Journal of Statistics Applications & Probability Letters
An International Journal

http://dx.doi.org/10.12785/jsapl/010101

Relations for Moment Generating Functions of Lower Generalized Order Statistics from Doubly Truncated Continuous Distributions and Characterizations

R. U. Khan[1,*] *and Devendra Kumar*[2]

[1] Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh-202 002, India
[2] Department of Statistics, Amity Institute of Applied Sciences, Amity University, Noida-201301, India

Received: 17 May. 2013, Revised: 23 Sep. 2013, Accepted: 26 Sep. 2013
Published online: 1 Jan. 2014

Abstract: In this paper, we derive recurrence relations for moment generating function of lower generalized order statistics within a class of doubly truncated distributions. Doubly truncated inverse Weibull, exponentiated Weibull, power function, exponentiated Pareto, exponentiated gamma, generalized exponential, exponentiated log-logistic, generalized inverse Weibull, extended type I generalized logistic, logistic and Gumble distributions are given as illustrative examples. Further, recurrence relations for moment generating function of order statistics and lower record values are obtained as special cases of the lower generalized order statistics, also two theorems for characterizing the general form of distribution based on moment generating function of lower generalized order statistics are given.

Keywords: Lower generalized order statistics, order statistics, lower record values, moment generating function, recurrence relations and characterization

1 Introduction

Kamps [1] introduced the concept of generalized order statistics (gos). It is known that ordinary order statistics, upper record values and sequential order statistics are special cases of gos . In this paper we will consider the lower generalized order statistics $(lgos)$. It can be shown that order statistics, lower record values are special cases of $lgos$. A statistic $X^*(r,n,m,k)$ is said to be the $r-$th $lgos$ based on a random sample of size n drawn from a population whose distribution function (df) is $F(x)$ and probability density function (pdf) is $f(x)$, if its pdf is given by

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)), \tag{1}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i\ , \quad r = 1,2,\ldots,n-1, \quad \gamma_r = k+(n-r)(m+1), \quad k \geq 1, \quad m \geq -1,$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}\, x^{m+1}, & m \neq -1 \\ -\ln x, & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1).$$

We shall also take $X^*(0,n,m,k) = 0$. If $m = 0$, $k = 1$, then $X^*(r,n,m,k)$ reduces to the $(n-r+1)-$th order statistic, $X_{n-r+1:n}$ from the sample $X_1, X_2, \ldots, X_n$ and when $m = -1$, then $X^*(r,n,m,k)$ reduces to the $r-$ th lower k record value

* Corresponding author e-mail: aruke@rediffmail.com

© 2014 NSP
Natural Sciences Publishing Cor.

(Pawlas and Szynal [2]). The work of Burkschat *et al*. [3] may also refer for *lgos*.

Recurrence relations for marginal and joint moment generating functions of *gos* from power function distribution are derived by Saran and Singh [4]. Al-Hussaini *et al*. [5], [6] have established recurrence relations for moments and conditional moment generating functions of *gos* and joint moment generating functions of *gos* based on mixed population, respectively. Khan *et al*. [7] have established recurrence relations for moment generating function of *gos* from Gompertz distribution among others.
Characterizations based on *gos* have been studied by some authors. Keseling [8] characterized some continuous distributions based on conditional distributions of *gos* . Bieniek and Szynal [9] characterized some distributions via linearity of regression of *gos*. Cramer *et al*. [10] gave a unifying approach on characterization via linear regression of ordered random variables. Khan *et al*. [11] characterized some continuous distributions through conditional expectation of functions of *gos*.
Kamps [12] investigated the importance of recurrence relations of order statistics in characterization.
In the present study, we have obtained some recurrence relations for marginal moment generating functions of *lgos* from doubly truncated a general form of distribution and its various deductions and particular cases are discussed. Further two theorems for characterizing this distribution are stated and proved.
Now if for given P_1 and Q_1

$$\int_{-\infty}^{Q_1} f_1(x)dx = Q \quad \text{and} \quad \int_{-\infty}^{P_1} f_1(x)dx = P, \tag{2}$$

where $f_1(x)$ is the pdf of X, then the truncated pdf is given by

$$f(x) = \frac{f_1(x)}{P-Q}, \;\; x \in (Q_1, P_1)$$

with the corresponding df

$$F(x) = \frac{1}{P-Q}[F_1(x) - Q], \;\; x \in (Q_1, P_1).$$

Suppose the distribution function $F_1(x)$ is of the following general form

$$F_1(x) = e^{-ah(x)}, \qquad \alpha \leq x \leq \beta, \tag{3}$$

where $a > 0$ is a constant and $h(x)$ is continuous, monotonic and differentiable function of x in the interval $[\alpha,\ \beta]$.
Then truncated $pdf\ f(x)$ is given by

