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Estimation of Insolvency Probability Under Systemic Risk

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Abstract: This paper is devoted to the estimation of the insolvency probability. In addition we utilize dependence models that evaluate Systemic Risk (SR), as we contribute by proposing Euler contributions of risk in an environment that is regulated by a risk measure. Moreover the framework we are utilizing assumes that a component of the environment is in distress. Finally, we calculate the Insolvency Probability due to Systemic Risk and we suggest certain distribution classes under which our results are valid.

Keywords: Systemic Risk, Euler Contributions, Risk Measures, Insolvency Probability, Ruin Probability

1 Introduction and Motivation

Systemic Risk (SR) is certainly considered of high interest in academia. In addition, Systemic Risk refers to the instability of a financial system component that can lead to its entire collapse [24]. Aforementioned statement suggests that *SR* is closely related to the concept of Dependence. As there is not yet a consensus among academics on a common way to evaluate the *SR*, there is a prolific scholar discussion towards that direction. [18] proposed some sufficient conditions for two random vectors to be ordered by the so-called Conditional distortion risk measures and demonstrated how these risk measures are quantifying *SR*. [8] specifies a framework for *SR* measures via multidimensional acceptance sets and aggregation functions. In addition, while usually *SR* measures are mostly interpreted as the minimal amount of cash needed before aggregating individual risks, their approach suggests that *SR* measures are minimal amount of cash needed before aggregating individual risks. While our effort exhibits similarities with all the aforementioned yet there are some fundamental differences. We consider that under the assumption that one component (entity) of an economic environment is in distress it is evaluated the Expected Capital Shortfall of this financial market (more on the relevant subsection).

[3] states that the main purpose concerning systemic risk is to evaluate the financial distress of an economy as a consequence of the failure of one of its components. They also point out the importance of Extreme Value Theory (EVT) in the analysis of systemic risk, and we also will make extensive use of it. In addition [10] introduces SRISK to measure the capital shortfall of a firm conditional on a severe market decline. This approach is quite similar with the one we also utilize. Also, in regard of Insolvency probability, which may also be addressed as Ruin Probability under the Insurance theory perspective, we are primarily motivated and we follow methodologically up to an extent the paper of [17]. In addition, similar to our research interest, we also mention the work of [5], [12] or even [23].

Having all the above in mind we contribute by calculating the Insolvency Probability due to Systemic Risk. Moreover, we set that variation variable is less than one. The structure of paper is the following: Section 2 contains a literature review on the basic concepts of risk measures, the Euler Allocation Principle and the Extreme Value Theory. Section 3 is devoted to the theoretical framework that concern the *SR* and Risk Contributions. Section 4 includes our contribution to the theoretical framework and finally section 5 concludes.

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2 Preliminaries

2.1 Risk Measures

Let consider Ω be a sample space. In addition, consider an investment over a single period of time, from 0 to T, where $X : \Omega \to \mathbb{R}$ is the money outcome of the investment. Moreover, consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $\mathscr{G}$ is the set of real valued functions on Ω.

Definition 2.1. A risk measure is a function from the set $\mathscr{G}$ of risks X to the real numbers $\rho : X \to \mathbb{R}$.

For the last twenty years up to now risk measure concept (based on measure theory) is used widely for regulatory purposes (see for instance [7, p.20]). Such a measure can determine the amount of currency (or other assets) a financial institution should keep in reserve depending on the financial risks it is exposed.

2.2 Euler Allocation Principle

For this subsection we consider how Euler risk contributions can be estimated for risk measures and in order to decompose portfolio-wide capital into a sum of risk contributions by sub-portfolios of solitary exposures. Also, Euler allocation principle's utilization can be found in [4]. There is stated that it calculates the excess needed capital of a sub-portfolio for generating some extra return. (for a thorough read on Euler allocation principle see [26, Section 6.3]).

Let consider an economic environment (for instance a portfolio) with $k \in \mathbb{N}$ assets. In addition those assets' profits/losses are the real r.v. $X_1, X_2, ..., X_k$ and X is the economic environment's profit/loss, where

$$S_k = \sum_{i=1}^{k} X_i. \tag{1}$$

In addition the capital that is required by this economic environment is determined with a risk measure $\rho(X)$ (see relevant subsection).

Alternatively, by introducing positive real numbers $u = (u_1, u_2, ..., u_k)$ it occurs a useful representation of (1):

$$S_k(u) = X(u_1, u_2, ..., u_k) = \sum_{i=1}^{k} u_i X_i. \tag{2}$$

For its usefulness for instance, u_i can stand for the amount of capital that is invested in the asset which has X_i profit/loss. In a more solid mathematical framework (and the one we utilize) we also consider [28] where u_i are arbitrarily positive real numbers independent of X_i, that capture dependence. Additionally, we consider some variations of the u and therefore we introduce the following function:

$$f_{\rho, S_k}(u) = \rho\left(S_k(u)\right). \tag{3}$$

By dropping S_k, then the left side of (3) can equivalently be written $f_\rho(u)$.

2.3 Extreme Value Theory

For the best possible understanding of the contribution section we need to present the concept of Extreme Value Theory (EVT). For doing so, we consult the work of [14] where it is considered that there is a sequence of $X_1, ..., X_k$ independent r.v. with common distribution function V. The interest is focused on $M_k = \bigvee_{i=1}^{k} X$ where $k \to \infty$. Also a linear normalization of M_k is needed and so $M_k^* = \frac{M_k - b_k}{\alpha_k}$ for sequences of constants $\{\alpha_n > 0\}$ and $\{b_k \in (-\infty, \infty)\}$. With this approach we are interested in the correct selections of $\{\alpha_k\}$ and $\{b_k\}$ rather than M_k^*, that will allow us to seek limit distributions for M_k^*.

