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Extended Exponential Distribution With Application On Financial Data

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Abstract: In this paper, we present a novel approach to extend the applicability of probability density functions (PDFs) for the exponential distribution, enabling the creation of a versatile family of PDFs with diverse properties. Our method utilizes fundamental statistical parameters, such as the rate parameter, to facilitate this expansion. The central contribution of this research is the development and proof of a powerful Generalization Theorem for Exponential PDFs. This theorem allows for the nth-generation generalization of exponential PDFs, each iteration introducing unique characteristics. We apply our Generalization Theorem specifically to exponential PDFs, displaying its wide-ranging utility within this domain. Additionally, we conduct estimation and simulation studies to assess the performance of the generalized exponential PDFs in comparison to their original counterparts. We hereby christen this groundbreaking theorem as the "Exponential PDF Generalization Theorem." This paper marks a significant advancement in the manipulation and adaptation of exponential probability density functions, ushering in new avenues for statistical modeling and analysis within the realm of exponential distributions.

Keywords: Exponential distribution, Moments, Hazard rate function, Order statistics, Estimation of parameter, Goodness of fit

1 Introduction

Probability density functions (PDFs) are fundamental tools in the field of statistics, providing a mathematical framework for describing the distribution of random variables. They are essential in various domains such as finance, engineering, biology, and social sciences [1,2,3]. Researchers have consistently sought to enhance the versatility of PDFs, aiming to generalize them and create new families of distributions with diverse properties.

The exponential distribution, characterized by its memoryless property and parameterized by the rate parameter (λ), holds a prominent position in statistical theory and practice [4]. It finds widespread use in areas including reliability engineering [5], queueing theory [6], and survival analysis [7]. However, the need to expand the scope of the exponential distribution has become increasingly evident in modern statistical modeling.

Numerous researchers have explored the generalization of probability distributions, drawing inspiration from a rich body of statistical literature. The foundational works of Johnson and Kotz [1] in "Continuous Univariate Distributions", Montgomery and Runger's (2018) "Applied Statistics and Probability for Engineers" [2], and Ross's (2014) "Introduction to Probability Models" [3] have laid the groundwork for understanding the essentials of probability theory.

Cox (1984) delved into the intricacies of risk and risk measures in the "Journal of the Royal Statistical Society. Series A (General)" [4], while Nelson (1982) contributed to the field with "Applied Life Data Analysis" [5]. Kleinrock's exploration of queueing systems in "Queueing Systems, Volume I: Theory" [6] further enriched the domain of statistical modeling. The foundational works of Kaplan and Meier (1958) in "Nonparametric estimation from incomplete observations" in the "Journal of the American Statistical Association" [7] have provided significant contributions to this field.

Roy and Adnan (2012) introduced the concept of wrapped weighted exponential distributions [8]. Within the realm of circular distributions, Mardia's work on "Statistics of Directional Data" [9] has been instrumental

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in understanding directional statistics. Building upon these foundations, researchers have introduced novel distributions. Alkhazaalh and Al-Zoubi (2021) presented the Epanechnikov-exponential distribution, offering unique properties and applications [10].

The transmutation of distributions has also been explored, with Al-Omari, Al-khazaleh, and Alzoubi (2021) proposing the transmuted Janardan distribution, a generalization of the Janardan distribution [11]. Additionally, Al-Khazaleh (2016) extended the Burr type XII distribution to introduce the Transmuted Burr type XII distribution [12]. The exponential distribution, known for its utility and memoryless property, has been studied extensively. Jowett (1958) discussed the exponential distribution and its applications [13], while Lukacs (1960) explored characteristic functions [14]. Sahoo's (2013) work on "Probability and Mathematical Statistics" has contributed to the field [15], and Fisher (1997) introduced an absolute criterion for fitting frequency curves [16]. In the realm of statistical computing, the R programming language and environment, developed by the R Core Team in 2019, provide essential tools for statistical analysis and modeling [17].

In this paper, we present an innovative approach to extend the capabilities of probability density functions, with a specific focus on the exponential distribution. Drawing inspiration from fundamental statistical parameters like the rate parameter, we introduce a novel methodology that facilitates the creation of a versatile family of PDFs, each possessing unique characteristics.

The central contribution of this research is the formulation and rigorous proof of the "Exponential PDF Generalization Theorem." This theorem serves as the linchpin for the expansion of exponential PDFs, enabling the development of nth-generation PDFs, each offering distinct attributes and applications. This theorem represents a significant leap forward in the field of exponential distributions and statistical modeling, with far-reaching implications for practitioners and researchers alike.

To illustrate the practicality and adaptability of our approach, we apply the "Exponential PDF Generalization Theorem" to diverse scenarios and conduct a series of comprehensive estimation and simulation studies. These investigations provide valuable insights into the performance of generalized exponential PDFs in comparison to their original counterparts.

The introduction of the "Exponential PDF Generalization Theorem" in this paper herald's new possibilities in statistical modeling and analysis within the realm of exponential distributions. It equips researchers and practitioners with a potent tool to manipulate and adapt exponential PDFs to suit specific needs, opening doors to innovative applications across various fields where the exponential distribution serves as a cornerstone. In the subsequent sections, we delve into the intricacies of our methodology, theorem, and empirical findings, demonstrating the transformative potential of this pioneering research.

2 Extended Exponential Distribution

The main aim of this paper is to present interesting extensions of Exponential Distribution. We shall first define Extended Exponential Distribution in terms of a new parameter $\alpha > 0$ and call it α-Extended Exponential Distribution (α-EED). The probability distribution of the time between events in a Poisson point process is the exponential distribution (ED). The probability density and cumulative distribution functions of the exponential random variable $(X \sim \mathrm{Exp}(\lambda))$ are defined by Jowett (1958) [13] as respectively:

$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \lambda > 0,$

$F(x) = 1 - e^{-\lambda x}, \quad x > 0, \lambda > 0.$

The mean of the exponential distribution is $\mu = \frac{1}{\lambda}$, the variance is $\sigma^2 = \mathrm{var}(X) = \frac{1}{\lambda^2}$, skewness = 2, and kurtosis = 6.

Theorem 1. *Let X be a random variable of continuous type and let $\lambda > 0, \alpha > 0$, be the parameters; then the function*

$$f(x;\lambda,\alpha) = \frac{\lambda}{\Gamma[\alpha+1,1]} \mathrm{e}^{-\lambda x - 1} (\lambda x + 1)^{\alpha}; \quad x > 0, \quad (1)$$

is the pdf of random variable X of continuous type.

Proof.

$$\begin{aligned}
&\int_0^\infty f(x,\lambda,\alpha)dx = \int_0^\infty \frac{\lambda}{\Gamma[\alpha+1,1]} \mathrm{e}^{-\lambda x-1} (\lambda x+1)^{\alpha} dx \\
&= \frac{1}{e\Gamma[\alpha+1,1]} \int_0^\infty \lambda \mathrm{e}^{-\lambda x} (\lambda x+1)^{\alpha} dx \\
&= \frac{1}{e\Gamma[\alpha+1,1]} \int_0^\infty \lambda^{(\alpha+1)} \mathrm{e}^{-y} \frac{(y+1)^{\alpha}}{(\lambda)^{\alpha+1}} dy \\
&= \frac{1}{e\Gamma[\alpha+1,1]} \int_0^\infty (y+1)^{\alpha} \mathrm{e}^{-y} dy \\
&= \frac{1}{e\Gamma[\alpha+1,1]} \mathrm{e}\Gamma[\alpha+1,1] = 1.
\end{aligned}$$

Let $X \sim \alpha - \mathrm{EED}(\lambda,\alpha)$. Then the distribution function for the random variable X is

$$F(x) = 1 - \frac{\Gamma[1+\alpha, 1+x\lambda]}{\Gamma[1+\alpha, 1]},$$

and the expectation and variance are given by $E(x) = \frac{1}{\lambda}$, $Var(X) = \frac{1}{\lambda^2}$.

The last theorem provides a generalized probability density function for the exponential distribution with respect to the parameters λ and α, utilizing the expectation and variance of the exponential distribution.