$$f(x) = \frac{ah'(x)}{P-Q} e^{-ah(x)}, \;\; x \in (Q_1, P_1) \tag{4}$$

and the corresponding truncated $df\ F(x)$ by

$$F(x) = -Q_2 - \frac{f(x)}{ah'(x)}, \;\; x \in (Q_1, P_1), \tag{5}$$

where

$$Q_2 = \frac{Q}{P-Q}.$$

2 Relations for marginal moment generating function

Let us denote the marginal moment generating function of $j-$th power of the $r-$th *lgos*, $X^*(r,n,m,k)$ by $M^{(j)}_{X^*(r,n,m,k)}(t)$.
Theorem 2.1. For the distribution given in (5), $n \in N,\ \ 2 \leq r \leq n,\ \ k \geq 1,\ \ k+m > 0$ and $\ m > -1$

$$M^{(j)}_{X^*(r,n,m,k)}(t) = M^{(j)}_{X^*(r-1,n,m,k)}(t) + \frac{jt}{a\gamma_r} E[\Psi(X^*(r,n,m,k))]$$
$$-Q_2 K \left\{ M^{(j)}_{X^*(r,n-1,m,k+m)}(t) - M^{(j)}_{X^*(r-1,n-1,m,k+m)}(t) \right\} \tag{6}$$

© 2014 NSP
Natural Sciences Publishing Cor.

and for $m=-1$

$$M^{(j)}_{Z_r^{(k)}}(t)=M^{(j)}_{Z_{r-1}^{(k)}}(t)+\frac{jt}{ak}E\left[\Psi(Z_r^{(k)})\left(1-e^{a[h(Z_r^{(k)})-h(Q_1)]}\right)\right], \tag{7}$$

where

$$\Psi(x)=\frac{x^{j-1}e^{tx^j}}{h'(x)},\ K=\frac{C_{r-2}}{C_{r-2}^{(n-1,k+m)}}=\prod_{i=1}^{r-1}\left(\frac{\gamma_i}{\gamma_i-1}\right),\ C_{r-1}^{(n-1,k+m)}=\prod_{i=1}^{r}\gamma_i^{(n-1,k+m)},$$

$$\gamma_r^{(n-1,k+m)}=k+m+(n-1-r)(m+1).$$

Proof. From (1), we have

$$M^{(j)}_{X^*(r,n,m,k)}(t)=E\left[e^{tX^{*j}(r,n,m,k)}\right]$$

$$=\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1}e^{tx^j}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \tag{8}$$

Integrating (8) by parts treating $[F(x)]^{\gamma_r-1}f(x)$ for integration and the rest of the integrand for differentiation, we get

$$M^{(j)}_{X^*(r,n,m,k)}(t)=\frac{(r-1)C_{r-2}}{(r-1)!}\int_{Q_1}^{P_1}e^{tx^j}[F(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx$$

$$-\frac{jtC_{r-2}}{(r-1)!}\int_{Q_1}^{P_1}x^{j-1}e^{tx^j}[F(x)]^{\gamma_r}g_m^{r-1}(F(x))dx$$

$$=M^{(j)}_{X^*(r-1,n,m,k)}(t)-\frac{jtC_{r-2}}{(r-1)!}\int_{Q_1}^{P_1}x^{j-1}e^{tx^j}[F(x)]^{\gamma_r}g_m^{r-1}(F(x))dx, \tag{9}$$

the constant of integration vanishes since the integral considered in (8) is a definite integral. On using (5), we obtain when $m>-1$ that

$$M^{(j)}_{X^*(r,n,m,k)}(t)=M^{(j)}_{X^*(r-1,n,m,k)}(t)$$

$$+\frac{jtC_{r-2}}{a(r-1)!}\int_{Q_1}^{P_1}\frac{x^{j-1}e^{tx^j}}{h'(x)}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

$$+\frac{Q_2\,jtC_{r-2}}{(r-1)!}\int_{Q_1}^{P_1}x^{j-1}e^{tx^j}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))dx$$

