Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/190114

Simultaneous and Alternating Models via Queuing System

Salsabeel M. Abd El-Salam [1], *Shimaa Atef* [1,*], *Essam El-Seidy* [1], *A. Elmasry* [1] *and Amira R. Abdel-Malek* [2]

[1] Department of Mathematics, Faculty of Science, Ain Shams University, Abbasia, Cairo, Egypt
[2] Department of Mathematical and Natural Sciences, Faculty of Engineering, Egyptian Russian University, Badr, Egypt

Received: 4 Aug. 2024, Revised: 16 Sep. 2024, Accepted: 26 Oct. 2024
Published online: 1 Jan. 2025

Abstract: It is a privilege for us to examine this unique strategic queuing problem in queuing systems in this research. This area focuses on multiple decision-making queuing entities, such as servers and consumers. This is not the case with the conventional queuing theory, which sees them as passive, non-judgmental entities that are endogenously determined. The multiple agents in a queuing system have conflicting interests, which must be addressed through the use of game theory principles and analytical techniques. Thus, strategic queuing may be defined as the study of queuing systems from a game-theoretical standpoint. We examine a single-queue system in which human servers can decide how diligently to process orders that arrive concurrently or in different orders. We discuss the implications for managers and owners of businesses who are trying to improve service delivery systems. In this paper, we examine $M/M/2/\infty$ through various game theory modes, including strictly alternating, random alternating, and simultaneous games. We also derive the expected waiting time for some of these models.

Keywords: High and Low Effort, Iterated Prisoner's Dilemma Game (IPD), Simultaneous Game, Strictly Alternating Game, Random Alternating Game, Queue Visibility, Single-Queue Systems, Waiting Time

1 Introduction

Queuing theory is a branch of mathematics that examines how waiting areas, and operate. It is also known as waiting queue theory and queuing theory. A queue scenario essentially consists of two components: someone who makes a service request; commonly referred to as the customer and one who delivers the services is as the server. For instance, keep in mind that the clients in a bank's line-up are individuals who wish to deposit or withdraw money, and the servers are the bank tellers.

The complete queuing system is examined by queuing theory, including factors like the frequency of customer arrivals, the number of servers and clients, the capacity of the waiting space, the average turnaround time for services, and the orderliness of the queue. The queue's discipline is determined by the first-in, first-out, prioritized, or serve in random order principles.

Agner Krarup Erlang, a Danish engineer and mathematician, developed the queuing theory in the early 20th century. While working there, Erlang tried to assess and enhance the operation of the Copenhagen Telephone Exchange. He attempted to determine how many circuits were necessary to provide an acceptable quality of telephone service in order to prevent having clients "on hold" (or in a phone queue) for an inordinate amount of time. He was also curious about the amount of telephone operators needed to manage a specific volume of calls. His mathematical research resulted in his 1920 publication "Telephone Waiting Times," which included some of the first queuing models and set the foundation for applied queuing theory.

The majority of economic activity is driven by queuing systems made up of servers executing a series of (randomly) coming jobs. There are many examples, such as the health care sector, where service providers treat patients, and networks, where each server or computer performs some of the orders given to it, and the retail sector, where individuals and businesses sell goods that consumers want to buy, the manufacturing sector, where raw materials are transformed into final things by a blend

* Corresponding author e-mail: shimaa.atef@sci.asu.edu.eg & shimaa.atef_90@yahoo.com

of human and nonhuman employees. Let's look at two instances to provide context. A bank's customers are those who want to deposit or withdraw money, and the bank teller employees are the servers in this scenario. The requests that have been sent to the printer are the customers, and the server is the printer when examining the queuing condition of a printer. Thus, it should come as no surprise that queuing theory has a rich history in a variety of fields, such as mathematics [1,2], operations research [3,4], management [5,6], and economics [7,8].

The frequent interaction between servers in multi-server queuing systems is an essential but frequently underappreciated aspect. The opportunity for reputation-building and reciprocity presented by recurrent engagement allows for more sophisticated decision-making on the part of servers. The best that we can tell, these problems have not been researched in the context of queuing systems, even though such strategies have been addressed in the theoretical and experimental literature on repeated games [9]. The stochastic character of customer arrivals and the dynamic impact of servers' decisions are what distinguish the queuing scenario. Particularly, when servers put in a lot of effort, more customer requests are handled and the wait is probably going to get shorter. The short-term incentives for the servers are impacted by the change in the number of open orders, making the minimal effort more alluring. On the other side, when servers work inefficiently, few customer orders are fulfilled and the line is likely to become longer, making the incentives to work inefficiently less appealing.