As special forms of the α-EED for $\alpha = 2,3,4,5$ and 6 the pdfs and CDFs as follows respectively in Table 1

Table 1: pdf and CDF of α-EED for different values of α

pdfs	CDFs
$\frac{1}{5}e^{-x\lambda}\lambda\,(1+x\lambda)^2$	$1+\frac{1}{5}e^{-x\lambda}\,(-5-x\lambda\,(4+x\lambda))$
$\frac{1}{16}e^{-x\lambda}\lambda\,(1+x\lambda)^3$	$1-\frac{1}{16}e^{-x\lambda}\,(16+15x\lambda+6x^2\lambda^2+x^3\lambda^3)$
$\frac{1}{65}e^{-x\lambda}\lambda\,(1+x\lambda)^4$	$1-e^{-x\lambda}-\frac{1}{65}e^{-x\lambda}x\lambda\,(64+x\lambda\,(30+x\lambda\,(8+x\lambda)))$
$\frac{1}{326}e^{-x\lambda}\lambda\,(1+x\lambda)^5$	$1-e^{-x\lambda}-\frac{1}{326}e^{-x\lambda}x\lambda\,(325+x\lambda\,(160+x\lambda\,(50+x\lambda\,(10+x\lambda))))$
$\frac{\lambda}{1957}\,(1+x\lambda)^6 e^{-x\lambda}$	$1-e^{-x\lambda}-\frac{e^{-x\lambda}x\lambda(1956+x\lambda(975+x\lambda(320+x\lambda(75+x\lambda(12+x\lambda)))))}{1957}$

The graphs of the pdf and cdf of the α-EED for varying values of λ and α are presented in Figure 1 and Figure 2, respectively.