$$=M^{(j)}_{X^*(r-1,n,m,k)}(t)+\frac{jt}{a\gamma_r}E[\Psi(X^*(r,n,m,k))]$$

$$+\frac{Q_2K\,jtC_{r-2}^{(n-1,k+m)}}{(r-1)!}\int_{Q_1}^{P_1}x^{j-1}e^{tx^j}[F(x)]^{\gamma_r^{(n-1,k+m)}}g_m^{r-1}(F(x))dx$$

as $\gamma_r-1=\gamma_r^{(n-1,k+m)},\ C_{r-1}=\gamma_rC_{r-2}$.
On using relation in (9), the result (6) can be established.
When $m=-1$, then $X^*(r,n,-1,k)=Z_r^{(k)}$, we have from (9)

$$M^{(j)}_{Z_r^{(k)}}(t)=M^{(j)}_{Z_{r-1}^{(k)}}(t)-\frac{jtk^{r-1}}{(r-1)!}\int_{Q_1}^{P_1}x^{j-1}e^{tx^j}[F(x)]^kg_{-1}^{r-1}(F(x))dx \tag{10}$$

$$=M^{(j)}_{Z_{r-1}^{(k)}}(t)+\frac{jtk^{r-1}}{a(r-1)!}\int_{Q_1}^{P_1}\frac{x^{j-1}e^{tx^j}}{h'(x)}[F(x)]^{k-1}f(x)g_{-1}^{r-1}(F(x))dx$$

$$+\frac{Q_2\,jtk^{r-1}}{(r-1)!}\int_{Q_1}^{P_1}x^{j-1}e^{tx^j}[F(x)]^{k-1}g_{-1}^{r-1}(F(x))dx$$

© 2014 NSP
Natural Sciences Publishing Cor.

upon using the relation in (5). Now substituting for $f(x)$ from (4), we find that

$$M^{(j)}_{Z^{(k)}_r}(t) = M^{(j)}_{Z^{(k)}_{r-1}}(t) + \frac{jt}{ak}E[\Psi(Z^{(k)}_r)] + \frac{Q_2 jtk^{r-1}}{(r-1)!}$$

$$\times \int_{Q_1}^{P_1} x^{j-1}e^{tx^j}[F(x)]^{k-1}\Big\{-\frac{(P-Q)e^{ah(x)}}{ah'(x)}f(x)\Big\}g^{r-1}_{-1}(F(x))dx$$

$$= M^{(j)}_{Z^{(k)}_{r-1}}(t) + \frac{jt}{ak}E[\Psi(Z^{(k)}_r)] - \frac{Qjtk^{r-1}}{a(r-1)!}$$

$$\times \int_{Q_1}^{P_1} \frac{x^{j-1}e^{tx^j+ah(x)}}{h'(x)}[F(x)]^{k-1}f(x)g^{r-1}_{-1}(F(x))dx.$$

Making use of (2), we get

$$M^{(j)}_{Z^{(k)}_r}(t) = M^{(j)}_{Z^{(k)}_{r-1}}(t) + \frac{jt}{ak}E[\Psi(Z^{(k)}_r)] - \frac{jt}{ak}e^{-ah(Q_1)}E[\Psi(Z^{(k)}_r)e^{ah(Z^{(k)}_r)}].$$

The relation in (7) is derived simply by rewriting the above equation.
By differentiating both sides of equations (6) and (7) with respect to t and then setting $t = 0$, we obtain the recurrence relation for moments of *lgos* when $m > -1$

$$E[X^{*j}(r,n,m,k)] = E[X^{*j}(r-1,n,m,k)] + \frac{j}{a\gamma_r}E[\phi(X^*(r,n,m,k))]$$

$$-Q_2K\Big\{E[X^{*j}(r,n-1,m,k+m)] - E[X^{*j}(r-1,n-1,m,k+m)]\Big\} \tag{11}$$

and when $m = -1$

$$E[(Z^{(k)}_r)^j] = E[(Z^{(k)}_{r-1})^j] + \frac{j}{ak}E\Big[\phi(Z^{(k)}_r)\Big(1-e^{a[h(Z^{(k)}_r)-h(Q_1)]}\Big)\Big], \tag{12}$$

where

$$\phi(x) = \frac{x^{j-1}}{h'(x)}.$$

Special cases

i) Putting $m = 0,\ k = 1$ in (6) and (11), we can get the relations for marginal moment generating function and moments of order statistics as

$$M^{(j)}_{X_{n-r+1:n}}(t) = M^{(j)}_{X_{n-r+2:n}}(t) + \frac{jt}{a(n-r+1)}E[\Psi(X_{n-r+1:n})]$$

$$-\frac{nQ_2}{(n-r+1)}\Big\{M^{(j)}_{X_{n-r:n-1}}(t) - M^{(j)}_{X_{n-r+1:n-1}}(t)\Big\}, \tag{13}$$

$$E[X^j_{n-r+1:n}] = E[X^j_{n-r+2:n}] + \frac{j}{a(n-r+1)}E[\phi(X_{n-r+1:n})]$$

$$-\frac{nQ_2}{(n-r+1)}\Big\{E[X^j_{n-r:n-1}] - E[X^j_{n-r+1:n-1}]\Big\}. \tag{14}$$