More specifically, unless the strategic nature of its many agents is taken into consideration, an economic evaluation of a queuing system cannot be realistic. This viewpoint was first put forth in a seminal study by Naor [10], who investigated the join-or-balk problem for consumers in the M/M/1 queue when its queue length is visible, about 50 years ago. He also gave thought to the issue of a monopolist and a social planner who, by taking into account the strategic behavior of their customers, maximize their profits and, respectively, the welfare of society. By taking into account the same issues for the unobservant form of the system, Edelson and Hildebrand added to Naor's [11] work by taking into account the identical issues for the system's unobservable form. They assume that the system has attained a stochastic stable state and restrict the ability of the arriving consumers to monitor the system's customer count, forcing them to base their join-or-balk decisions entirely on its operational and economic aspects. Since then, a lot more research has been done on strategic behavior in queuing systems.

There are many game models, such as the simultaneous model, in which players make their decisions without being aware of one another's choices. Along with the alternating model, which is used, for example, in the game of chess, and allows players to make decisions in turns. This model comes in two varieties: strictly alternating models, and random alternating models. We take into account two players and two options per player for a game in strictly alternating models. The player who begins the round with his pick is referred to as the leader, while the other player is referred to as the recipient. Each player in the random model has a predetermined probability of becoming the leader. In our paper, we will study the three models mentioned and we will discuss theses models through All D $S_0 = (0,0,0,0)$, Grim $S_8 = (1,0,0,0)$, TFT $S_{10} = (1,0,1,0)$ or All C $S_{15} = (1,1,1,1)$ strategies.

2 Invisible Queuing

Consider a single-queue system $(M/M/2/\infty)$ with two-server. Suppose that λ is the arrival customer rate to the queue and μ is the departure rate from system. We are interested in situations when servers encounter a social conundrum. We take into account a situation where servers have control over how much effort they put out, we'll assume that each server has a choice between two levels of effort. For example, each server has the option to "cooperate" by selecting a high effort or "defect" by selecting a low effort. Let $e \in (h,l)$ denote the effort selected by the server.

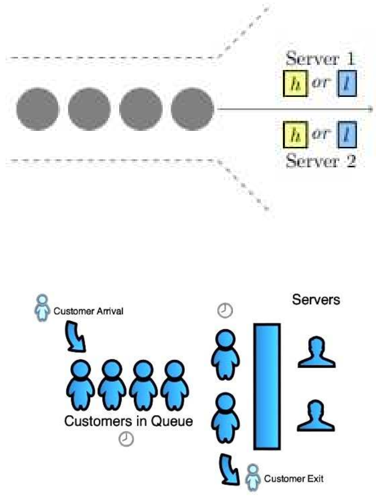


Figure 1: A Single Queue System with Two-Server

In our paper, we will study a system with two servers and an infinite number of customers (the queue is not visible). We will discuss if the servers always do their work at low effort, begin with high effort then work by low effort, determine the effort based on the effort that the other server will make or always do high effort. From a game theory point of view, for two-player Iterated Prisoner's Dilemma games, we have two options for each player where each player takes his decision C or D. So, we have four outcomes $(C,C),(C,D),(D,C)$ or (D,D). Therefore, there are 2^4 different strategies denoted by $S_0,S_1,...,S_{15}$ which can be categorized by (a_1,a_2,a_3,a_4) of zeros and ones where a_i = 0 or 1; if the player plays D or C respectively. Some of these strategies have some special features such as All D (always plays defect no matter what the opposing player decides), Grim (begin with cooperate then plays defect no matter what the opposing player decides), Pavlov "Win Stay Lose Shift" (always plays the role of cooperation if the opposing player makes the same decision and plays the role of disadvantage if the opposing player makes the opposite decision), Tit-For-Tat (plays cooperation if the opposing player also cooperates, and he plays the role of defect if the opposing player decides to play defect, meaning that he takes his decision based on the decision of the opposing player) and All C (always plays cooperate no matter what the opposing player decides) strategies and other strategies [12,13,14].

Furthermore, The two players get a reward, $(\mathscr{R},\mathscr{R})$ for the (C,C) profile, $(\mathscr{S},\mathscr{T})$ for the (C,D) profile, $(\mathscr{T},\mathscr{S})$ for the (D,C) profile and they get Punishment, $(\mathscr{P},\mathscr{P})$, for the (D,D) profile. So, the payoff matrix of (2P-IPD) [12,13] is given by

$$\begin{array}{ccc} & C & D \\ C & \mathscr{R} & \mathscr{S} \\ D & \mathscr{T} & \mathscr{P} \end{array} \tag{1}$$

where

$$\mathscr{S} < \mathscr{P} < \mathscr{R} < \mathscr{T} \text{ and } \mathscr{R} > \frac{\mathscr{T}+\mathscr{S}}{2}. \tag{2}$$