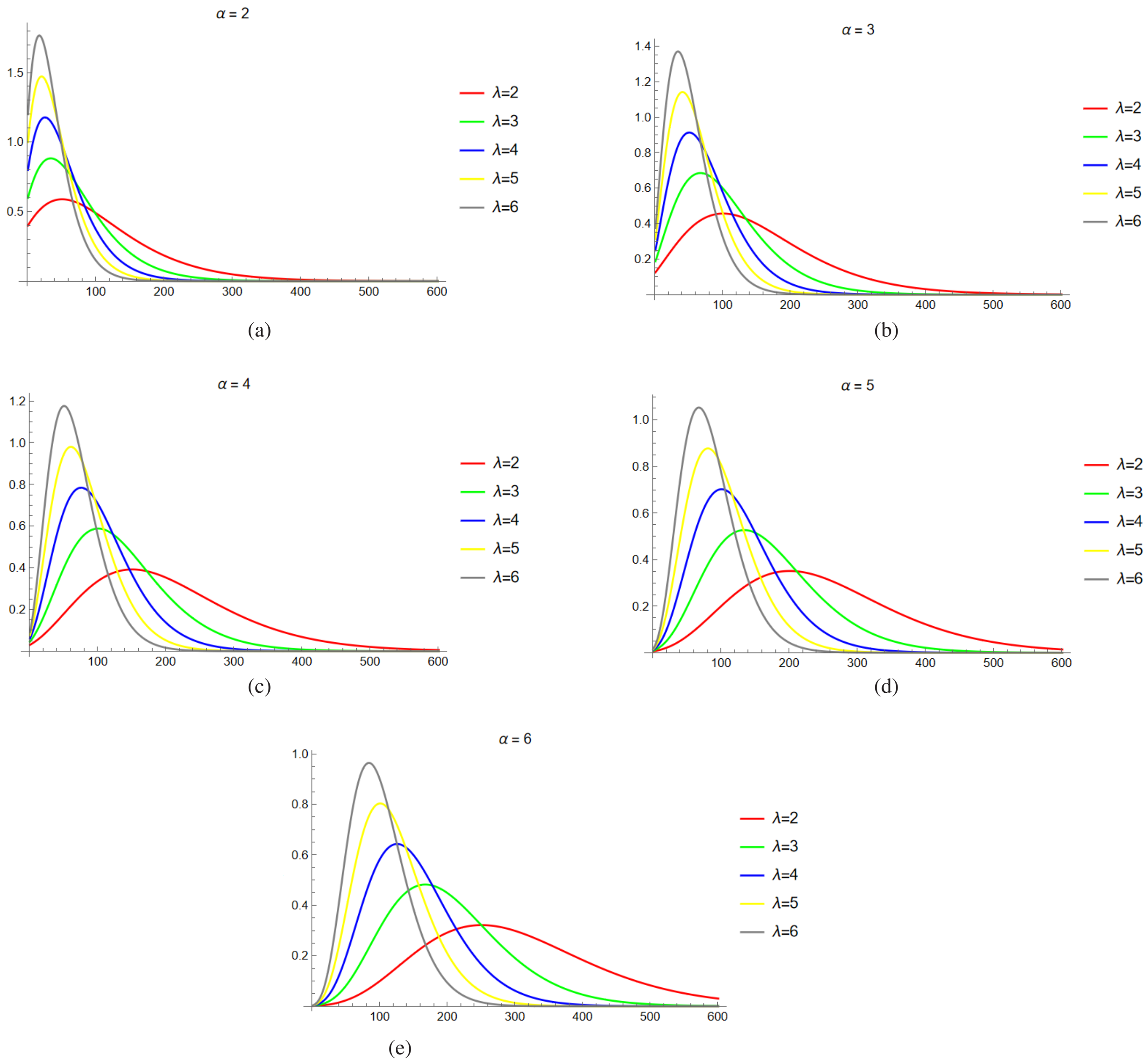

Figure 1: pdf of α-EED

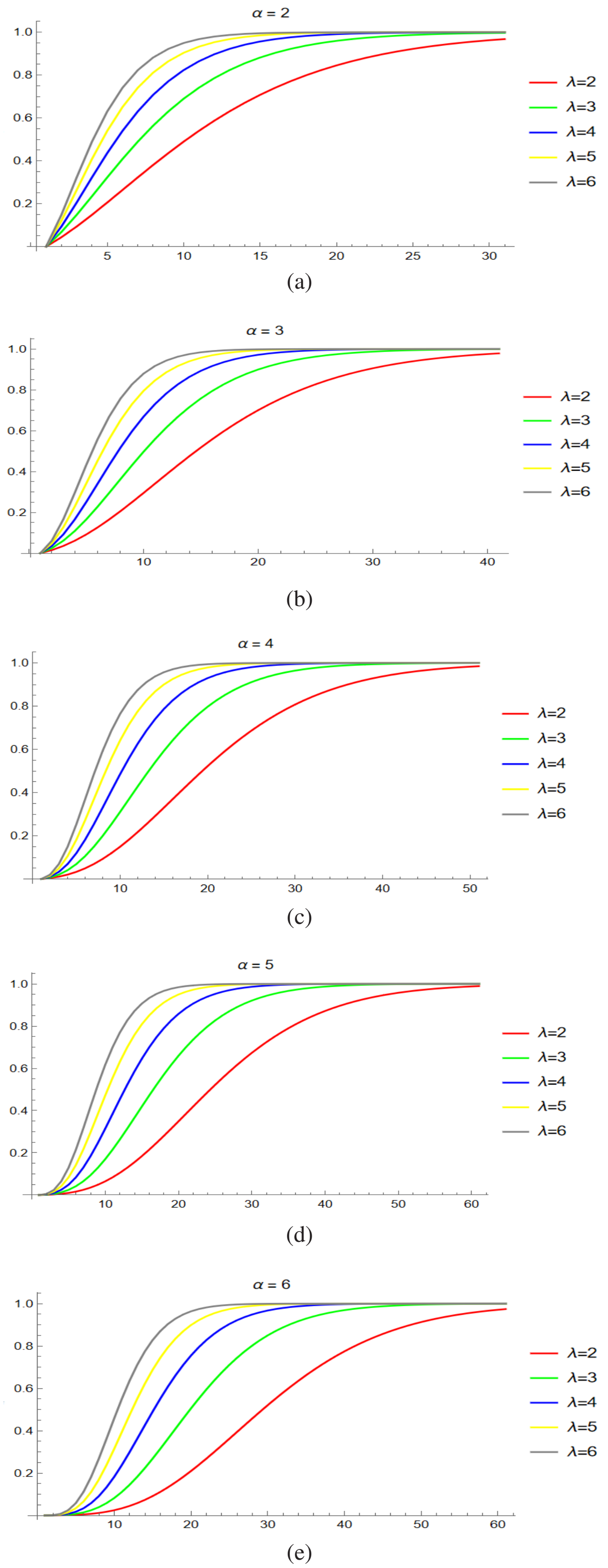

Figure 2: CDF of α-EED

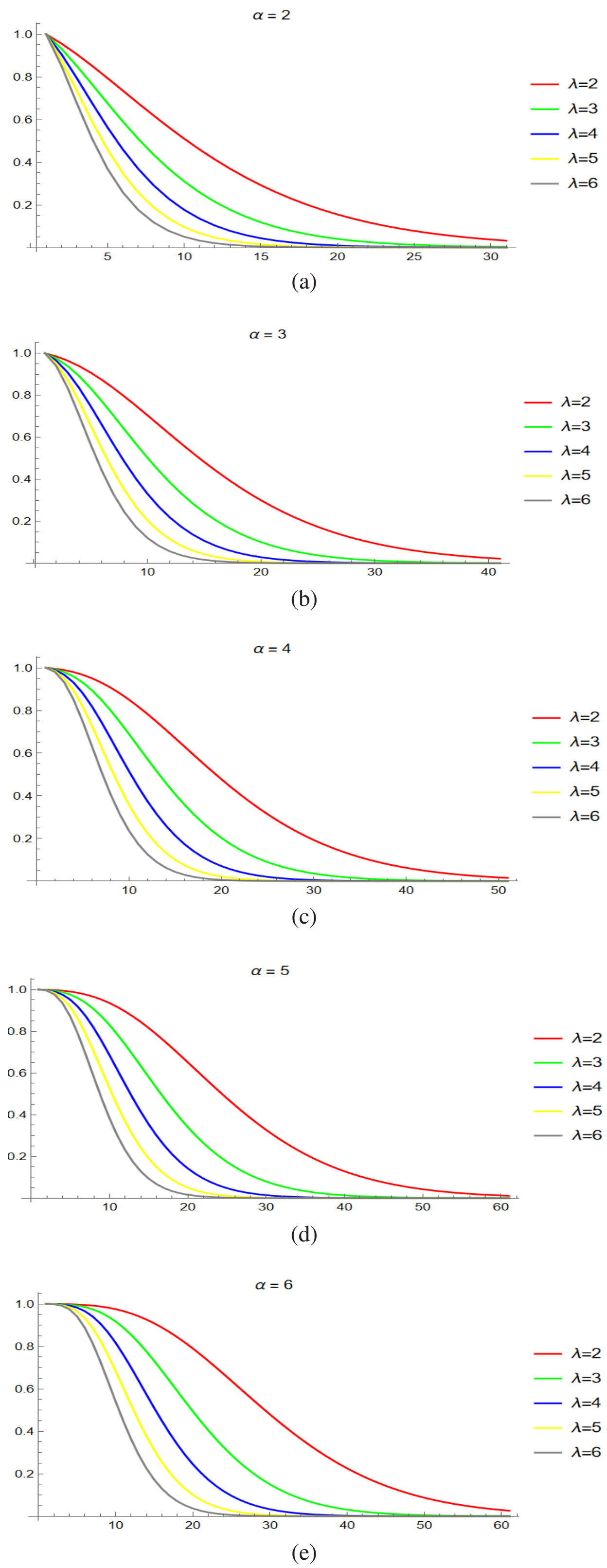

Figure 3: The graph of different forms of $R(x;\lambda,\alpha)$

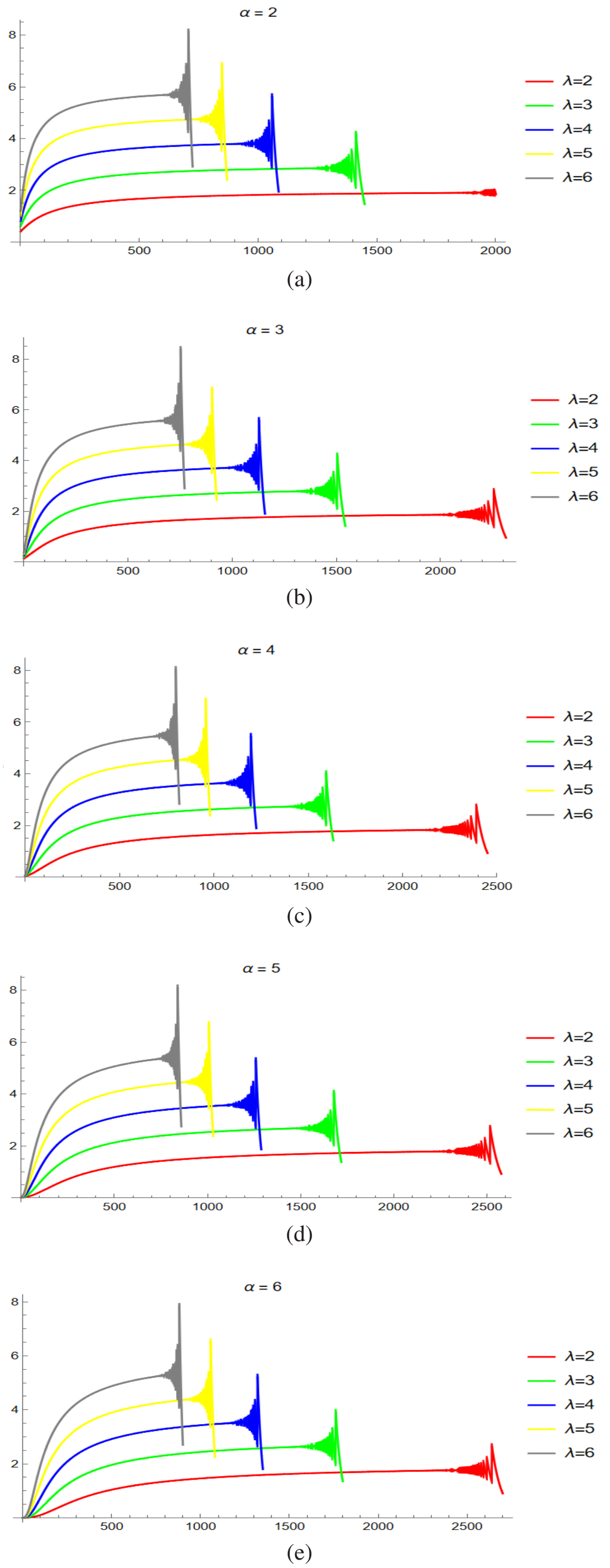

Figure 4: The graph of different forms of $h(x;\lambda,\alpha)$

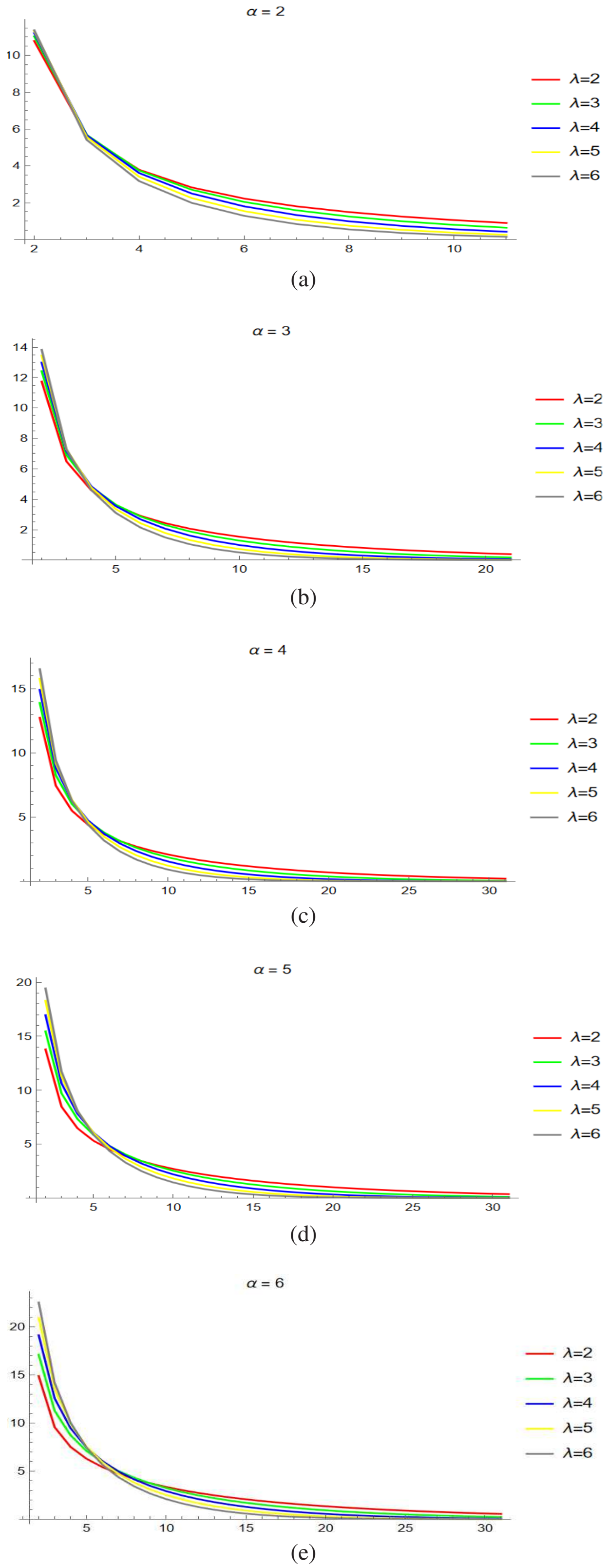

Figure 5: The graph of different forms of $r(x;\lambda,\alpha)$

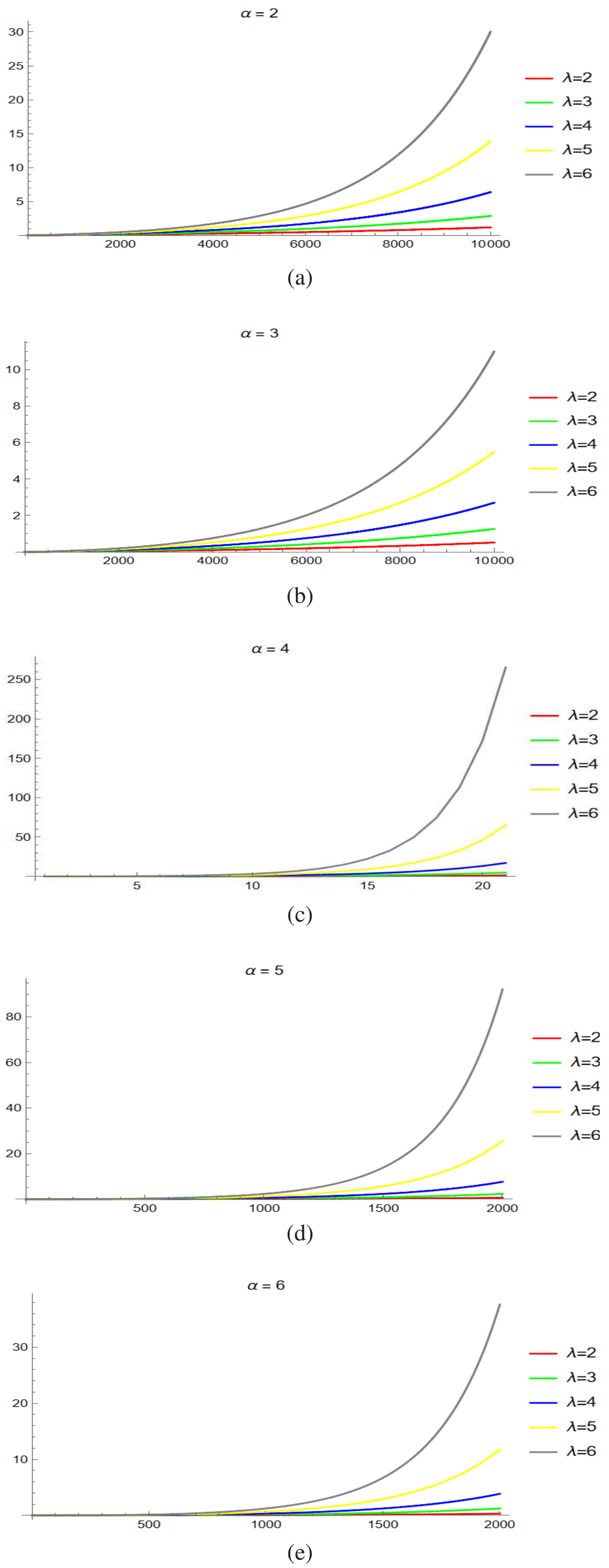

Figure 6: The graph of different forms of $O(x;\lambda,\alpha)$

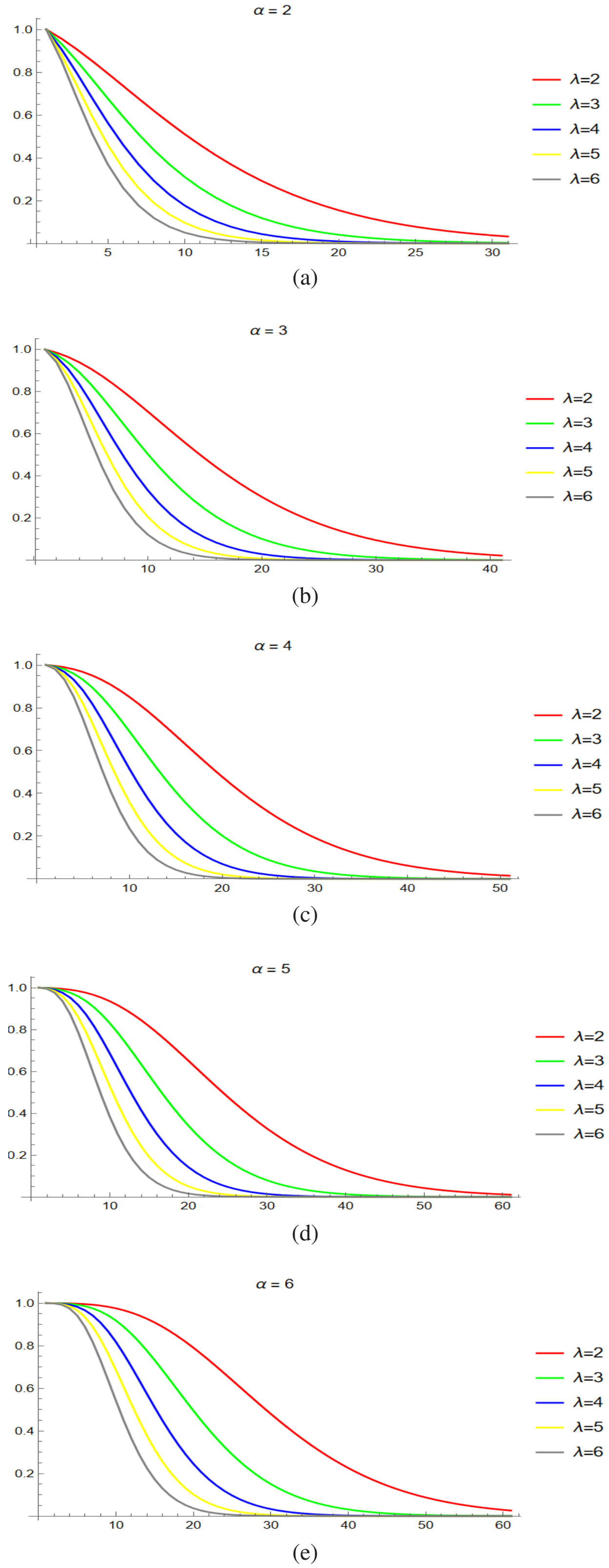

Figure 7: The graph of different forms of $R(x;\lambda,\alpha)$