That is

$$M^{(j)}_{X_{r:n}}(t) = M^{(j)}_{X_{r-1:n}}(t) - \frac{jt}{a(r-1)}E[\Psi(X_{r-1:n})] + \frac{nQ_2}{(r-1)}\Big\{M^{(j)}_{X_{r-2:n-1}}(t) - M^{(j)}_{X_{r-1:n-1}}(t)\Big\},$$

$$E[X^j_{r:n}] = E[X^j_{r-1:n}] - \frac{j}{a(r-1)}E[\phi(X_{r-1:n})] + \frac{nQ_2}{(r-1)}\Big\{E[X^j_{r-2:n-1}] - E[X^j_{r-1:n-1}]\Big\}.$$

ii) Setting $k = 1$ in (7) and (12), relations for lower records can be obtained as

$$M^{(j)}_{X_{L(r)}}(t) = M^{(j)}_{X_{L(r-1)}}(t) + \frac{jt}{a}E\Big[\Psi(X_{L(r)})\Big(1-e^{a[h(X_{L(r)})-h(Q_1)]}\Big)\Big], \tag{15}$$

© 2014 NSP
Natural Sciences Publishing Cor.

$$E[X_{L(r)}^{(j)}] = E[X_{L(r-1)}^{(j)}] + \frac{j}{a} E\left[\phi(X_{L(r)})\left(1 - e^{a[h(X_{L(r)})-h(Q_1)]}\right)\right]. \tag{16}$$

Remark 2.1. At $Q = 0$ and $P = 1$, (non-truncated case) relations (6), (7), (11), and (12) reduce, respectively, to

$$M_{X^*(r,n,m,k)}^{(j)}(t) = M_{X^*(r-1,n,m,k)}^{(j)}(t) + \frac{jt}{a\gamma_r} E[\Psi(X^*(r,n,m,k))],$$

$$M_{Z_r^{(k)}}^{(j)}(t) = M_{Z_{r-1}^{(k)}}^{(j)}(t) + \frac{jt}{ak} E[\Psi(Z_r^{(k)})],$$

$$E[X^{*j}(r,n,m,k)] = E[X^{*j}(r-1,n,m,k)] + \frac{j}{a\gamma_r} E[\phi(X^*(r,n,m,k))],$$

$$E[(Z_r^{(k)})^j] = E[(Z_{r-1}^{(k)})^j] + \frac{j}{ak} E[\phi(Z_r^{(k)})],$$

the order statistics and lower record values cases are given from (13), (14), (15) and (16), as

$$M_{X_{n-r+1:n}}^{(j)}(t) = M_{X_{n-r+2:n}}^{(j)}(t) + \frac{jt}{a(n-r+1)} E[\Psi(X_{n-r+1:n})],$$

$$E[X_{n-r+1:n}^j] = E[X_{n-r+2:n}^j] + \frac{j}{a(n-r+1)} E[\phi(X_{n-r+1:n})].$$

That is

$$M_{X_{r:n}}^{(j)}(t) = M_{X_{r-1:n}}^{(j)}(t) - \frac{jt}{a(r-1)} E[\Psi(X_{r-1:n})],$$

$$E[X_{r:n}^j] = E[X_{r-1:n}^j] - \frac{j}{a(r-1)} E[\phi(X_{r-1:n})]$$

and

$$M_{X_{L(r)}}^{(j)}(t) = M_{X_{L(r-1)}}^{(j)}(t) + \frac{jt}{a} E[\Psi(X_{L(r)})],$$

$$E[X_{L(r)}^{(j)}] = E[X_{L(r-1)}^{(j)}] + \frac{j}{a} E[\phi(X_{L(r)})].$$

© 2014 NSP
Natural Sciences Publishing Cor.