2.1 Simultaneous Model

A simultaneous game, often known as a static game, is one in which each server (player) makes his effort (decision) independently of the efforts made by the other servers. In a simultaneous game, both servers typically take action at the same moment. In simultaneous two-player Iterated Prisoner's Dilemma game, we suppose that player I with $A = (a_1,a_2,a_3,a_4)$ strategy matches the opponent using the $B = (b_1,b_2,b_3,b_4)$ strategy, where a_i or b_i the probability for playing C after the state i. The Markov transition matrix for simultaneous games (M_S) [12,13] is given by

$$M_S = \begin{bmatrix} a_1b_1 & a_1(1-a_1) & (1-a_1)b_1 & (1-a_1)(1-b_1) \\ a_2b_3 & a_2(1-b_3) & (1-a_2)b_3 & (1-a_2)(1-b_3) \\ a_3b_2 & a_3(1-b_2) & (1-a_3)b_2 & (1-a_3)(1-b_2) \\ a_4b_4 & a_4(1-b_4) & (1-a_4)b_4 & (1-a_4)(1-b_4) \end{bmatrix} \tag{3}$$

Assuming that eigenvalue 1's corresponding left eigenvector for the transition matrix is Π, then

$$\Pi \, M_S = \Pi \tag{4}$$

where

$$\Pi = (\pi_1,\pi_2,\pi_3,\pi_4) \tag{5}$$

$$\sum_{i=1}^{4} \pi_i = 1 \tag{6}$$

The payoff for the player using A against an opponent using B is given by

$$E(A,B) = \pi_1\mathscr{R} + \pi_2\mathscr{S} + \pi_3\mathscr{T} + \pi_4\mathscr{P} \tag{7}$$

Now, we will show an example, If two servers play with S_{10}

- Case 1: If the two servers play with high effort

ServerI	C	C	C	C	C	C		
SeverII	C	C	C	C	C	C		
	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\mathscr{R}$	$\longrightarrow$	$\mathscr{R}$

- Case 2: If the first server starts with high effort and the second one with low effort

ServerI	C	D	C	D	C	D		
ServerII	D	C	D	C	D	C		
	$\mathscr{S}$	$\mathscr{T}$	$\mathscr{S}$	$\mathscr{T}$	$\mathscr{S}$	$\mathscr{T}$	$\longrightarrow$	$\frac{\mathscr{T}+\mathscr{S}}{2}$

- Case 3: If the fist server starts with low effort and the second one with high effort

ServerI	D	C	D	C	D	C		
ServerII	C	D	C	D	C	D		
	$\mathscr{T}$	$\mathscr{S}$	$\mathscr{T}$	$\mathscr{S}$	$\mathscr{T}$	$\mathscr{S}$	$\longrightarrow$	$\frac{\mathscr{T}+\mathscr{S}}{2}$

- Case 4: If the two servers play with low effort

ServerI	D	D	D	D	D	D		
SeverII	D	D	D	D	D	D		
	$\mathscr{P}$	$\mathscr{P}$	$\mathscr{P}$	$\mathscr{P}$	$\mathscr{P}$	$\mathscr{P}$	$\longrightarrow$	$\mathscr{P}$

We have three regimes based on the prior situations as

$R_1 = \mathscr{R},$

$R_2 = \dfrac{\mathscr{T}+\mathscr{S}}{2},$

$R_3 = \mathscr{P},$

The result of the disturbance is as follows:

Table 1: The perturbation

Regimes	Perturbation	Results
Regime R_1	server I plays D instead of C server II plays D instead of C	$R_1 \longrightarrow R_2$ $R_1 \longrightarrow R_2$
Regime R_2	column 1 server I plays D instead of C server II plays C instead of D column 2 server I plays C instead of D server II plays D instead of C	$R_2 \longrightarrow R_3$ $R_2 \longrightarrow R_1$ $R_2 \longrightarrow R_1$ $R_2 \longrightarrow R_3$
Regime R_3	server I plays C instead of D server II plays C instead of D	$R_3 \longrightarrow R_2$ $R_3 \longrightarrow R_2$

Thus, According to Table 1, the associated transition matrix changes

$$P = \begin{bmatrix} & R_1 & R_2 & R_3 \\ R_1 & 0 & 1 & 0 \\ R_2 & 1/2 & 0 & 1/2 \\ R_3 & 0 & 1 & 0 \end{bmatrix} \tag{8}$$