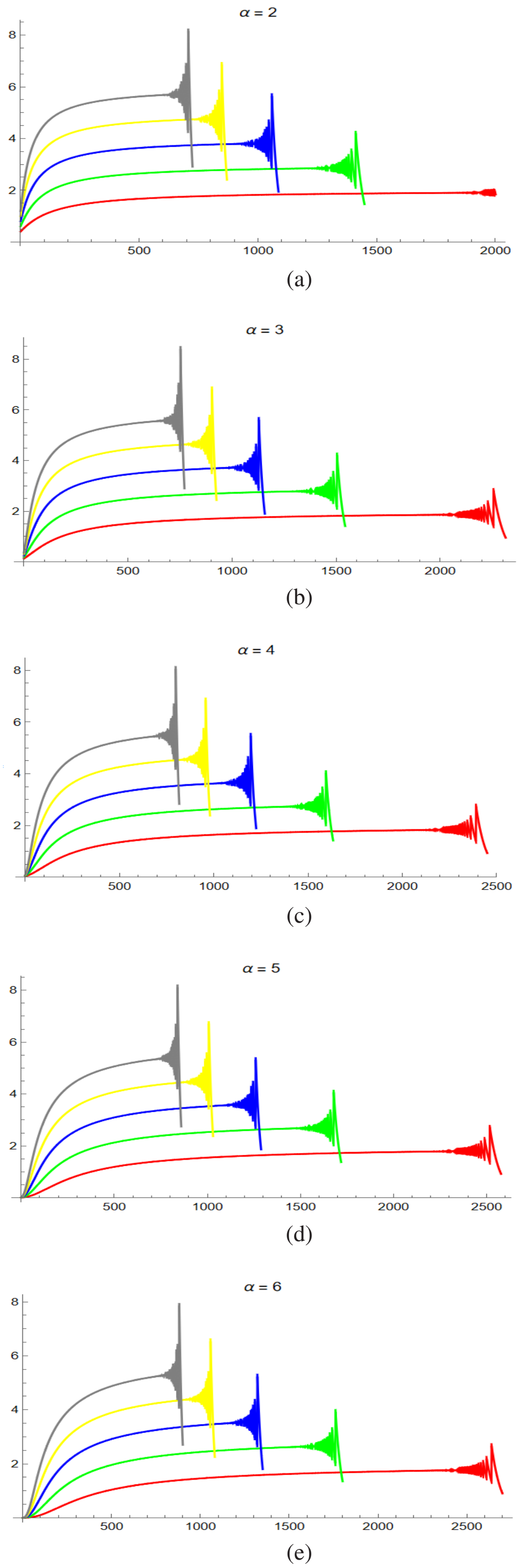

Figure 8: The graph of different forms of $h(x;\lambda,\alpha)$

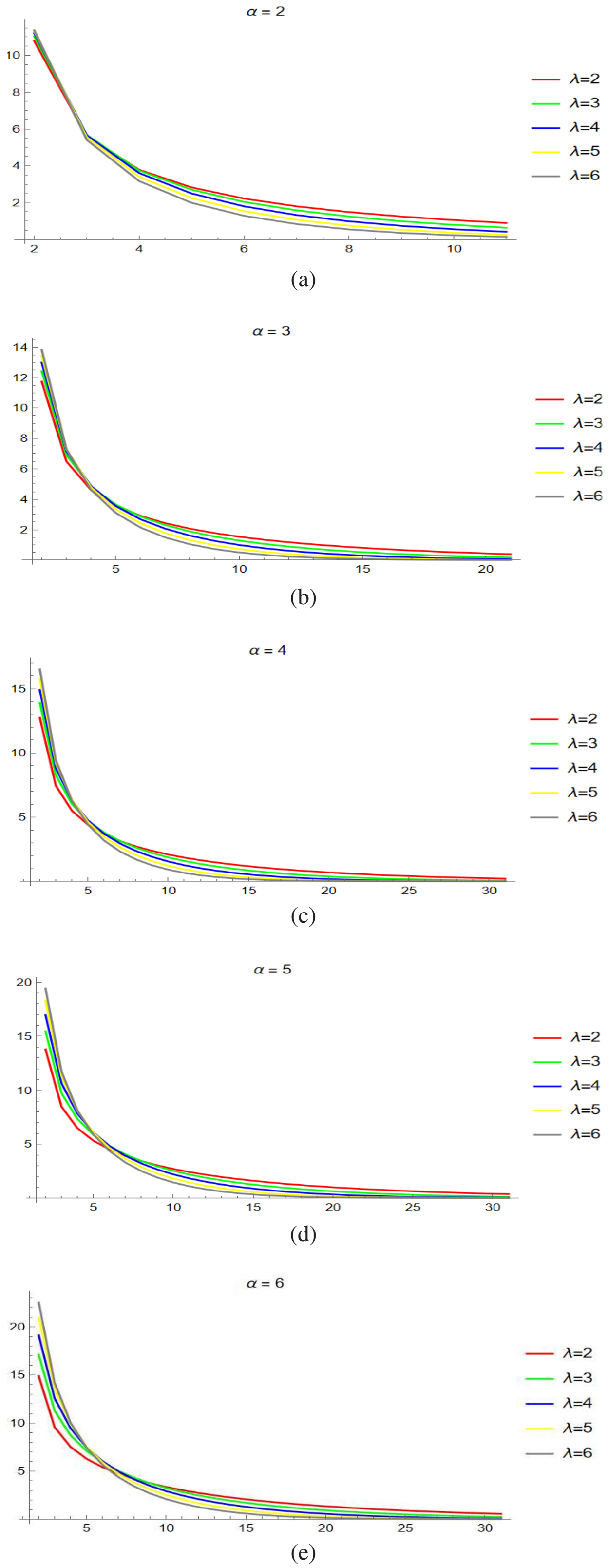

Figure 9: The graph of different forms of $r(x;\lambda,\alpha)$

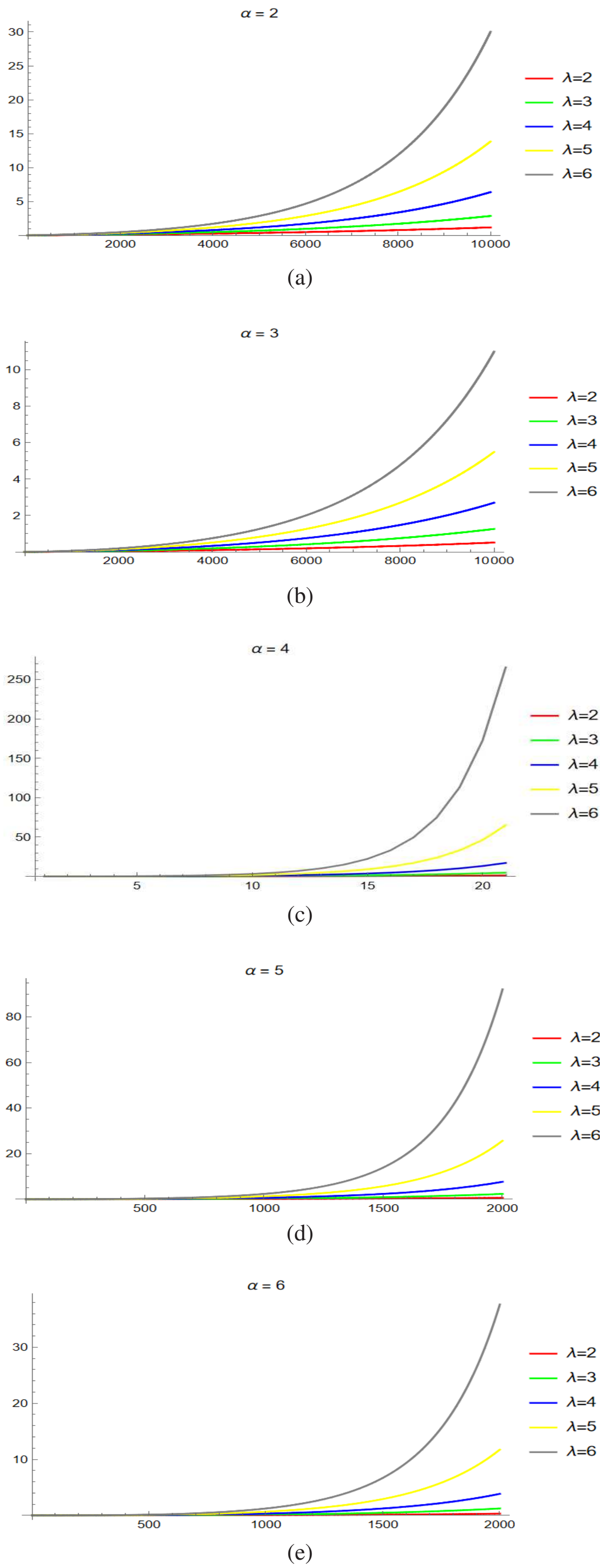

Figure 10: The graph of different forms of $O(x;\lambda,\alpha)$

The limit of the cdf of α-EED

$$\lim_{x\to\infty} F(x;\lambda) = \lim_{x\to\infty}\left[\frac{\Gamma[1+\alpha,1]-\Gamma[1+\alpha,1+x\lambda]}{\Gamma[1+\alpha,1]}\right] = 1$$
$$\lim_{x\to 0} F(x;\lambda) = \lim_{x\to 0}\left[\frac{\Gamma[1+\alpha,1]-\Gamma[1+\alpha,1+x\lambda]}{\Gamma[1+\alpha,1]}\right] = 0$$

3 Statistical Measures

In this section, we present and discuss key concepts related to reliability measures, specifically focusing on the Survival, Hazard, and Cumulative Hazard Functions. The Reliability Function, a fundamental component of reliability analysis, is defined as the complement of the cumulative distribution function. Forbes et al. (2011) precisely elucidate this definition, where the Reliability Function serves as a critical concept in assessing the reliability and survival characteristics of a system or process.

$$\begin{aligned} R(x) &= \int_x^{\infty} f(t)\,dt = 1 - F(x) \\ &= 1 - \left[1 - \frac{\Gamma(1+\alpha,1+x\lambda)}{\Gamma(1+\alpha,1)}\right] \\ &= \frac{\Gamma(1+\alpha,1+x\lambda)}{\Gamma(1+\alpha,1)} \end{aligned}$$