Under that concept we consider the [14, Theorem 3.1] where it presents the well documented Fisher-Tippett theorem (see [21]) and suggests that for $\{\alpha_k > 0\}$ and $\{b_k \in (-\infty, \infty)\}$ sequences where

$$\lim_{k \to \infty} \mathbb{P}\left[\frac{M_k - b_k}{\alpha_k} \leq y\right] \to G(y) \tag{4}$$

and G in (4) is a non-degenerate distribution function, then G belongs either to the Gumbel family, the Fréchet family or the Weibull family.

Also these three distribution families can be grouped into the Generalized Extreme Value (GEV) distribution. A presentation of (GEV) is achievable if we consider that it has three parameters μ, σ, ξ. Then we have that $x = \frac{y-\mu}{\sigma}$, we set that ξ is the shape parameter and we get that

$$G(x) = \exp\left(-\left[1 + \frac{x}{\alpha}\right]^{-\alpha}\right)$$

is the Fréchet family where we have that $\xi = \frac{1}{\alpha} > 0$.

Moreover V is in the domain of attraction of Fréchet if and only if

$$\lim_{t\to\infty} \frac{\overline{V}(xt)}{\overline{V}(t)} = x^{-\alpha}, \tag{5}$$

for some $\alpha > 0$ in (5) and regularly decay in the tail of V which is symbolized as $V \in \mathscr{R}_{-\alpha}$ in [3].

3 Theoretical Framework

3.1 Systemic Risk

For our analysis we choose the *SR* definition, as proposed by [1] and can also be found in [10]. There, under the assumption that one component (entity) of an economic environment is in distress it is evaluated the Expected Capital Shortfall of this financial market. For instance, the term component (or the term entity) may refer to a company of a particular business sector, or a bank that is regulated by a regulatory organization. It can even be an investment as part of a portfolio. Consequently an economic environment can be perceived as a business sector or a financial market where its banking system is regulated by a regulatory organization. It can as well be a portfolio that its risk is controlled by a portfolio manager who is utilizing a risk measure. Similar to this effort is the one proposed by [2]. There is presented an economic model of systemic risk where the whole business sector is under-capitalized and consequently it harms the real economy. In terms of similarity we also mention [11]. In this paper is developed a framework for measuring, allocating and managing systemic risk.

For the best possible understanding of the distress of a component one can consider a situation where a risk measure is utilized for controlling risk of that component. Then distress in the component occurs when its price, dictated by the risk measure is exceeded. In other words, the risk measure gives the level of loss that should not exceed. If this happens, then financial distress emerges. In a formal mathematical framework we consider the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and the set of positive r.v. with infinite upper points $L_+(\mathbb{P})$. In addition let $X, Z \in L_+(\mathbb{P})$ and the equation [3, equation 1.1], or $\rho_{X,Z}(q) := \mathbb{E}\left[\left(X - t_1(q)\right)_+ |Z > t_2(q)\right]$, where $t_1(q)$ and $t_2(q)$ are two positive functions with $q \in (0,1)$ and $\lim_{q\to 1} t_2(q) = \infty$. Clearly, $t_1(q)$ and $t_2(q)$ can be perceived as risk measures that a regulator (or an investor) is utilizing for controlling the levels of risk. The economical interpretation of the components of this conditional probability is quite straightforward as $\left(X - t_1(q)\right)_+$ represents the total liabilities of the financial market (or investment or bank sector) minus total capital allocated to the market and can be determined by risk measure. Also $Z > t_2(q)$ represents the fact that there is crisis in a component of the market as it should be clear that $t_2(q)$ represents the capital, via a risk measure that is allocated to the entity that bears the liability Z. In regard how we utilize all the above, we refer the reader to Euler contribution for distortion risk measures subsection.

4 Contribution

4.1 Euler Contributions for Risk Measures

Initially we establish that Euler Allocation is applicable in an economic environment that is regulated by a Distortion Risk Measure. Moreover we consider that *SR* has a significant impact on that economic environment. Let consider that there are k economic entities with $X_1, X_2, \ldots X_k$ random variables, $F_1, F_2, \ldots F_k$ marginal distribution functions and $k \in \mathbb{N}$. In addition, consider that the regulator allocates for every X_i entity C_i capital. Recall that in Euler Allocation Principle subsection is stated that C_i is the capital that is required by the economic entity X_i and is determined with a risk measure (see relevant sub-section). Moreover the total capital that is allocated to all the economic entities would be $\sum_{i=1}^{k} C_i$ and we also set that $S_k := \sum_{i=1}^{k} X_i$ with I distribution function. By assuming that the first economic entity exhibits financial distress, then without loss of generality SR is defined to have the following value

$$SR := \mathbb{E}\big[\big(S_k - \sum_{i=1}^{k} C_i\big)_+ | X_1 > C_1\big] \tag{6}$$

in terms of aggregate risk and is depicted in [3] and [1]. Moreover SR contribution to the n^{th} economic entity is defined to have the following value $SR_n := \mathbb{E}\big[\big(X_n - C_n\big)_+ | X_1 > C_1\big]$, where $n \in \{1,2,....,k\}$ and is also depicted in [3].