We obtain the following equations by computing the left eigenvectors for the eigenvalue 1:

$$-v_1 + \frac{1}{2}v_2 = 0, \tag{9}$$

$$v_1 - v_2 + v_3 = 0, \tag{10}$$

$$\frac{1}{2}v_2 - v_3 = 0 \tag{11}$$

By solving (9), (10) and (11) with $v_1+v_2+v_3 = 1$ as a linear system of equations, then we obtain

$$v = (v_1, v_2, v_3) = (1/4, 1/2, 1/4) \tag{12}$$

$$\begin{aligned} E(S_{10}, S_{10}) &= v_1.R_1 + v_2.R_2 + v_3.R_3 \\ &= \frac{1}{4}\mathscr{R} + \frac{1}{2}\frac{\mathscr{T}+S}{2} + \frac{1}{4}\mathscr{P} \\ &= \frac{1}{4}\mathscr{R} + \frac{1}{4}\mathscr{S} + \frac{1}{4}\mathscr{T} + \frac{1}{4}\mathscr{P} \end{aligned} \tag{13}$$

Furthermore, the payoff vector is equal to $(1/4,1/4,1/4,1/4) \equiv (1,1,1,1)$. Using the same approach, we obtained all payoff vectors as shown in the following table.

Table 2: The payoff for server I against server II

	All D	Grim	TFT	All C
All D	$(0,0,0,1)$	$(0,0,0,1)$	$(0,0,0,1)$	$(0,0,1,0)$
Grim	$(0,0,0,1)$	$(0,0,0,1)$	$(1,1,1,1)$	$(1,0,0,0)$
TFT	$(0,0,0,1)$	$(0,0,0,1)$	$(1,1,1,1)$	$(1,0,0,0)$
All C	$(0,1,0,0)$	$(1,2,0,0)$	$(1,0,0,0)$	$(1,0,0,0)$

2.1.1 Waiting Time for Simultaneous Model

According to the queuing theory, a line can be studied in terms of six different components: the arrival process, the service process, the departure process, the number of servers available, the queue discipline, the queue capacity, and the number of people serviced. The causes of the congestion can be found and addressed by building a model of the complete procedure from start to finish. So, in this sub-subsection, we will deduce the waiting time in the system and queuing for simultaneous case depending on the arrival and departure rates [15]. Let L_Q, W_Q, L_S and L_Q be the expected numbers of customers and expected waiting time in queuing and in the system, respectively.

Let $\Lambda = \frac{\lambda}{\mu}$, we obtain the probability of n-customer in the system (steady state), where λ is the arrival customer rate to the queue and μ is the departure rate from system.

$$P_n = \begin{cases} \frac{\Lambda^n}{n!} P_0 & if \ \ 0 \le n \le c \\ \frac{\Lambda^n}{c^{n-c}\ c!} P_0 & if \ \ n \ge c \end{cases} \tag{14}$$

$$\begin{aligned} P_0 &= \left[1 + \sum_{n=1}^{\infty} P_n\right]^{-1} \\ &= \left[\sum_{n=0}^{c-1} \frac{\Lambda^n}{n!} + \sum_{n=c}^{\infty} \frac{\Lambda^n}{c^{n-c}\ c!}\right]^{-1} \\ &= \left[\sum_{n=0}^{c-1} \frac{\Lambda^n}{n!} + \frac{\Lambda^n}{c!} \sum_{n=c}^{\infty} \frac{\Lambda}{c}^{n-c}\right]^{-1} \\ &= \left[\sum_{n=0}^{c-1} \frac{\Lambda^n}{n!} + \frac{\Lambda^n}{c!} \frac{1}{1-\frac{\Lambda}{c}}\right]^{-1} \end{aligned} \tag{15}$$

where $\frac{\Lambda}{c} < 1$

Then,

$$
\begin{aligned}
P_2 &= \frac{\Lambda^2}{2} P_0 \\
&= \frac{\Lambda^2}{2} \left[1 + \Lambda + \frac{\Lambda^2}{2} \frac{1}{1-\frac{\Lambda}{2}}\right]^{-1} \\
&= \frac{\Lambda^2}{2} \left[1 + \Lambda + \frac{\Lambda^2}{2-\Lambda}\right]^{-1} \\
&= \frac{\Lambda^2}{2} \left[1 + \frac{2\Lambda}{2-\Lambda}\right]^{-1}
\end{aligned} \tag{16}
$$

The expected number of customers in queuing is given by

$$
\begin{aligned}
L_Q &= \sum_{n=c}^{\infty} (n-c)\, P_n \\
&= \sum_{n=c}^{\infty} (n-c)\, \frac{\Lambda^n}{c^{n-c}\, c!} P_0 \\
&= P_0 \frac{\Lambda^c}{c!} \frac{\Lambda}{c} \sum_{n=c}^{\infty} (n-c) \left(\frac{\Lambda}{c}\right)^{n-c-1} \\
&= P_0 \frac{\Lambda^{c+1}}{c!\, c} \frac{d}{d(\frac{\Lambda}{c})} \sum_{n=c}^{\infty} \left(\frac{\Lambda}{c}\right)^{n-c} \\
&= P_0 \frac{\Lambda^{c+1}}{c!\, c} \frac{d}{d(\frac{\Lambda}{c})} \frac{1}{1-\frac{\Lambda}{c}} \\
&= P_0 \frac{\Lambda^{c+1}}{(c-1)!} \frac{1}{(c-\Lambda)^2} \\
&= P_0 \frac{\Lambda^c}{c!} \frac{\Lambda\, c}{(c-\Lambda)^2} \\
&= P_c \frac{\Lambda\, c}{(c-\Lambda)^2}
\end{aligned} \tag{17}
$$