The hazard rate function, as expounded by Rinne and Horest (2014), constitutes a pivotal component of reliability analysis. This function offers valuable insights into the instantaneous failure rate or risk of failure at a specific point in time within a given system or process.

$$\begin{aligned} h(x) &= \frac{f(x)}{R(x)} = \frac{f(x)}{1-F(x)} \\ &= \frac{\frac{\lambda e^{-\lambda x}(\lambda x+1)^{\alpha}}{\Gamma(\alpha+1,1)}}{\frac{\Gamma(1+\alpha,1+x\lambda)}{\Gamma(1+\alpha,1)}} \\ &= \frac{\lambda e^{-\lambda x}(\lambda x+1)^{\alpha}}{\Gamma(1+\alpha,1+x\lambda)} \end{aligned}$$

The reversed hazard rate function, as rigorously defined by Desai et al. (2011), represents a crucial metric within reliability analysis. This function provides critical information regarding the probability distribution of the time until the next event in a system or process, offering valuable insights into its behaviour and reliability characteristics.

$$\begin{aligned} r(x) &= \frac{f(x)}{F(x)} \\ &= \frac{\frac{\lambda e^{-\lambda x}(\lambda x+1)^{\alpha}}{\Gamma(\alpha+1,1)}}{\frac{\Gamma(1+\alpha,1)-\Gamma(1+\alpha,1+x\lambda)}{\Gamma(1+\alpha,1)}} \\ &= \frac{\lambda e^{-\lambda x}(\lambda x+1)^{\alpha}}{\Gamma(1+\alpha,1)-\Gamma(1+\alpha,1+x\lambda)} \end{aligned}$$

The odds function, as meticulously articulated by Zimmer et al. (1998), constitutes a significant measure within the realm of probability and reliability analysis. This function represents the ratio of the cumulative distribution function to the reliability function, offering a comprehensive perspective on the likelihood of success or failure within a given system or process.

$$\begin{aligned} O(x) &= \frac{F(x)}{R(x)} \\ &= \frac{\frac{\Gamma(1+\alpha,1)-\Gamma(1+\alpha,1+x\lambda)}{\Gamma(1+\alpha,1)}}{1-\frac{\Gamma(1+\alpha,1)-\Gamma(1+\alpha,1+x\lambda)}{\Gamma(1+\alpha,1)}} \\ &= \frac{\frac{\Gamma(1+\alpha,1)-\Gamma(1+\alpha,1+x\lambda)}{\Gamma(1+\alpha,1)}}{\frac{\Gamma(1+\alpha,1+x\lambda)}{\Gamma(1+\alpha,1)}} \\ &= \frac{\Gamma(1+\alpha,1)-\Gamma(1+\alpha,1+x\lambda)}{\Gamma(1+\alpha,1+x\lambda)} \\ &= \frac{\Gamma(1+\alpha,1)}{\Gamma(1+\alpha,1+x\lambda)}-1 \end{aligned}$$

4 Moments and Generating Functions

The rth moment, mean, variance and standard deviation can be computed as follows:

$$(\lambda x+1)^{\alpha} = \sum_{k=0}^{\alpha}\binom{\alpha}{k}(\lambda x)^{\alpha-k}$$

$$\begin{aligned} E(x^r) &= \int_0^\infty x^r f(x;\lambda,\alpha)\,dx \\ &= \int_0^\infty x^r \frac{\lambda}{e\Gamma[\alpha+1,1]} e^{-\lambda x}(\lambda x+1)^{\alpha}\,dx \\ &= \frac{\lambda}{e\Gamma[\alpha+1,1]}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\lambda^{\alpha-k}\int_0^\infty e^{-\lambda x}x^{\alpha-k+r}\,dx \end{aligned}$$

where $u=\lambda x$ and $du=\lambda dx$ so $\frac{u}{\lambda}=x$

$$\begin{aligned} &= \frac{1}{e\Gamma[\alpha+1,1]}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\frac{\lambda^{\alpha-k+1}}{\lambda^{\alpha-k+r+1}}\int_0^\infty e^{-u}u^{\alpha-k+r}\,du \\ &= \frac{1}{e\Gamma[\alpha+1,1]}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\frac{1}{\lambda^r}\int_0^\infty e^{-u}u^{\alpha-k+r}\,du \\ E(X^r) &= \frac{1}{\lambda^r e\Gamma[\alpha+1,1]}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\Gamma(\alpha-k+r+1) \end{aligned}$$

The moment generating function is defined as (Rose and Smith, 2002):

$$\begin{aligned} M_X(t) &= E(e^{tx}) \\ &= \int_0^\infty e^{tx} f(x;\lambda,\alpha)\,dx \\ &= \int_0^\infty e^{tx}\frac{\lambda}{e\Gamma[\alpha+1,1]}e^{-\lambda x}(\lambda x+1)^{\alpha}\,dx \\ &= \frac{\lambda}{e\Gamma[\alpha+1,1]}\int_0^\infty e^{-(\lambda-t)x}(\lambda x+1)^{\alpha}\,dx \\ &= \frac{\lambda}{e\Gamma[\alpha+1,1]}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\lambda^{\alpha-k}\int_0^\infty e^{-x(\lambda-t)}x^{\alpha-k}\,dx \end{aligned}$$

Let $\begin{cases} u=x(\lambda-t),\ du=(\lambda-t)\,dx \\ x=\frac{u}{\lambda-t} \end{cases}$

$$\begin{aligned} &= \frac{\lambda}{e\Gamma[\alpha+1,1]}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\lambda^{\alpha-k}\int_0^\infty e^{-u}\left(\frac{u}{\lambda-t}\right)^{\alpha-k}\frac{du}{\lambda-t} \\ &= \frac{\lambda}{e\Gamma[\alpha+1,1]}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(\frac{\lambda}{\lambda-t}\right)^{\alpha-k+1}\int_0^\infty e^{-u}u^{\alpha-k}\,du \\ &= \frac{1}{e\Gamma[\alpha+1,1]}\sum_{k=0}^{\alpha}\binom{\alpha}{k}\left(\frac{\lambda}{\lambda-t}\right)^{\alpha-k+1}\Gamma(\alpha-k+1) \end{aligned}$$

5 Maximum Likelihood Estimate (MLE)

The likelihood function is defined as:

$$L(\theta;p;x) = \prod_{i=1}^{n} f(x_i) \tag{2}$$

Taking the natural logarithm of both sides of Equation (1), the log-likelihood function is:

$$\ell = \ln\left(L(\theta;p;x)\right) \tag{3}$$

$$\ell = \ln\left(\prod_{i=1}^{n} f(x_i)\right) \tag{4}$$

$$\ell = \sum_{i=1}^{n}\ln\left(f(x_i)\right) \tag{5}$$

Substituting the expression for $f(x_i)$:

$$\ell = \sum_{i=1}^{n}\ln\left(\frac{\lambda e^{-\lambda x_i}(\lambda x_i+1)^{\alpha}}{e\Gamma[\alpha+1,1]}\right) \tag{6}$$

$$\ell = \sum_{i=1}^{n}\left(\ln\frac{\lambda}{e\Gamma[\alpha+1,1]}+\ln\left((\lambda x_i+1)^{\alpha}\right)+\ln e^{-\lambda x_i}\right) \tag{7}$$

$$\ell = (n\ln\lambda - n - n\ln\Gamma[\alpha+1,1]) + \alpha\sum_{i=1}^{n}\ln(\lambda x_i+1) - \lambda\sum_{i=1}^{n}x_i \tag{8}$$

To find the maximum likelihood estimates, we need to take the partial derivatives of ℓ with respect to λ and α.