4.2 A Calculation of Systemic Risk's Insolvency Probability, when Variation Variable is less then 1

Another important topic that should be addressed is insolvency due to Systemic Risk. By recalling the definition of SR as depicted in (6), it is natural to consider that the probability of insolvency or Ψ can be defined as the following:

$$\Psi_{(k,n)} := \mathbb{P}\left[\bigvee_{i=1}^{n} S_k > \sum_{i=1}^{k} C_i\right] \text{ as } \sum_{i=1}^{k} C_i \to \infty, \tag{7}$$

where in accordance with [28, eq.1.2] and our Euler Allocation Principle subsection, we have that S_k is the randomly weighted sum that can be decomposed into primary n real-valued independent random variables $X_1,...,X_k$ and $u_1,u_2,...,u_k$ positive real numbers, independent of the primary. By setting $\sum_{i=1}^{k} C_i = x$, then 8 is depicted in the following form:

$$\Psi_{(k,n)} := \mathbb{P}\left[\bigvee_{i=1}^{n} S_k > x\right] \text{ as } x \to \infty. \tag{8}$$

Regardless of the indicators that may appear due the mathematical structure we consider that Ψ stands in this paper for the Insolvency Probability. In addition, by introducing positive real numbers $u = (u_1,u_2,...,u_k)$ as presented in (2), and by considering [28, eq.1.1 and eq.1.2] it occurs a useful representation of X and specifically:

$$\sum_{i=1}^{k} u_i X_i = S_k(u).$$

In accordance with [28], $u_1,u_2,...,u_k$ can be perceived as as positive real numbers that capture dependence. Although such interpretation is not per se useful in the sense that dependence is not our scope for this subsection, nevertheless $u_1,u_2,...,u_k$ will prove very useful for our mathematical framework. Having the above in mind, we consider [28, eq.1.3] and [27] where we get that if $X_1,...,X_k$ iid by a Sub-exponential distribution (see [16, Definition 2] for the definition of Sub-exponential distribution, denoted as $\mathscr{S}$) and $u_1,u_2,...,u_k$ are in $(0,b]$ for some constant b and $0 < b \leq \infty$, then

$$\mathbb{P}\left[\bigvee_{i=1}^{n} S_k(u) > x\right] \sim \mathbb{P}\big[S_k(u) > x\big] \sim \mathbb{P}\left[\bigvee_{i=1}^{n} u_i X_i > x\right] \sim \sum_{i=1}^{k} \mathbb{P}\big[u_i X_i > x\big]. \tag{9}$$

Remark 4.1. (9) suggests that the heavy tails of the $X_1,...,X_k$ random variables vanishes the dependence presented from $u_1,u_2,...,u_k$. Such notion is in tandem with the principle of a single big jump in the presence of random weights (one may consult [16] for subtleties).

In tandem with the rest of our contribution subsection, we have that there are k economic entities with $X_1,X_2,...X_k$ continuous non negative random variables, $F_1,F_2,...F_k$ marginal distribution functions that vary regularly unless we state otherwise and $k \in \mathbb{N}$.

Also, in accordance with [19] we define the positive truncated mean function, or

$$m_+(x) := \int_0^x \big[1 - F(y)\big] dy.$$

In addition, we define the integrand J_-, or

$$J_- := J_-(X) = \int_{-\infty}^{0^-} \frac{|\, x\,|}{m_+(|\, x\,|)} dF(x).$$

A key assumption is that $\bigvee_{i=1}^{n} S_k(u)$ is finite almost surely, which occurs iff $S_k(u) \to \infty$ as k $\to \infty$ with probability 1 [20, Chapter XII, Section 2, Theorem 1]. Also, we have from [19, Corollary 1 (a)], [19, Theorem 2 (c)] and the remark that follows in the same context that:

$$\text{if } \mathbb{E}|X_i| = \infty, \text{ then } S_k(u) \to \infty \text{ a.s. as } k \to \infty \text{ iff } J_- < \infty, \tag{10}$$

which is the [19, Corollary 1 (a)], as it will be utilized in this paper.

Remark 4.2. Initially (10) assures that $\bigvee_{i=1}^{n} S_k$ is finite and thus a proper r.v..Furthermore, by attempting to give an intuitive explanation of the (10) condition, it suggests that the right tail of F_i is heavier than the left one.

Definition 4.3. A distribution function F on $\mathbb{R}$ is dominatedly-varying tailed, or $F \in \mathscr{D}$ when its right tail satisfies $\overline{F}(xz) = O\big(\overline{F}(x)\big)$ for all $0 < z < 1$.

In addition, we consider [16, Definition 1] for the definition of Long-tailed distribution, denoted as $\mathscr{L}$. Also,we state from [6] the following:

Definition 4.4. A distribution function F on $\mathbb{R}$ has an extended rapidly varying tail or belongs to class $\mathscr{E}_{\mathscr{R}}$, when its right tail satisfies $\limsup\limits_{x\to\infty} \frac{\overline{F}(xz)}{\overline{F}(x)} < 1$ for some $z > 1$.

$\mathscr{E}_{\mathscr{R}}$ is also met in other works like [25, eq.4], where is addressed under the notation $\mathscr{P}\mathscr{D}$ (positively decreasing-tailed). Also, we set that $F_*(z) = \liminf \overline{F}(xz)/\overline{F}(x)$, $F^*(z) = \limsup \overline{F}(xz)/\overline{F}(x)$, and for a distribution function F with an ultimate right tail we have that the upper Matuszewska index, or α_F is defined as

$$\alpha_F = \inf\{-(\log F_*(z)/\log z) : z > 1\} \in [0,\infty],$$

and the lower Matuszewska index, or β_F is defined as

$$\beta_F = \sup\{-(\log F^*(z)/\log z) : z > 1\} \in [0,\infty].$$

For the remaining of the paper we assume that z is positive. Now, let us proceed with the following Lemma, which is based on [17, Corollary 1]:

Lemma 4.5. Let consider that $\mathbb{E}|X_i| = \infty$, [19, Corollary 1 (a)] holds, $\overline{F}(z)$ is regularly varying with varying variable β and $m(z)$ is regularly varying with varying variable $1-\alpha$, where $0 < \alpha \le 1$. If $\alpha < \beta$ and $\beta \in \mathbb{R}_+$, then

$$\Psi(x) \sim \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(2-\alpha)} \frac{x\overline{F(x)}}{m(x)}, \text{ as } x \to \infty,$$

where Γ is the Gamma function.