Then for our study, the expected numbers of customers in queuing will be given as

$$
\begin{aligned}
L_Q &= P_2 \frac{2\Lambda}{(2-\Lambda)^2} \\
&= \frac{\Lambda^2}{2} \left[1 + \frac{2\Lambda}{2-\Lambda}\right]^{-1} \frac{2\Lambda}{(2-\Lambda)^2} \\
&= \frac{\Lambda^3}{(2-\Lambda)^2} \left[1 + \frac{2\Lambda}{2-\Lambda}\right]^{-1}
\end{aligned} \tag{18}
$$

And the expected number of customers in the system (in queuing and in service) is given by

$$L_S = L_Q + \Lambda \tag{19}$$

Therefore, the expected waiting time in queuing is given by

$$W_Q = \frac{L_Q}{\lambda} \tag{20}$$

Then for our study, the expected waiting time in queuing will be given as

$$W_Q = \frac{\Lambda^2}{(2-\Lambda)^2} \left[1 + \frac{2\Lambda}{2-\Lambda}\right]^{-1} \tag{21}$$

Furthermore, the expected waiting time in the system (in both queuing and service) is given by

$$W_S = W_Q + \frac{1}{\mu} \tag{22}$$

Finally, we will deduce the expected number and waiting time for simultaneous case depending on the arrival and the departure rates.

For example: If we have a hospital with two receptionists who receive and serve patients, if each of them can receive 20 patients per hour and the patients arrive at a rate of 30 patients per hour.

Now, we have $\lambda = 30$ and $\mu = 20$, then $\Lambda = 3/2$ and $P_0 = 0.14$.
The expected number of patients waiting for service $L_Q \approx 2$ customer.
And the expected waiting time in the hospital is $W_S = W_Q + \frac{1}{\mu} = \frac{L_Q}{\lambda} + \frac{1}{\mu} = 0.11$ hours.

2.2 Strictly Alternating Model

In Alternating games, one of the two servers (players) makes his effort (decision) in a round, while the other server replays his serves in another round. In strictly alternating two-player Iterated Prisoner's Dilemma game, we suppose that player I with $A = (a_1, a_2, a_3, a_4)$ strategy matches the opponent using the $B = (b_1, b_2, b_3, b_4)$ strategy, where a_i or b_i the probability for playing C after the state i as simultaneous model. But the Markov transition matrix for strictly alternating games (M_{SA}) [16, 17] is given by

$$
M_{SA} = \begin{bmatrix}
a_1b_1 & a_1(1-b_1) & (1-a_1)b_2 & (1-a_1)(1-b_2) \\
a_2b_3 & a_2(1-b_3) & (1-a_2)b_4 & (1-a_2)(1-b_4) \\
a_3b_1 & a_3(1-b_1) & (1-a_3)b_2 & (1-a_3)(1-b_2) \\
a_4b_3 & a_4(1-b_3) & (1-a_4)b_4 & (1-a_4)(1-b_4)
\end{bmatrix} \tag{23}
$$

If we assume Π as the left eigenvector for the transition matrix corresponding to the eigenvalue 1, we get the equation (4) but for the matrix M_{SA} and the payoff for the player using A against an opponent using B as equation (7) in the previous sub-section 2.1.

ServerI	C	C	C	C	C	
SeverII		C	C	C	C	C

$\mathscr{R}\ \mathscr{R}\ \mathscr{R}\ \mathscr{R}\ \mathscr{R}\ \mathscr{R}\ \mathscr{R}\ \mathscr{R}\ \mathscr{R} \longrightarrow \mathscr{R}$

Now, we will show an example, If two servers play with S_{10}

• Case 1: If the two servers play with high effort
• Case 2: If the first server starts with high effort and the second one with low effort

ServerI	C	D	C	D	C	
ServerII		D	C	D	C	D

$\mathscr{S}\ \mathscr{P}\ \mathscr{T}\ \mathscr{R}\ \mathscr{S}\ \mathscr{P}\ \mathscr{T}\ \mathscr{R}\ \mathscr{S} \qquad \frac{\mathscr{S}+\mathscr{P}+\mathscr{T}+\mathscr{R}}{4}$

• Case 3: If the fist server starts with low effort and the second one with high effort