Table 2.1. Examples Based on Theorem 2.1

Distribution	$F(x)$	a	$h(x)$
Inverse Weibull	$e^{-(\theta/x)^p}$ $0<x<\infty$	θ^p	x^{-p}
Exponentiated Weibull	$[1-e^{-(\lambda x)^p}]^\tau$ $0<x<\infty$	τ	$-ln[1-e^{-(\lambda x)^p}]$
Power function	$(x/\lambda)^p$ $0<x<\lambda$	1	$-ln(x/\lambda)^p$
Exponentiated Pareto	$[1-(1+x)^{-\lambda}]^\theta$ $0<x<\infty$	θ	$-ln[1-(1+x)^{-\lambda}]$
Exponentiated gamma	$[1-e^{-x}(x+1)]^\theta$ $0<x<\infty$	θ	$-ln[1-e^{-x}(x+1)]$
Generalized exponential	$[1-e^{-\lambda x}]^\theta$ $0<x<\infty$	θ	$-ln[1-e^{-\lambda x}]$
Exponentiated log-logistic	$\left[\frac{(x/\sigma)^\beta}{1+(x/\sigma)^\beta}\right]^\theta$ $0<x<\infty$	θ	$-ln\left[\frac{(x/\sigma)^\beta}{1+(x/\sigma)^\beta}\right]$
Generalized inverse Weibull	$e^{-\theta(\alpha/x)^\beta}$ $0<x<\infty$	θ	$(\alpha/x)^\beta$
Extended type I generalized logistic	$\left(\frac{\lambda}{\lambda+e^{-x}}\right)^p$ $-\infty<x<\infty$	p	$-ln\left(\frac{\lambda}{\lambda+e^{-x}}\right)$
Logistic	$[1+e^{-x}]^{-1}$ $-\infty<x<\infty$	1	$ln(1+e^{-x})$
Gumbel	$e^{-e^{-x}}$ $-\infty<x<\infty$	1	e^{-x}

Similarly several recurrence relations based on Theorem 2.1 can be established with proper choice of a and $h(x)$.

3 Characterization

Theorem 3.1. Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, $m>-1$ then

$$M^{(j)}_{X^*(r,n,m,k)}(t) = M^{(j)}_{X^*(r-1,n,m,k)}(t) + \frac{jt}{a\gamma_r} E[\Psi(X^*(r,n,m,k))]$$

$$-Q_2 K\left\{M^{(j)}_{X^*(r,n-1,m,k+m)}(t) - M^{(j)}_{X^*(r-1,n-1,m,k+m)}(t)\right\} \tag{17}$$

if and only if

$$F(x) = -Q_2 - \frac{f(x)}{ah'(x)}, \quad Q_1 \le x \le P_1,$$

where $\Psi(x) = \frac{x^{j-1}e^{tx^j}}{h'(x)}$.

Proof. The necessary part follows immediately from equation (6). On the other hand if the recurrence relation in equation (17) is satisfied, then on using equations (8) and (9), we have

$$-\frac{jtC_{r-2}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}e^{tx^j}[F(x)]^{\gamma_r} g_m^{r-1}(F(x))dx$$

$$= \frac{jtC_{r-2}}{a(r-1)!}\int_{Q_1}^{P_1} \frac{x^{j-1}e^{tx^j}}{h'(x)}[F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x))dx$$

© 2014 NSP
Natural Sciences Publishing Cor.

$$+\frac{Q_2 jtC_{r-2}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}e^{tx^j}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))dx$$

which can be written as

$$\int_{Q_1}^{P_1} x^{j-1}e^{tx^j}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))\left\{F(x)+\frac{f(x)}{ah'(x)}+Q_2\right\}dx=0.$$

If $t=1$, the generalization of the Müntz-Szász Theorem (Hwang and Lin, [13]) can be applied to obtain

$$F(x)=-Q_2-\frac{f(x)}{ah'(x)},\ \ Q_1\le x\le P_1.$$

Theorem 3.2. Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, $m=-1$, then

$$M^{(j)}_{Z_r^{(k)}}(t)=M^{(j)}_{Z_{r-1}^{(k)}}(t)+\frac{jt}{ak}E\left[\Psi(Z_r^{(k)})\left(1-e^{a[h(Z_r^{(k)})-h(Q_1)]}\right)\right] \tag{18}$$

if and only if

$$F(x)=-Q_2-\frac{f(x)}{ah'(x)},\ \ Q_1\le x\le P_1.$$

Proof. The necessary part follows immediately from equation (7). On the other hand if the recurrence relation in equation (18) is satisfied, then on using equations (10), we have