Remark 4.6. Since [19, Corollary 1 (a)] holds, we have that due to the fact that $J_- < \infty$ that $\Psi(x) \sim \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(2-\alpha)} \frac{x\overline{F(x)}}{m(x)}$, as $x \to \infty$ is not meaningless.

Proof. Initially we consider [17, Theorem 2]. There is addressed the case where $m(z)$ is regularly varying and $\overline{F}$ is long tailed and index $1-\alpha \in [0,1]$. Specifically in [17, eq.6] we have that

$$\Psi(x) \sim \frac{\overline{G_a}(x)}{\Gamma(1+\alpha)\Gamma(2-\alpha)}, \tag{11}$$

where α is the varying variable of m. Moreover, we consider [17, (7)] which is a special case of $\overline{G_a}(x)$ once we have that $0 < \alpha \le 1$ and is

$$\overline{G_a}(x) = \min\left(1, \int_1^\infty \frac{\overline{F}(x+z)}{m(z)} dz\right). \tag{12}$$

By considering (11) and (12), we get that

$$\Psi(x) \sim \frac{\overline{G}(x)}{\Gamma(\alpha)\Gamma(2-\alpha)}. \tag{13}$$

By taking a closer look at (12) and (13) we realize that we primarily need to calculate $\int_1^\infty \frac{\overline{F}(x+z)}{m(z)} dz$. For doing so, we initially consider the case where both $m(x)$ and $\overline{F}$ are varying regularly. Now, let us fix $e > 0$, $E > 0$ where $E > e$. Moreover, we consider the partition of $[1,\infty]$ into $[1,e]$,$[e,E]$ and $[E,\infty]$. Accordingly we get from [17, eq.21] that

$$\int_1^{ex} \frac{\overline{F}(x+z)}{m(z)} dz \le \overline{F}(x) \int_1^{ex} \frac{1}{m(z)} dz \sim \frac{\overline{F}(x)}{\alpha} \frac{ex}{m(ex)}, \text{ as } x \to \infty, \tag{14}$$

while 14 stems from the fact that

$$\frac{d}{dx}\frac{z}{m(z)} = \frac{\alpha + o(1)}{m(z)} \text{ as } z \to \infty,$$
$$d\left(\frac{z}{m(z)}\right) \sim \frac{\alpha + o(1)}{m(z)}dz,$$
$$\int_1^{ex} d\left(\frac{z}{m(z)}\right) \sim \int_1^{ex} \frac{\alpha + o(1)}{m(z)}dz,$$
$$\frac{z}{m(z)}\bigg|_1^{ex} \sim \int_1^{ex} \frac{\alpha + o(1)}{m(z)}dz,$$
$$\frac{ex}{m(ex)}\frac{1}{\alpha} \sim \int_1^{ex} \frac{1}{m(z)}dz,$$
$$\frac{ex}{m(ex)}\frac{\overline{F}(x)}{\alpha} \sim \overline{F}(x)\int_1^{ex} \frac{1}{m(z)}dz.$$

By considering [19, Corollary 1 (a)] we get that $\int_1^{ex} \frac{\overline{F}(x+z)}{m(z)}dz$ is bounded by a finite quantity.

Remark 4.7. While (14) is initially met in [17, eq.21], yet we further elaborate on the proof of the statement this equation stands for.

We also have that

$$\int_{Ex}^{\infty} \frac{\overline{F}(x+z)}{m(z)}dz \leq \int_{Ex}^{\infty} \frac{\overline{F}(z)}{m(z)}dz \sim \frac{1}{\beta - \alpha}\frac{Ex\overline{F}(Ex)}{m(Ex)}, \text{ as } x \to \infty, \tag{15}$$

where (15) stems from the fact that we know that asymptotically $\overline{F}(z) = z^{-\beta}$ and $m(z) = z^{1-\alpha}$. Moreover:

$$\int_{Ex}^{\infty} \frac{\overline{F}(z)}{m(z)}dz,$$
$$\sim \int_{Ex}^{\infty} \frac{z^{-\beta}}{z^{1-\alpha}}dz,$$
$$\sim \int_{Ex}^{\infty} t^{-\beta+\alpha-1}dz,$$
$$\sim \frac{z^{-\beta+\alpha}}{-\beta+\alpha}\bigg|_{Ex}^{\infty},$$

and consequently

$$\int_{Ex}^{\infty} \frac{\overline{F}(z)}{m(z)}dz \sim \frac{1}{Ex^{\beta-\alpha}}\frac{1}{\beta - \alpha}. \tag{16}$$

Again, we consider (15) and we have the following:

$$\frac{1}{\beta - \alpha}\frac{Ex\overline{F}(Ex)}{m(Ex)}$$
$$\sim \frac{1}{\beta - \alpha}\frac{Ex(Ex)^{-\beta}}{Ex^{1-\alpha}},$$
$$\sim \frac{1}{\beta - \alpha}\frac{(Ex)^{-\beta+1}}{(Ex)^{1-\alpha}},$$
$$\sim \frac{1}{\beta - \alpha}\frac{1}{(Ex)^{1-\alpha}(Ex)^{-1+\beta}},$$

and consequently

$$\frac{1}{\beta-\alpha}\frac{Ex\overline{F}(Ex)}{m(Ex)} \sim \frac{1}{\beta-\alpha}\frac{1}{(Ex)^{\beta-\alpha}}. \tag{17}$$

It is clear that by considering (16) and (17) we get (15) and in tandem with Condition [19, Corollary 1 (a)] we get that $\int_{Ex}^{\infty}\frac{\overline{F}(x+z)}{m(z)}dz$ is bounded by a finite quantity.

Remark 4.8. (15) is again stated in [17, eq.22], yet we further elaborate on the proof of the statement this equation stands for.