ServerI	D	C	D	C	D	
ServerII		C	D	C	D	C

$\mathscr{T}\ \mathscr{R}\ \mathscr{S}\ \mathscr{P}\ \mathscr{T}\ \mathscr{R}\ \mathscr{S}\ \mathscr{P}\ \mathscr{T} \qquad \frac{\mathscr{S}+\mathscr{P}+\mathscr{T}+\mathscr{R}}{4}$

• Case 4: If the two servers play with low effort

ServerI	D	D	D	D	D	
SeverII		D	D	D	D	D

$\mathscr{P}\ \mathscr{P}\ \mathscr{P}\ \mathscr{P}\ \mathscr{P}\ \mathscr{P}\ \mathscr{P}\ \mathscr{P}\ \mathscr{P} \longrightarrow \mathscr{P}$

We have three regimes based on the prior situations as

$$R_1 = \mathscr{R},$$
$$R_2 = \frac{\mathscr{T}+\mathscr{R}+\mathscr{P}+\mathscr{S}}{4},$$
$$R_3 = \mathscr{P}.$$

The result of the disturbance is as follows:
Thus, according to Table 3, the corresponding transition matrix becomes

$$P = \begin{bmatrix} & R_1 & R_2 & R_3 \\ R_1 & 0 & 0 & 1 \\ R_2 & 1/2 & 0 & 1/2 \\ R_3 & 1 & 0 & 0 \end{bmatrix}$$

Furthermore, the payoff vector is equal to $(1/2,0,0,1/2) \equiv (1,0,0,1)$. Using the same approach, we obtained the results shown in the following table:

Table 3: The perturbation

Regimes	Perturbation	Results
Regime R_1	server I plays D instead of C server II plays D instead of C	$R_1 \longrightarrow R_3$ $R_1 \longrightarrow R_3$
Regime R_2	column 1 server I plays D instead of C server II plays C instead of D	$R_2 \longrightarrow R_3$ $R_2 \longrightarrow R_1$
	column 2 server I plays C instead of D server II plays C instead of D	$R_2 \longrightarrow R_1$ $R_2 \longrightarrow R_1$
	column 3 server I plays C instead of D server II plays D instead of C	$R_2 \longrightarrow R_1$ $R_2 \longrightarrow R_3$
	column 4 server I plays D instead of C server II plays D instead of C	$R_2 \longrightarrow R_3$ $R_2 \longrightarrow R_3$
Regime R_3	server I plays C instead of D server II plays C instead of D	$R_3 \longrightarrow R_1$ $R_3 \longrightarrow R_1$

Table 4: The payoff for server I against server II

	All D	Grim	TFT	All C
All D	(0,0,0,1)	(0,0,0,1)	(0,0,0,1)	(0,0,1,0)
Grim	(0,0,0,1)	(0,0,0,1)	(1,0,0,2)	(1,0,2,0)
TFT	(0,0,0,1)	(0,0,0,1)	(1,0,0,1)	(1,0,0,0)
All C	(0,1,0,0)	(1,2,0,0)	(1,0,0,0)	(1,0,0,0)

2.3 Randomly Alternating Model

In this model, each player has the same chance to be the leader in every round (i.e. with probability $\frac{1}{2}$) for each one. The chance of the player to be a leader in the next round is independent of the players' decision. In random alternating two-player Iterated Prisoner's Dilemma game, we suppose that player I with $A = (a_1, a_2, a_3, a_4)$ strategy matches the opponent using the $B = (b_1, b_2, b_3, b_4)$ strategy, where a_i or b_i the probability for playing C after the state i as simultaneous model. But the Markov transition matrix for random alternating games (M_{RA}) [18] is given by

$$M_{RA} = \frac{1}{2}\begin{bmatrix} a_1 & b_2 & (1-a_1) & (1-b_2) \\ a_2 & b_1 & (1-a_2) & (1-b_1) \\ a_3 & b_4 & (1-a_3) & (1-b_4) \\ a_4 & b_3 & (1-a_4) & (1-b_3) \end{bmatrix} \qquad (24)$$

If we assume Π as the left eigenvector for the transition matrix corresponding to the eigenvalue 1, we get the equation (4) but for the matrix M_{RA} and the payoff for the player using A against an opponent using B as equation (7) in the sub-section 2.1.