The partial derivative of ℓ with respect to λ is:

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} + \alpha\sum_{i=1}^{n}\frac{x_i}{\lambda x_i+1} - \sum_{i=1}^{n}x_i \tag{9}$$

The partial derivative of ℓ with respect to α is:

$$\frac{\partial\ell}{\partial\alpha} = \frac{\partial}{\partial\alpha}(n\ln\lambda - n - n\ln\Gamma[\alpha+1,1]) + \frac{\partial}{\partial\alpha}\left(\alpha\sum_{i=1}^{n}\ln(\lambda x_i+1)\right) - \frac{\partial}{\partial\alpha}\left(\lambda\sum_{i=1}^{n}x_i\right)$$
$$= -n\frac{\partial\ln\Gamma[\alpha+1,1]}{\partial\alpha} + \sum_{i=1}^{n}\ln(\lambda x_i+1)$$

$$\frac{\partial\ln\Gamma[\alpha+1,1]}{\partial\alpha} = \frac{\Gamma'[\alpha+1,1]}{\Gamma[\alpha+1,1]} = \psi(\alpha+1) - \psi(1) \tag{10}$$

where $\psi(\cdot)$ is the digamma function. So,

$$\frac{\partial\ell}{\partial\alpha} = -n(\psi(\alpha+1) - \psi(1)) + \sum_{i=1}^{n}\ln(\lambda x_i+1) \tag{11}$$

6 Distribution of Order Statistics

The pdf of the ith order statistic is given by (Craig and Hogg, 2013):

$$f_{(i)}(x) = \frac{n!}{(i-1)!\,(n-i)!}f(x)(F(x))^{i-1}(1-F(x))^{n-r}, \quad 1<i<n$$

$$f_{(i)}(x) = \frac{n!}{(i-1)!(n-i)!}\left[\frac{\lambda e^{-\lambda x-1}(\lambda x+1)^{\alpha}}{\Gamma(\alpha+1,1)}\right] \times\left(\frac{\Gamma(1+\alpha,1)-\Gamma(1+\alpha,1+x\lambda)}{\Gamma(1+\alpha,1)}\right)^{i-1} \times\left(1-\frac{\Gamma(1+\alpha,1)-\Gamma(1+\alpha,1+x\lambda)}{\Gamma(1+\alpha,1)}\right)^{n-i}$$
$$= n!\left(\lambda e^{-\lambda x-1}(\lambda x+1)^{\alpha}\right)\left((\Gamma(1+\alpha,1+x\lambda))^{n-i}\right) \times\frac{(\Gamma(1+\alpha,1)-\Gamma(1+\alpha,1+x\lambda))^{i-1}}{(i-1)!(n-i)!\,(\Gamma(1+\alpha,1))^{n}}$$

$$f_{(1)}(x) = \frac{n\left(\lambda e^{-\lambda x-1}(\lambda x+1)^{\alpha}\right)(\Gamma[1+\alpha,1+x\lambda])^{n-1}}{(\Gamma[\alpha+1,1])^{n}}$$

$$f_{(n)}(x) = nf(x)(F(x))^{n-1} = n\lambda e^{-\lambda x-1}(\lambda x+1)^{\alpha} \times\frac{(\Gamma[1+\alpha,1]-\Gamma[1+\alpha,1+x\lambda])^{n-1}}{(\Gamma[1+\alpha,1])^{n}}$$

7 Renyi and Tsallis Entropy Measure

The importance of Rényi entropy was introduced by Rényi (1961), which lies in statistics as an indicator of diversity and in quantitative information. The Rényi entropy is defined as:

$$f_{\text{ALKhaz}}(x;\lambda,\alpha) = \frac{\lambda\lambda^{\alpha}}{e\Gamma[\alpha+1,1]}e^{-\lambda x}\left(x+\frac{1}{\lambda}\right)^{\alpha} = \frac{\lambda}{e\Gamma[\alpha+1,1]}e^{-\lambda x}(\lambda x+1)^{\alpha};$$

$x>0,\ \lambda>0, \alpha>0$

$$E_R = \frac{1}{1-\beta}\log\int_x (f(x))^{\beta}dx,$$

where

$$\beta>0,\ \beta\neq 1.$$

$$E_R = \frac{1}{1-\beta}\log\frac{\lambda}{e\Gamma[\alpha+1,1]}\int_0^{\infty}\left(e^{-\lambda x}(\lambda x+1)^{\alpha}\right)^{\beta}dx$$
$$= \frac{1}{1-\beta}\log\frac{\lambda}{e\Gamma[\alpha+1,1]}\int_0^{\infty}\left(e^{-\lambda\beta x}(\lambda x+1)^{\alpha\beta}\right)dx$$
$$= \frac{1}{1-\beta}\log\frac{\lambda}{e\Gamma[\alpha+1,1]} \times\int_0^{\infty}e^{-\lambda\beta x}\sum_{k=0}^{\alpha\beta}\binom{\alpha\beta}{k}(\lambda)^{\alpha\beta-k}(x)^{\alpha\beta-k}dx$$

Let $u=\lambda\beta x$, $du=\lambda\beta dx$ and $(\lambda x+1)^{\alpha\beta} = \sum_{k=0}^{\alpha\beta}\binom{\alpha\beta}{k}(\lambda x)^{\alpha\beta-k}$

$$E_R = \frac{1}{1-\beta}\log\frac{\lambda}{e\Gamma[\alpha+1,1]}\sum_{k=0}^{\alpha\beta}\binom{\alpha\beta}{k}(\lambda)^{\alpha\beta-k} \times\int_0^{\infty}e^{-u}(u)^{\alpha\beta-k}\left(\frac{1}{\lambda\beta}\right)^{\alpha\beta-k+1}du$$
$$= \frac{1}{1-\beta}\log\frac{\lambda}{e\Gamma[\alpha+1,1]}\sum_{k=0}^{\alpha\beta}\binom{\alpha\beta}{k}\left(\frac{1}{\lambda\beta}\right) \times\int_0^{\infty}e^{-u}(u)^{\alpha\beta-k}du$$
$$= \frac{1}{1-\beta}\log\frac{1}{e\Gamma[\alpha+1,1]} \times\sum_{k=0}^{\alpha\beta}\binom{\alpha\beta}{k}\left(\frac{1}{\beta}\right)\Gamma(\alpha\beta-k+1)$$

Tsallis (1988) suggested Tsallis entropy. It is given by:

$$E_T = \frac{1}{\alpha-1}\left[1-\int_x (f(x))^{\alpha}dx\right], \alpha>0, \alpha\neq 1.$$

$$E_T = \frac{1}{\rho-1}\left[1-\left(\frac{\lambda}{e\Gamma(\alpha+1,1)}\right)^{\rho} \times \int_0^{\infty}\left(e^{-\lambda\rho x}(\lambda x+1)^{\alpha\rho}\right)dx\right]$$
$$= \frac{1}{\rho-1}\left[1-\left(\frac{\lambda}{e\Gamma(\alpha+1,1)}\right)^{\rho}\sum_{k=0}^{\alpha\rho}\binom{\alpha\rho}{k}\lambda^{\alpha\rho-k} \times \int_0^{\infty}\left(e^{-\lambda\rho x}x^{\alpha\rho-k}\right)dx\right]$$

Let $u=\lambda\rho x$, $x=\frac{u}{\lambda\rho}$, $du=\lambda\rho\,dx$, and $dx=\frac{du}{\lambda\rho}$. Then

$$E_T = \frac{1}{\rho-1}\left[1-\left(\frac{\lambda}{e\Gamma(\alpha+1,1)}\right)^{\rho} \times \sum_{k=0}^{\alpha\rho}\binom{\alpha\rho}{k}\lambda^{\alpha\rho-k}\left(\frac{1}{\lambda\rho}\right)^{\alpha\rho-k+1}\int_0^{\infty}e^{-u}u^{\alpha\rho-k}du\right]$$
$$= \frac{1}{\rho-1}\left[1-\left(\frac{\lambda}{e\Gamma(\alpha+1,1)}\right)^{\rho} \times \sum_{k=0}^{\alpha\rho}\binom{\alpha\rho}{k}\lambda^{\alpha\rho-k}\left(\frac{1}{\lambda\rho}\right)^{\alpha\rho-k+1}\Gamma(\alpha\rho-k+1)\right]$$
$$= \frac{1}{\rho-1}\left[1-\left(\frac{1}{e\Gamma(\alpha+1,1)}\right)^{\rho}\lambda^{\rho-1} \times \sum_{k=0}^{\alpha\rho}\binom{\alpha\rho}{k}\left(\frac{1}{\rho}\right)^{\alpha\rho-k+1}\Gamma(\alpha\rho-k+1)\right]$$

8 Lorenz Curves, Bonferroni and Gini Index

Lorenz (1905) introduced what is now known as the Lorenz curve. This curve serves as a visual tool for illustrating income inequality or wealth disparity and is commonly employed in the field of economics. It provides a graphical representation of how unequal the distribution of income or wealth is within a population.