Now, in regard of $[e,E]$ recall that we initially examine the case of Regular Variation. Moreover, we set that there is an interval $n \in [e,E]$ and we have that $\int_{ex}^{Ex}\frac{\overline{F}(x+z)}{m(z)}dz = \frac{\overline{F}(x)}{m(x)}\int_{ex}^{Ex}\frac{\overline{F}(x+z)}{\overline{F}(x)}\frac{m(x)}{m(z)}dz$, $\int_{ex}^{Ex}\frac{\overline{F}(x+z)}{m(z)}dz = \frac{x\overline{F}(x)}{m(x)}\int_{e}^{E}\frac{\overline{F}(x(1+n))}{\overline{F}(x)}\frac{m(x)}{m(xn)}dn$,

and, by considering Uniform Convergence Theorem for r.v. functions (see [9, Theorem 1.5.2]) we have that

$$\frac{\overline{F}(x(1+n))}{\overline{F}(x)}\frac{m(x)}{m(xn)} \to \frac{(1+n)^{-\beta}}{n^{1-\alpha}}, \tag{18}$$

as $x \to \infty$ uniformly in $n \in [e,E]$ and finally

$$\int_{ex}^{Ex}\frac{\overline{F}(x+z)}{m(z)}dz \sim \frac{x\overline{F}(x)}{m(x)}\int_{e}^{E}\frac{(1+n)^{-\beta}}{n^{1-\alpha}}dn \text{ as } x \to \infty. \tag{19}$$

Once we let $[1,e]$ become negligible in the sense that $e \to 0$, let $[E,\infty]$ also become negligible in the sense that $E \to \infty$, then from (13), (14) and (19) we get that $\Psi(x) \sim \frac{\overline{G}(x)}{\Gamma(\alpha)\Gamma(2-\alpha)}$, $\frac{\overline{G}(x)}{\Gamma(\alpha)\Gamma(2-\alpha)} = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \times \frac{x\overline{F}(x)}{m(x)}\int_0^{\infty}\frac{(1+n)^{-\beta}}{n^{1-\alpha}}dn$ and finally

$$\Psi(x) \sim \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \times \frac{x\overline{F}(x)}{m(x)}\int_0^{\infty}(1+n)^{-\beta}n^{\alpha-1}dn. \tag{20}$$

In addition, we consider that $\int_0^{\infty}(1+n)^{-\beta}n^{\alpha-1}dn$ is the representation of the Beta function $B(\alpha,\beta-\alpha) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}dt$, once we put $t = \frac{n}{1+n}$, and so

$$\Psi(x) \sim \frac{B(\alpha,\beta-\alpha)}{\Gamma(\alpha)\Gamma(2-\alpha)} \times \frac{x\overline{F}(x)}{m(x)}. \tag{21}$$

Now we consider the relation of B function and Γ function where

$$B(\alpha,\beta-\alpha) = \frac{\Gamma(\beta-\alpha)\Gamma(\alpha)}{\Gamma(\beta)}. \tag{22}$$

Finally, we plug (22) into (21) and so,

$$\Psi(x) \sim \frac{\Gamma(\beta-\alpha)\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2-\alpha)} \times \frac{x\overline{F}(x)}{m(x)}$$

and thus,

$$\Psi(x) \sim \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(2-\alpha)} \times \frac{x\overline{F}(x)}{m(x)} \text{ as } x \to \infty,$$

and proof is complete. □

Now, let us expand our work when $\overline{F}(z) \in \mathscr{D}\cap\mathscr{E}_{\mathscr{R}}$ and $m(z) \in \mathscr{D}\cap\mathscr{E}_{\mathscr{R}}$, and proceed with the following Theorem

Theorem 4.9. Let consider that $\mathbb{E}|X_i| = \infty$, [19, Corollary 1 (a)] holds, $\overline{F}(z) \in \mathscr{D}\cap\mathscr{E}_{\mathscr{R}}$ and $m(z) \in \mathscr{D}\cap\mathscr{E}_{\mathscr{R}}$, where $0 < \alpha_l \leq 1$ and $0 < \alpha_u \leq 1$. If $\alpha_u < \beta_u$ and $\alpha_l < \beta_l$, then there exist constants C and D such that

$$\frac{\Gamma(\beta_l-\alpha_l)}{\Gamma(\beta_l)\Gamma(2-\alpha_l)}\frac{Dx\overline{F}(x)}{m(x)} \lesssim \Psi(x) \lesssim \frac{\Gamma(\beta_u-\alpha_u)}{\Gamma(\beta_u)\Gamma(2-\alpha_u)}\frac{Cx\overline{F}(x)}{m(x)},$$

as $x \to \infty$, where Γ is the Gamma function.

Proof. Initially, we consider for $\mathscr{D}$ [13, eq.3.2] and [15, Theorem 3, A, a1 and a2 of unpublished Appendix titled Positive increase and bounded increase for non-decreasing functions]. It is not difficult to realize that in regard of $\mathscr{D}$ we get an upper bound for $\frac{\overline{F}(xz)}{\overline{F}(x)}$ which is depicted in (23) and also an upper bound for $\frac{m(xz)}{m(x)}$, depicted in (24).