We have

$$\pi_3 = \frac{1}{2} - \pi_1 \tag{25}$$

$$\pi_4 = \frac{1}{2} - \pi_2 \tag{26}$$

Then $\Pi = (\pi_1, \pi_2, \frac{1}{2} - \pi_1, \frac{1}{2} - \pi_2)$ and $0 < \pi_1 \pi_2 < \frac{1}{2}$

We get

$$2\pi_1(2 - a_1 + a_3) + 2\pi_2(a_4 - a_2) = a_3 + a_4 \tag{27}$$

$$2\pi_1(b_4 - b_2) + 2\pi_2(2 - b_1 + 2b_3) = b_3 + b_4 \tag{28}$$

Therefore, the payoff of Player I (A-player) can be written in the form

$$\frac{\mathscr{P}}{2} + \pi_1(\mathscr{R} - \mathscr{T}) + \pi_2(\mathscr{T} - \mathscr{P}) \tag{29}$$

By the same approach, the payoff of Player I (A-player) against itself can be written in the form

$$(\frac{1}{2} - \pi_1)\mathscr{P} + \pi_1\mathscr{R} \tag{30}$$

Using the same approach, we obtained the results shown in the following table:

Table 5: The payoff for server I against server II

	All D	Grim	TFT	All C
All D	$(0,0,1,1)$	$(0,0,1,1)$	$(0,1,2,1)$	$(0,1,1,0)$
Grim	$(0,0,1,1)$	$(0,0,1,1)$	$(0,1,2,1)$	$(0,1,1,0)$
TFT	$(0,1,2,1)$	$(0,1,2,1)$	$(1,1,1,1)$	$(1,2,1,0)$
All C	$(0,1,1,0)$	$(0,1,1,0)$	$(1,2,1,0)$	$(1,1,0,0)$

3 The payoff Using Numerical Values

In this section, we use Axelrod's values $\mathscr{S} = 0, \mathscr{P} = 1, \mathscr{R} = 3$ and $\mathscr{T} = 5$ to expect payoff for server I against server II for simultaneous case as in Table 6, for strictly alternating case in Table 7 and for randomly alternating case in Table 8 as

4 Domination

We shall talk about the dominance in this section, we will discuss the domination [13]. If both $a_{nm} > a_{nn}$ and $a_{mm} > a_{nm}$, where a_{nn}, a_{nm}, a_{mn} and a_{mm} are elements of the payoff matrix, then S_n is outcompeted by S_m. We write $S_n << S_m$, if the strategy S_n is outperformed by S_m. Furthermore, the domination is given as

Table 6: The payoff with Axelrod's values **for the simultaneous model**

	All D	Grim	TFT	All C
All D	1	1	1	5
Grim	1	1	1.25	3
TFT	1	1	1.25	3
All C	0	1	3	3

Table 7: The payoff with Axelrod's values **for strictly alternating model**

	All D	Grim	TFT	All C
All D	1	1	1	5
Grim	1	1	1.666	4.333
TFT	1	1	2	3
All C	0	1	3	3

Table 8: The payoff with Axelrod's values **for randomly alternating model**

	All D	Grim	TFT	All C
All D	3	3	2.75	2.5
Grim	3	3	2.75	2.5
TFT	2.75	2.75	2.25	2
All C	2.5	2.5	2	1.5

Table 9: A list of strategies outcompeting S_n **for simultaneous model**

All D	$S_0 <<$	-
Grim	$S_8 <<$	-
TFT	$S_{10} <<$	S_{15}
All C	$S_{15} <<$	S_0

Table 10: A list of strategies outcompeting S_n **for strictly alternating model**

All D	$S_0 <<$	S_{10}
Grim	$S_8 <<$	S_{10}
TFT	$S_{10} <<$	S_{15}
All C	$S_{15} <<$	S_0

Table 11: A list of strategies outcompeting S_n **for randomly alternating model**

All D	$S_0 <<$	-
Grim	$S_8 <<$	-
TFT	$S_{10} <<$	S_0, S_8
All C	$S_{15} <<$	S_0, S_8, S_{10}

5 Conclusion

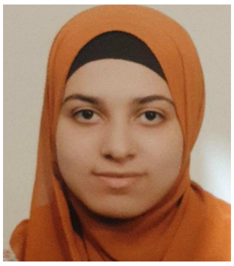
After checking the results, we concluded that if we have an invisible queue and only two servers work to accomplish this work, then for simultaneous or strictly alternating models, it is better for any server to start with a low effort and then limit its effort to what the other server is doing. If the competing server (the second server) always does with high effort or always does with low effort, it will continue with low effort, and if the competing server determines its effort based on the other server's effort, it should switch its effort to high. But for the randomly alternating model, it is better for any server to do low effort at all.

Conflicts of Interest

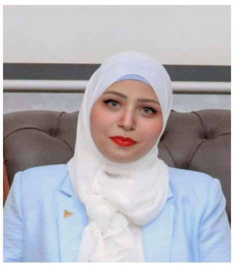
Concerning the publishing of this research, the authors say they have no competing interests.