$$\begin{aligned}-\frac{jtk^{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}e^{tx^j}[F(x)]^k g_{-1}^{r-1}(F(x))dx\\ =\frac{jtk^{r-1}}{a(r-1)!}\int_{Q_1}^{P_1}\frac{x^{j-1}e^{tx^j}}{h'(x)}[F(x)]^{k-1}f(x)g_{-1}^{r-1}(F(x))dx\\ +\frac{Q_2 jtk^{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}e^{tx^j}[F(x)]^{k-1}g_{-1}^{r-1}(F(x))dx\end{aligned}$$

which can be written as

$$\int_{Q_1}^{P_1} x^{j-1}e^{tx^j}[F(x)]^{k-1}g_{-1}^{r-1}(F(x))\left\{F(x)+\frac{f(x)}{ah'(x)}+Q_2\right\}dx=0. \tag{19}$$

Now, applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, [13]) to equation (19) we get

$$F(x)=-Q_2-\frac{f(x)}{ah'(x)},\ \ Q_1\le x\le P_1.$$

Acknowledgements

The authors acknowledge with thanks to Professor Mahmoud Abdel-Aty, Sohag University, Egypt and University of Bahrain, Kingdom of Bahrain and the learned referee for his valuable time to evaluate this manuscript.

References

[1] U. Kamps, A Concept of Generalized Order Statistics. B.G. Teubner Stuttgart, Germany, (1995).
[2] P. Pawlas and D. Szynal, Recurrence relations for single and product moments of lower generalized order statistics from the inverse Weibull distribution. Demonstratio Math., **XXXIV**, 353-358 (2001).
[3] M. Burkschat, E. Cramer and U. Kamps, Dual generalized order statistics. Metron, **LXI**, 13-26 (2003).
[4] J. Saran and A. Pandey, Recurrence relations for marginal and joint moment generating functions of generalized order statistics from power function distribution. Metron, **LXI**, 27-33 (2003).

© 2014 NSP
Natural Sciences Publishing Cor.

[5] E.K. Al-Hussaini, A.A. Ahmad and M.A. Al-Kashif, Recurrence relations for moment and conditional moment generating functions of generalized order statistics. Metrika, **61**, 199-220 (2005).
[6] E.K. Al-Hussaini, A.A. Ahmad and M.A. Al-Kashif, Recurrence relations for joint moment generating functions of generalized order statistics based on mixed population. J. Statist. Theory Appl., **6**, 134-155 (2007).
[7] R.U. Khan, Benazir, Zia and H. Athar, On moment generating function of generalized order statistics from Gompertz distribution and its characterization. J. Statist. Theory Appl., **9**, 363-373 (2010).
[8] C. Keseling, Conditional distributions of generalized order statistics and some characterizations. Metrika, **49**, 27-40 (1999).
[9] M. Bieniek and D. Szynal, Characterizations of distributions via linearity of regression of generalized order statistics. Metrika, **58**, 259-271 (2003).
[10] E. Cramer, U. Kamps and C. Keseling, Characterization via linear regression of ordered random variables: a unifying approach. Comm. Statist. Theory Methods, **33**, 2885-2911 (2004).
[11] A.H. Khan, R.U. Khan and M. Yaqub, Characterization of continuous distributions through conditional expectation of functions of generalized order statistics. J. Appl. Probab. Statist., **1**, 115-131 (2006).
[12] U. Kamps, Characterizations of distributions by recurrence relations and identities for moments of order statistics. In: N. Balakrishnan and C.R. Rao, eds., Hand book of statistics, 16, Order Statistics: Theory & Methods, North Holand, Amsterdam. 291-311(1998).
[13] J. S. Hwang and G.D. Lin, On a generalized moments problem II. Proc. Amer. Math. Soc., **91**, 577-580 (1984).

R. U. Khan is an Associate Professor in the Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh, India. He awarded his Ph.D. degree in Statistics in 1984. His area of research is order statistics, records and generalized order statistics. He has got published several research papers in the journal of international repute. He is also member of refereeing board of various journals.

Devendra Kumar received M. Phil. and Ph. D. in Statistics from Aligarh Muslim University, Aligarh, India. Presently he is an Assistant Professor in the Department of Statistics, Amity Institute of Applied Sciences, Amity University, Noida, India. He has nearby more than fifty research papers in International journals. His research interests include Generalized Order Statistics, Order Statistics and Record values.

© 2014 NSP
Natural Sciences Publishing Cor.