_{\mathscr{H}}(g) + Hw(g) \quad g \in= [0,1],\\ &v(0) = v_0. \end{aligned}\tag{12}$$

Where $\rho = \frac{1}{2}$, $v(g) = \begin{pmatrix} v_1(g)\\ v_2(g)\end{pmatrix}$, for $g\in[0,1]$,
$A = \begin{pmatrix} 0 & -0.5\\ 0.5 & 0\end{pmatrix}$; $H = \begin{pmatrix} 0\\ 1\end{pmatrix}$; $\Delta(g,v(g)) = \begin{pmatrix} e^{-g}\, sing\, v_1(g)\\ (e^g+1)\, sing\, v_2(g)\end{pmatrix}$
the solution of (12) is given by

$$\begin{aligned} v(g) = v_0 &+ \frac{1-0.5}{B(0.5)}A(g,v(g)) + \frac{0.5}{B(0.5)\Gamma(0.5)}\int_0^g A(s,v(s))(g-s)^{0.5-1}ds\\ &+ \frac{1-0.5}{B(0.5)}\Delta(g,v(g)) + \frac{0.5}{B(0.5)\Gamma(0.5)}\int_0^g \Delta(s,v(s))(g-s)^{0.5-1}dz_{\mathscr{H}}(s)\\ &+ \frac{1-0.5}{B(0.5)}Hw(g) + \frac{0.5}{B(0.5)\Gamma(0.5)}\int_0^g Hw(s)(g-s)^{0.5-1}ds \end{aligned}$$

we have the control of the system (12) as

$$\begin{aligned} w(g) = E\{\Theta^{-1}[v_1 - v_0 &- \frac{1-0.5}{B(0.5)}A(g,v(g)) - \frac{0.5}{B(0.5)\Gamma(0.5)}\int_0^b A(s,v(s))(b-s)^{0.5-1}ds\\ &- \frac{1-0.5}{B(0.5)}\Delta(g,v(g)) - \frac{0.5}{B(0.5)\Gamma(0.5)}\int_0^b \Delta(s,v(s))(b-s)^{0.5-1}dz_{\mathscr{H}}(s)]/\mathscr{F}_s\} \end{aligned}$$

By computation, we have Σ which is defined in $(\mathscr{C}_{13})$ as $\Sigma = 0.4924 < 1$ $(for\ h_1 = h_2 = 0.1)$, .
All the assumptions of theorem (4) are verified and hence the system (12) is completely controllable on [0,1].

8 Conclusion

This study looks at the analysis of the averaging principle for the fractional equations of ABC derivative using the Rosenblatt process. We looked into the possibility of and the uniqueness of the system of stochastic fractional differential equations. More efforts have been made to create adequate conditions for the analysis of the averaging principle. Suitable examples have been used to illustrate the entire analysis. As a conclusion, the stochastic differential equation with Rosenblatt process and the ABC fractional derivative can be employed as effective tools for analysing the dynamical patterns of a variety of real-world issues. In the future, we shall construct the ABC fractional derivative driven by Rosenblatt's stochastic fractional integro-differential equation's averaging principle. Efficacy of the technique was represented using control theory and expressed the effectiveness with the mathematical tool MATLAB.

Acknowledgments

This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2024/R/1445). The authors are thankful to the Deanship of Graduate Studies and Scientific Research at University of Bisha for supporting this work through the Fast-Track Research Support Program.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

[1] Abdon Atangana and Dumitru Baleanu , New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, *Thermal Science*, 20(2), (2016), 764-769.
[2] Behzad Ghanbari, Abdon Atangana, A new application of fractional Atangana-Baleanu derivatives: designing ABC-fractional masks in image processing, *Physica A: Statistical Mechanics and Its Applications*, 542, (2020), 123516.
[3] Ciprian A. Tudor, "Analysis of the Rosenblatt process", *ESAIM, Probability and Statistics*, 12, (2008), 230-257.
[4] Dumitru Baleanu and Arran Fernandez, "On some new properties of fractional derivative with Mittag-Leffler kernel", *Nonlinear Science and Numerical Simulation*, 59, (2018), 444-462.
[5] Friedman A., "Stochastic differential equations and applications", *Dover publications*, (2004).
[6] Guangjun Shen, Ruidong Xiao and Xiuwei Yin., "Averaging principle and stability of hybrid stochastic fractional differential equations driven by Levy noise", *International Journal of Systems Science*, 51(12), (2020), 2115-2133.
[7] Hamdv M.Ahmed and Quanxin Zhu, "The averaging principle of Hilfer fractional stochastic delay differential equations with Poisson jumps", *Applied Mathematics Letters*, 112, (2021), 106755.
[8] Liu J and Xu W, "An averaging result for impulsive fractional neutral stochastic differential equations", *Applied Mathematics Letters*, 114, (2021), 106892.
[9] Luo Danfeng, Quanxin Zhu, and Z higuo Luo, "An averaging principle for stochastic fractional differential equations with time-delays", *Applied Mathematics Letters*, 105, (2020), 106290.
[10] Naima Hakkar, Lavanya Muruganantham, Amar Debbouche and Sundara Vadivoo, "Nonlinear fractional order neutral-type stochastic integro-differential system with Rosenblatt process - A controllability exploration", *Proceedings of the Institute of Mathematics and Mechanics National Academy of Sciences of Azerbaijan*, 48, (2022), 68-83.
[11] Pengju Duan, Hao Li, Jie Li and Pei Zhang, "Averaging principle for Caputo fractional stochastic differential equations driven by fractional Brownian motion with delays", *Complexity*, 8, (2021), 6646843.
[12] Podlubny I., "Fractional Differential Equations", volume 198 of Mathematics in Science and Engineering, *Technical University of Kosice, Kosice, Slovak Rebublic*, (1999).
[13] Vadivoo B.S., Jothilakshmi G, Almalki Y, Debbouche A, and Lavanya M, "Relative controllability analysis of fractional order differential equations with multiple time delays", *Applied Mathematics and Computation, Elsevier*, 428, (2022), 127192.
[14] Wenjing Xu, Jinqiao Duan, and Weixu, "An averaging principle for fractional stochastic differential equations with Levy noise", *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 30, (2020), 083126.
[15] Yong Xu, Bin Pei, and Yongge Li, "An averaging principle for stochastic differential delay equations with fractional Brownian motion", *Abstract and Applied Analysis*, 10, (2014), 479195.