Also for $\mathscr{E}_{\mathscr{R}}$ we consult [15, Theorem 3, B, b1 and b2 of unpublished Appendix titled Positive increase and bounded increase for non-decreasing functions]. The aforementioned we can get a lower bound for $\frac{\overline{F}(xz)}{\overline{F}(x)}$ which is depicted in (23) and also a lower bound for $\frac{m(xz)}{m(x)}$, depicted in (23). In addition, by considering [6, p.124] we have for $\overline{F}(z)$ that there exist a C_F for each $Z > 1$ and for some $D_F > 0$ and all $Z > 1$ as so that:

$$D_F z^{-\beta_l} \lesssim \frac{\overline{F}(xz)}{\overline{F}(x)} \lesssim C_F z^{-\beta_u} \text{ uniformly in } z \in [1, Z], \tag{23}$$

as $x \to \infty$. Similarly, for m we have that:

$$C_m z^{1-\alpha_l} \lesssim \frac{m(xz)}{m(x)} \lesssim C_m z^{1-\alpha_u} \text{ uniformly in } z \in [1, Z], \tag{24}$$

as $x \to \infty$. By considering (18), we get that the upper bound is

$$\frac{\overline{F}(x(1+n))}{\overline{F}(x)} \times \frac{m(x)}{m(xn)} \lesssim \frac{(1+n)^{-\beta_u} C_F}{n^{1-\alpha_u} C_m}, \tag{25}$$

as $x \to \infty$. Notice that m's asymptotic behavior is depicted in the denominator of (18). Furthermore, we set that $C_F / C_m = C$, and by plugging (25) into (20) we get

$$\Psi(x) \lesssim \frac{1}{\Gamma(\alpha_m)\Gamma(2-\alpha_m)} \times \frac{Cx\overline{F}(x)}{m(x)} \int_0^\infty (1+n)^{-\beta_u} n^{\alpha_u - 1} dn,$$

as $x \to \infty$. Working as in the previous Lemma, we get that $\int_0^\infty (1+n)^{-\beta_u} n^{\alpha_u - 1} dn$, is the representation of the Beta function $B(\alpha_u, \beta_u - \alpha_u)$, once we put $t = \frac{n}{1+n}$, and in tandem with (21) we get

$$\Psi(x) \lesssim \frac{B(\alpha_u, \beta_u - \alpha_u)}{\Gamma(\alpha_u)\Gamma(2-\alpha_u)} \times \frac{Cx\overline{F}(x)}{m(x)}, \tag{26}$$

as $x \to \infty$. Now we consider the relation of B function and Γ function, similarly with that depicted in (22), and by plugging it to (26) we get that

$$\Psi(x) \lesssim \frac{\Gamma(\beta_u - \alpha_u)}{\Gamma(\beta_u)\Gamma(2-\alpha_u)} \times \frac{Cx\overline{F}(x)}{m(x)}, \tag{27}$$

as $x \to \infty$, which is the upper bound of $\Psi(x)$.

Similarly we have to work for the lower bound. Specifically, we consider (23), (24) and (18) and we have that

$$\frac{(1+n)^{-\beta_l} D_F}{n^{1-\alpha_l} D_m} \lesssim \frac{\overline{F}(x(1+n))}{\overline{F}(x)} \times \frac{m(x)}{m(xn)}, \tag{28}$$

as $x \to \infty$. Notice once again that m's asymptotic behavior is depicted in the denominator. Furthermore, we set that $D_F / D_m = D$, and by plugging (28) into (20) we get

$$\frac{1}{\Gamma(\alpha_l)\Gamma(2-\alpha_l)} \times \frac{Dx\overline{F}(x)}{m(x)} \int_0^\infty (1+n)^{-\beta_l} n^{\alpha_l - 1} dn \lesssim \Psi(x),$$

as $x \to \infty$ which is the lower bound of $\Psi(x)$. Working as in the previous Lemma, we get that $\int_0^\infty (1+n)^{-\beta_l} n^{\alpha_l - 1} dn$, is the representation of the Beta function $B(\alpha_l, \beta_l - \alpha_l)$, once we put $t = \frac{n}{1+n}$, and in tandem with (21) we get

$$\frac{B(\alpha_l, \beta_l - \alpha_l)}{\Gamma(\alpha_l)\Gamma(2-\alpha_l)} \times \frac{Dx\overline{F}(x)}{m(x)} \lesssim \Psi(x) \tag{29}$$

as $x \to \infty$. Now we consider the relation of B function and Γ function, similarly with that depicted in (22), and by plugging it to (29) we get that

$$\frac{B(\alpha_l, \beta_l - \alpha_l)}{\Gamma(\alpha_l)\Gamma(2-\alpha_l)} \times \frac{Dx\overline{F}(x)}{m(x)} \lesssim \Psi(x) \tag{30}$$

as $x \to \infty$, which is the lower bound of $\Psi(x)$. By combining (27) and (30) we get that

$$\frac{\Gamma(\beta_l - \alpha_l)}{\Gamma(\beta_l)\Gamma(2-\alpha_l)} \frac{Dx\overline{F}(x)}{m(x)} \lesssim \Psi(x) \lesssim \frac{\Gamma(\beta_u - \alpha_u)}{\Gamma(\beta_u)\Gamma(2-\alpha_u)} \frac{Cx\overline{F}(x)}{m(x)},$$

as $x \to \infty$, and the proof is complete. □

Let us state also the following Lemma, which is heavily influence by [28, Theorem 3.1].

Lemma 4.10. If $u_1, u_2, \ldots, u_k$ are bounded from above, $i = 1, 2, \ldots, k$ and $F \in \mathscr{L} \cap \mathscr{D}$, then

$$\mathbb{P}\left[\bigvee_{i=1}^{n} S_k(u) > x\right] \sim \mathbb{P}\left[S_k(u) > x\right] \sim \mathbb{P}\left[\bigvee_{i=1}^{n} u_i X_i > x\right] \sim \sum_{i=1}^{k} \mathbb{P}\left[u_i X_i > x\right], \text{ as } x \to \infty.$$