References

[1] **Kolmogorov, A.**, *Sur le probleme d?attente*, Mathematicheskiy Sbornik, 38(1-2), 101?106, (1931).
[2] **Kendall, D. G.**, *Some problems in the theory of queues*, Journal of the Royal Statistical Society: Series B (Methodological), 13(2), 151?173, (1951).
[3] **Cobham, A.**, *Priority assignment in waiting line problems*, Journal of the Operations Research Society of America, 2(1), 70?76, (1954).
[4] **Little, J. D.**, *A proof for the queuing formula:* $L = \lambda W$, Operations research, 9(3), 383?387, (1961).
[5] **Kao, E. P., and G. G. Tung**, *Bed allocation in a public health care delivery system*, Management Science, 27(5), 507?520, (1981).
[6] **Graves, S. C.**, *The application of queueing theory to continuous perishable inventory systems*, Management Science, 28(4), 400?406, (1982).
[7] **Sah, R. K.**, *Queues, rations, and market: comparisons of outcomes for the poor and the rich*, The American Economic Review, pp. 69?77, (1987).
[8] **Polterovich, V.**, *Rationing, queues, and black markets*, Journal of the Econometric Society, pp. 1?28, (1993).
[9] **Dal Bo, P., and G. R. Frechette**, *The evolution of cooperation in infinitely repeated games: Experimental evidence*, The American Economic Review, 101(1), 411?429, (2011).
[10] **Naor, P.**, *The regulation of queue size by levying tolls*, Econometrica 37, 15?24 (2009).
[11] **Edelson, N.M., Hildebrand, K.**, *Congestion tolls for Poisson queueing processes*, Econometrica 43, 81?92 (1975).
[12] **M.A. Nowak, K. Sigmund and E. El-Sedy**, *Automata, repeated games and noise*, Journal of Mathematical Biology, 33, 703-722, (1995).
[13] **S. Atef, A. R. Abdel-Malek and E. El-Seidy**, Monitor Reaction of Win Stay-Lose Shift Strategies in Iterated Three-Player Prisoner's Dilemma Game, *Information Sciences Letters*, 4, 1-15 (2023).
[14] **E. El-Seidy, K.M. Soliman**, Iterated symmetric three-player prisoner's dilemma game, *Applied Mathematics and Computation*, 282, 117-127 (2016).
[15] **Hamdy A. Taha**, *Research: An Itroduction*, (2016).
[16] **M. A. Nowak and K. Sigmund**, *The alternating prisoner?s dilemma*, Journal of Mathematical Biology, 168,219?226 (1994).
[17] **P. S. Park, M.A. Nowak, K. Sigmund and Ch. Hilbe**, *Cooperation in alternating interactions with memory constraints*, Nature Communications, 13, 737 (2022).
[18] **E. El-Sedy**, *The Adaptive Dynamics For The Randomly Alternating Prisoner's Dilemma Game*, Revista Dela Union Matematica Argentina, 44, 99-108 (2003).



Salsabeel Mohamed is Teaching Assistant of Pure Mathematics in Faculty of computer and information sciences, Ain Shams University, Cairo, Egypt. She received her B.Sc. and M.Sc. degrees in mathematics from Ain Shams University, Cairo, Egypt. Presently, she is working at Mathematics department, Faculty of computer and information sciences, Ain Shams University.



Shimaa Atef is Teaching Assistant of Pure Mathematics in Faculty of Science, Ain Shams University, Cairo, Egypt. She received her B.Sc. and M.Sc. degrees in mathematics from Ain Shams University, Cairo, Egypt. Presently, she is working at Mathematics department, Faculty of Science, Ain Shams University.

Essam Elseidy is Professor of Pure Mathematics in Faculty of Science, Ain Shams University, Cairo, Egypt. He received his B.Sc. and M.Sc. degrees in mathematics from Ain Shams University, Cairo, Egypt. He was awarded his Ph.D. degree in game theory from the same university and university of vienna. Presently, he is working at Mathematics department, Faculty of Science, Ain Shams University. His current interests include Population Game Dynamic, Symmetric and Asymmetric Games, Differential Games.

Ayat Elmasry is lecturer of Mathematical Statistics in Faculty of Science, Ain Shams University, Cairo, Egypt. She received her B.Sc. and M.Sc. degrees in mathematics from Ain Shams University, Cairo, Egypt. She was awarded her Ph.D. degree in Inferential statistics from the same university. Presently, she is working at Mathematics department, Faculty of Science, Ain Shams University. Her current interests is distribution theory.

Amira Ragab is lecturer of Mathematical and Natural Sciences, Faculty of Engineering, Egyptian Russian University, Badr, Egypt. She received her B.Sc. and M.Sc. degrees in mathematics from Ain Shams University, Cairo, Egypt. She was awarded her Ph.D. degree in differential equations from the same university. Presently, she is working at Mathematical and Natural Sciences, Faculty of Engineering, Egyptian Russian University. Her current interests include Game Theory, Differential Games, Queuing Theory, Topological Games, and Fractional Differential Equations.