Assuming that X is a continuous non-negative random variable, and then the Lorenz curve is given by:

$$L(\rho) = \frac{1}{\mu}\int_0^{q}xf(x)dx = \frac{1}{\mu}\left(\mu-\int_q^{\infty}xf(x)dx\right)$$
$$= \frac{1}{\mu}\left(\mu-\int_q^{\infty}\frac{\lambda}{e\Gamma[\alpha+1,1]}xe^{-\lambda x}(\lambda x+1)^{\alpha}dx\right)$$
$$= 1-\frac{\Gamma[2+\alpha,1+q\lambda]-\Gamma[1+\alpha,1+q\lambda]}{\mu\lambda\Gamma[1+\alpha,1]}$$

where, $\rho\in(0,1]$, $q=F^{-1}(x)$ and μ is the expected value.

In 1930, another notable figure named Bonferroni introduced the Bonferroni curve, which is another type of inequality curve. In simpler terms, the Bonferroni curve provides a graphical representation that is extensively employed, especially in economic analysis. It is regarded as a significant curve in this context and is defined by certain mathematical properties.

$$B(\rho) = \frac{1}{\rho\mu}\int_0^{q}xf(x)dx$$
$$= \frac{1}{\rho\mu}\left(\mu-\int_q^{\infty}xf(x)dx\right)$$
$$= \frac{1}{\rho\mu}\left(\mu-\int_q^{\infty}xf(x)dx\right)$$
$$= \frac{1}{\rho}-\frac{\Gamma[2+\alpha,1+q\lambda]-\Gamma[1+\alpha,1+q\lambda]}{\rho\mu\lambda\Gamma[1+\alpha,1]}$$

where μ is the population mean.

Another measure of inequality of income or wealth in economic is suggested by Gini (1912) and is called Gini index. Mathematically, Gini index is usually determined and evaluated using Lorenz curve. The Gini index is given by:

$$G = 1-\frac{1}{\mu}\int_0^{\infty}(1-F(x))^2dx$$
$$= 1-\frac{1}{\mu}\left(\frac{1}{\Gamma[1+\alpha,1]}\right)^2\int_0^{\infty}(\Gamma[1+\alpha,1+x\lambda])^2dx$$

9 Simulation Study

A simulation study is accompanied using α-EED samples with sizes of 50, 100, 150, 200, and 500. The samples were taken for $(\alpha = 2,3,4 \text{and} \lambda = 2,3,4)$. For the parameter $(\alpha \text{and} \lambda)$, the maximum likelihood estimator is obtained. The technique has been carried out 10,000 times, and the estimator's bias (Bias) and mean square error (MSE) have all been obtained. The propensity of a statistic to overstate or underestimate a parameter is known as bias. Suppose is the parameter estimator. The statistic has a bias, which is bias. $(bais(\lambda) = E\left(\widehat{\lambda}\right)-\lambda)$ where $(E\left(\widehat{\lambda}\right))$ is the statistics' anticipated value. If bais $(\widehat{\lambda})=0$, then $(E\left(\widehat{\lambda}\right)=\lambda)$.

So, is an impartial estimate of the real parameter The average squared difference between the estimated and actual values is what is referred to as the mean squared error (MSE).

Table 2: The simulation results for the α-EED

n		$\alpha=2$	$\lambda=2$	$\alpha=3$	$\lambda=3$	$\alpha=4$	$\lambda=4$
50	$\hat{\alpha},\hat{\lambda}$	2.4458	2.1361	3.8753	3.6755	4.3288	4.3935
	Bias	0.5171	0.3841	0.5313	0.4998	0.6071	0.6161
	MSE	0.4289	0.2551	0.4606	0.4258	0.6259	0.6547
100	$\hat{\alpha},\hat{\lambda}$	1.7245	1.8177	2.3878	2.5837	3.4998	3.8397
	Bias	0.3809	0.2844	0.3541	0.3394	0.4081	0.4054
	MSE	0.2288	0.1308	0.2069	0.1890	0.2784	0.2719
150	$\hat{\alpha},\hat{\lambda}$	1.7603	1.6908	3.5197	3.6023	3.8694	3.7177
	Bias	0.2949	0.2228	0.2964	0.2663	0.3355	0.3469
	MSE	0.1353	0.0800	0.1387	0.1161	0.1752	0.1906
200	$\hat{\alpha},\hat{\lambda}$	2.2668	2.0345	2.9474	2.8575	4.4570	4.3835
	Bias	0.2562	0.1938	0.2522	0.2321	0.2837	0.2909
	MSE	0.1080	0.0614	0.1026	0.0890	0.1313	0.1375
500	$\hat{\alpha},\hat{\lambda}$	1.9719	1.9926	3.0653	3.0414	4.5768	4.5241
	Bias	0.1591	0.1175	0.1591	0.1507	0.1810	0.1808
	MSE	0.0397	0.0215	0.0392	0.0351	0.0504	0.0509

$$MSE = \frac{\sum_{i=1}^{n}\left(\lambda - \widehat{\lambda}_i\right)^2}{n}$$

How closely a regression line resembles a particular set of points is shown by the MSE value.

In Table 2, the simulation results for α-EED are shown for various samples and parameter settings. the findings demonstrated that as sample size increased, the values of bias and MSE are decreasing as the sample size increased. This demonstrates that the estimations of the two parameters are consistent.

10 Application on financial data

A Treasury bill (T-Bills) defined as a short-term U.S. government debt commitment financed by the Department of Treasury with one year or less. TBs are typically vended in values of one thousands of dollars. However, some can extent an extreme value of five millions of dollars in non-competitive bids. These kinds of securities represent a low-risk and safe investments. Its value created on the average of all the competitive bids established. The data is 60 monthly stock returns, with dividends, for 45 randomly selected companies for the period January 1988 through December 1992 (McDonald, 1996). The data as follows 1.00294, 1.00456, 1.00441, 1.00462, 1.00505, 1.00485, 1.00507, 1.00594, 1.00617, 1.0061, 1.00566, 1.00634, 1.00551, 1.00613, 1.00671, 1.00675, 1.00787, 1.00709, 1.00696, 1.00739, 1.00655,1.00677, 1.00687, 1.00607, 1, 1, 1, 1, 1.00677, 1.00625, 1.00677, 1.00657, 0.005984, 1.00682, 1.00565, 1.00599, 1.00518, 1.00477, 1.00439, 1.00534, 1.00472, 1.00417, 1.00488, 1.00461, 1.00456, 1.00425, 1.00392, 1.00379, 1.00339, 1.00283, 1.00338, 1.00325, 1.00276, 1.0032, 1.00308, 1.00261, 1.00257, 0.002286, 1.00235, 1.00282.

We present the fitting of the new generalized Exponential Distribution in Table together with other distributions. In order to compare distributions, - 2lnL, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected) and BIC (Bayesian Information Criterion).

Table 3: Fitting Treasury bill data using new generalized Exponential Distribution and other distributions

Distributions	parameters	estimate	-2ll	AIC	AICc	BIC
Extended Exponential	$\hat{\alpha}$	8.19184	31.444	35.444	35.61944	35.0003
	$\hat{\lambda}$	8.31989				
Exponential	$\hat{\lambda}$	1.02947	116.514	118.514	118.6519	118.2922
Gamma	$\hat{\alpha}$	3.42439	80.3728	84.3728	84.54824	83.9291
	$\hat{\beta}$	0.283662				
Lindely	$\hat{\lambda}$	1.44971	106.679	108.679	108.8169	108.4572

Table 4: Fitting A financial consultant at a brokerage firm records the portfolio values for 20 clients using new Extended Exponential Distribution and other distributions

Distributions	parameters	estimate	-2ll	AIC	AICc	BIC
Extended Exponential	$\hat{\alpha}$	5.42667	300.886	304.886	305.592	303.488
	$\hat{\lambda}$	0.005345				
Exponential	$\hat{\lambda}$	0.0009853	316.904	318.904	319.126	318.205
Gamma	$\hat{\alpha}$	4.32497	301.108	305.108	305.814	303.71
	$\hat{\beta}$	234.672				
Lindely	$\hat{\lambda}$	0.00196861	306.281	308.281	306.5032	307.582

It can be simply verified from Table 3 that the α-EED gives better fit than the Exponential, Gamma and Lindley distributions for modeling real financial data-sets, α-EED should be favored to the Exponential, Gamma and Lindley distributions A financial consultant at a brokerage firm records the portfolio values for 20 clients, as shown in Table 4 [18], where the portfolio values are shown in thousands of dollars. 278, 318, 422, 577, 618, 735, 798, 864, 903, 944, 1052, 1099, 1132, 1180, 1279, 1365, 1471, 1572, 1787, 1905.

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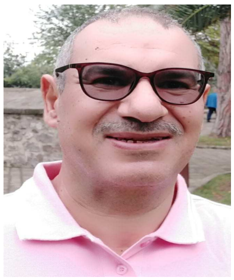

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