Proof. We already know from (9) that $u_1, u_2, \ldots, u_k$ are in $(0, b]$ for some constant b and $0 < b \leq \infty$. By having that $u_1, u_2, \ldots, u_k$ are bounded from above, we consider that there are [27, Type II bound of r.v.]. Now, one can consult [27, Corollary 3.1] and it is easy to understand that relation of [27, Theorem 3.1] also holds for the setting, suggested in the Lemma in the sense that in both case $F \in \mathscr{L} \cap \mathscr{D}$. For proving the above we initially observe that we have the following basic ordering:

$$\mathbb{P}\left[\bigvee_{i=1}^{k} S_k(u) > x\right] \leq \mathbb{P}\left[S_k(u) > x\right],$$

$$\mathbb{P}\left[S_k(u) > x\right] \leq \mathbb{P}\left[\bigvee_{i=1}^{k} u_i X_i > x\right],$$

$$\mathbb{P}\left[\bigvee_{i=1}^{k} u_i X_i > x\right] \leq \sum_{i=1}^{k} \mathbb{P}\left[u_i X_i > x\right]$$

is also true due to Bonferroni's inequality where $\mathbb{P}\left[\bigcup_{i=1}^{k} u_i X_i > x\right] \leq \sum_{i=1}^{k} \mathbb{P}\left[u_i X_i > x\right]$ and also because $\mathbb{P}\left[\bigvee_{i=1}^{k} u_i X_i > x\right] \leq \mathbb{P}\left[\bigcup_{i=1}^{k} u_i X_i > x\right]$. Now for achieving that (9) is true as $x \to \infty$ we have to prove that

$$\mathbb{P}\left[S_k > x\right] \geq \sum_{i=1}^{k} \mathbb{P}\left[u_i X_i > x\right], \tag{31}$$

and

$$\mathbb{P}\left[\sum_{i=1}^{k} u_i X_i > x\right] \leq \sum_{i=1}^{k} \mathbb{P}\left[u_i X_i > x\right], \tag{32}$$

as $x \to \infty$ are true. We already have that $u_1, u_2, \ldots, u_n$ are bounded from above so without loss of generality we assume that they are bounded from above by 1. Under that setting we can verify that

$$\sum_{1 \leq j \neq k \leq m} \mathbb{P}(u_j X_j > x, u_m X_m > x) = o(1) \sum_{i=1}^{k} \mathbb{P}(u_i X_i > x). \tag{33}$$

Also, (33) stems from the fact that from the left side we have independence suggests that we are dealing with a product of r.v., that are bounded. To that end the left side becomes negligible compare to the quantity on the right hand side, as $x \to \infty$.

For proving (31) we recall initially that $X_i \in L_+^1$. Now, we only have to consider that $S_k(u) \geq \bigvee_{i=1}^{k} u_i X_i$, the fact that (33) holds and thus we get that also (31) holds.

Now, let us prove that (32) holds. For that, we first consider the case where $u_1, u_2, \ldots, u_n$ are positive. Initially we consider an arbitrary subset $I \subset \{1, \ldots, k\}$, we have $I^c = \{1, \ldots, k\} \setminus I$ and also $\Omega_I^{\varepsilon}(u) = \{\omega : u_i > \varepsilon$ for $i \in I$ and $u_j \leq \varepsilon$ for $j \in I^c\}$ for $0 < \varepsilon < 1$. To that end we consider that $\sum_{i=1}^{k} u_i X_i = S_k(u)$ and obtain the following inequality

$$\mathbb{P}\left(\sum_{i=1}^{k} u_i X_i > x\right) \leq \sum_{I \subset \{1,\ldots,k\}} \mathbb{P}\left(\sum_{i \in I} u_i X_i + \sum_{j \in I^c} \varepsilon X_j > x, \Omega_I^{\varepsilon}(u)\right). \tag{34}$$

By considering [28, Lemma 5.1] we have that the right side of (34) equals

$$\sum_{i \in I} \mathbb{P}(u_i X_i > x, \Omega_I^{\varepsilon}(u)) + \sum_{j \in I^c} \mathbb{P}(\varepsilon X_j > x) \mathbb{P}(\Omega_I^{\varepsilon}(u))$$

$$= \sum_{i \in I} \mathbb{P}(u_i X_i > x, \Omega_I^{\varepsilon}(u)) + \sum_{j \in I^c} \mathbb{P}(\varepsilon X_j > x, u_j > \varepsilon) \frac{\mathbb{P}(\Omega_I^{\varepsilon}(u))}{\mathbb{P}(u_j > \varepsilon)}.$$

By plugging the above to (34) and interchanging summation order we get

$$\mathbb{P}\left(\sum_{i=1}^{k} u_i X_i > x\right)$$

$$\leq \sum_{i=1}^{k} \sum_{I: i \in I \subset \{1,\ldots,k\}} \mathbb{P}(u_i X_i > x, \Omega_I^{\varepsilon}(u)) + \sum_{j=1}^{k} \sum_{I: j \notin I \subset \{1,\ldots,k\}} \mathbb{P}(u_j X_j > x) \frac{\mathbb{P}(\Omega_I^{\varepsilon}(u))}{\mathbb{P}(u_j > \varepsilon)},$$

$$= \sum_{i=1}^{k} \mathbb{P}(u_i X_i > x, u_i > \varepsilon) + \sum_{j=1}^{k} \mathbb{P}(u_j X_j > x) \frac{\mathbb{P}(u_j \leq \varepsilon)}{\mathbb{P}(u_j > \varepsilon)},$$

$$\leq \left(1 + \max_{1 \leq j \leq k} \frac{\mathbb{P}(u_j \leq \varepsilon)}{\mathbb{P}(u_j > \varepsilon)}\right) + \sum_{j=1}^{k} \mathbb{P}(u_j X_j > x).$$

Recall that we have that u_j is positive, and as $\varepsilon \to 0$ we obtain (32). Now when $u_1, \ldots, u_k$ may take value 0 with positive probability, we recall I,I^c and also $\Omega_I^0(u) = \{\omega : u_i > 0$ for $i \in I$ and $u_j = 0$ for $j \in I^c\}$, and consequently we obtain

$$\mathbb{P}\left(\sum_{i=1}^{k} u_i X_i > x\right) = \sum_{\phi \neq I \subset \{1,\ldots,k\}} \mathbb{P}(\sum_{i \in I} u_i X_i > x, \Omega_I^0(u))$$

$$\mathbb{P}\left(\sum_{i=1}^{k} u_i X_i > x\right) \leq \sum_{\phi \neq I \subset \{1,\ldots,k\}} \sum_{i \in I} \mathbb{P}(u_i X_i > x, \Omega_I^0(u))$$

$$\mathbb{P}\left(\sum_{i=1}^{k} u_i X_i > x\right) \leq \sum_{i \in I}^{k} \mathbb{P}(u_i X_i > x),$$

(32) is true and the proof is complete. □

We also need to state the following Definition

Definition 4.11. A distribution function F on $\mathbb{R}$ belongs to class $\mathscr{A}$ when it belongs to $\mathscr{S}$ and its right tail satisfies $\limsup_{x\to\infty} \frac{\overline{F}(xz)}{\overline{F}(x)} < 1$ for some $z > 1$.

Now, we can state the following Theorem.

Theorem 4.12. Let consider that $\mathbb{E}|X_i| = \infty$, $u_1, u_2, \ldots, u_k$ are bounded from above, $i = 1, 2, \ldots, k$, [19, Corollary 1 (a)] holds, $\overline{F}(z) \in \mathscr{D} \cap \mathscr{A}$ and $m(z) \in \mathscr{D} \cap \mathscr{A}$, where $0 < \alpha_F \leq 1$ and $0 < \alpha_m \leq 1$. If $\alpha_F < \alpha_m$ and $\beta_F < \beta_m$, then there exist constants C and D such that

$$\Psi(x) \sim \mathbb{P}\left[\bigvee_{i=1}^{n} S_k(u) > x\right] \sim \mathbb{P}\left[S_k(u) > x\right] \sim \mathbb{P}\left[\bigvee_{i=1}^{n} u_i X_i > x\right], \text{ as } x \to \infty,$$

where $\Psi(x)$ is bounded by the quantities determined in Theorem 4.9.

Proof. Initially, we consider that $\mathscr{S} \subset \mathscr{L}$. With that in mind, straightforwardly from Lemma 4.10 we have that If $u_1, u_2, ..., u_k$ are bounded from above, $i = 1, 2, ..., k$ and $F \in \mathscr{S} \cap \mathscr{D}$, then

$$\mathbb{P}\left[\bigvee_{i=1}^{n} S_k(u) > x\right] \sim \mathbb{P}\left[S_k(u) > x\right] \sim \mathbb{P}\left[\bigvee_{i=1}^{n} u_i X_i > x\right] \sim \sum_{i=1}^{k} \mathbb{P}\left[u_i X_i > x\right], \text{ as } x \to \infty.$$

We also have from Definition that $\mathscr{A}$ belongs to $\mathscr{S}$, from which we get that $\mathscr{A} \subset \mathscr{S} \subset \mathscr{L}$ which implies that If $u_1, u_2, ..., u_k$ are bounded from above, $i = 1, 2, ..., k$ and $F \in \mathscr{A} \cap \mathscr{D}$, then

$$\mathbb{P}\left[\bigvee_{i=1}^{n} S_k(u) > x\right] \sim \mathbb{P}\left[S_k(u) > x\right] \sim \mathbb{P}\left[\bigvee_{i=1}^{n} u_i X_i > x\right] \sim \sum_{i=1}^{k} \mathbb{P}\left[u_i X_i > x\right], \text{ as } x \to \infty.$$

Finally we have from the Definitions that $\mathscr{A} \subset \mathscr{E}$, which allows us to utilize the previous Theorem in order to conclude that

$$\frac{\Gamma(\beta_l - \alpha_l)}{\Gamma(\beta_l)\Gamma(2-\alpha_l)} \times \frac{Dx\overline{F}(x)}{m(x)} \lesssim \Psi(x) \sim \mathbb{P}\left[\bigvee_{i=1}^{k} S_k(u) > x\right] \sim \mathbb{P}\left[S_k(u) > x\right] \sim \mathbb{P}\left[\bigvee_{i=1}^{k} u_i X_i > x\right] \lesssim \frac{\Gamma(\beta_u - \alpha_u)}{\Gamma(\beta_u)\Gamma(2-\alpha_u)} \times \frac{Cx\overline{F}(x)}{m(x)}$$

holds as $x \to \infty$ and the proof is complete. □

Remark 4.13. Aforementioned Theorem has a profound importance since we are interested in calculating the Insolvency probability as $x \to \infty$.

Remark 4.14. In the model we suggest Insolvency Probability can be calculated without any restriction in terms of the Risk Measure that is responsible for regulating/mitigating risk.

Remark 4.15. Our research could be utilized in many cases, like for instance when we deal with Pareto type distributions and the mean is undefined.

Remark 4.16. Also, findings of initial Theorem complies with the principle of a single big jump as presented in [16].

5 Conclusion

The main interest of this paper is the estimation of Insolvency Probability, under the fact that Systemic Risk is present. To that end, we proceed be setting risk contributions in an economic environment that is regulated by a Distortion Risk Measure. By considering that many important classes of Risk Measures, like the class of Spectral Risk Measures under some conditions, can be expressed in terms of Distortion Risk Measures we can also conclude that some important generalizations can be extracted out of our initial findings. Moreover, we mostly contribute be concluding the so called Insolvency Probability (Ruin Probability in Insurance Theory) due to the fact that systemic Risk is present and the variation variable is less than one.

Apparently, it is of great convenience the fact that in the model we suggest Insolvency Probability can be calculated without any restriction in terms of the Risk Measure that is responsible for regulating/mitigating risk. Finally we point out that our findings, which are consistent with the principle of a single big jump could be utilized in many cases, like for instance when we deal with Pareto type distributions and the mean is undefined.

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