Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/180620

Investigation of the Solution of a Smoking Model with Conformable Derivative

Shatha Hasan[1,2,*], *Mohammad Al Zurayqat*[1], *Mohammad Dradka*[1] *and Shaher Momani*[3,4]

[1]Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan
[2]Jadara Research Center, Jadara University, Irbid 21110, Jordan
[3]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE
[4]Department of Mathematics, Faculty of Science, The University of Jordan, Amman, Jordan

Received: 12 Sep. 2024, Revised: 2 Oct. 2024, Accepted: 3 Oct. 2024
Published online: 1 Nov. 2024

Abstract: This paper is designed to generalize one of the important biomathematical models that studies and analyses smoking spread in a given population within the framework of conformable derivative. The use of conformable operator instead of the classical integer order generalizes the epidemic model and gives us the opportunity of obtaining variety of results and expectations. Due to the difficulty of solving non-linear fractional systems, we use two analytical-approximate techniques to solve this model. The first technique is the residual power series method, which depends on minimizing the residual error of the equation under study. The second method is the Laplace decomposition method, which forms an efficient combination between Laplace transform and the well-known Adomian decomposition method. Both methods effectively solve different types of problems, including ordinary and partial differential equations of classical integer or fractional orders, with high accuracy and without the need for linearization or discretization.
In this paper, we apply both techniques to the conformable smoking model (CSM). Some numerical and graphical results for different values of the conformable derivative allow us to notice the effect of this conformable operator to the solution curves of the CSM. Moreover, some numerical results for the residual error are tabulated to assess the accuracy of our outcomes. The results show the simplicity, efficiency, rapid convergence and accuracy of both analytical methods. In fact, for the same number of iterations, both methods produce identical results.

Keywords: Adomian polynomials, conformable derivative, epidemic model, fractional power series, Laplace transform, residual power series

1 Introduction

It is well-known that smoking can be considered as one of the main causes of various preventable death in most countries around the world because its harmful effect on different body organs, resulting in many respiratory problems such as strokes, heart diseases and lung cancers. Recent figures from the World Health Organization (WHO) estimate that among the world's 1.2 billion smokers, more than 8 million people die prematurely yearly from tobacco use. This is the latest available WHO estimate as of November 2023 [1]. For this, this issue must be controlled globally. Consequently, many experts in different fields of science play great roles to help in controlling this dangerous problem.
Mathematics can also play a crucial role in this effort by providing different mathematical epidemic models that analyses the impact of smoking on people. For example, a generalized epidemiological model was derived as a description of the dynamics of drug use among teenagers, particularly tobacco use. Specific models are derived by taking into account other factors such as peer pressure, relapse, counseling and treatment [2]. In [3], a study that showed mathematically that interventions concerning with stop smoking tobacco can result in a reduction in tuberculosis infections. A nonlinear mathematical models to study the effect of media on smoking cessation was carried out in 2015 [4]. A four compartment model was formulated and analyzed to study the dynamics of smoking and its impact on society in [5], while in [6], the authors derived a smoking model which can be considered as a generalization to the giving up smoking model, and taking into account hospitalized smokers and smoke quitters. The dynamics of smoking habit under the

* Corresponding author e-mail: shatha@bau.edu.jo

impact of educational and media programs is formulated in [7] and a mathematical model with the assumption that the increase in tax reduces the incidence rate and increases the quitting rate is presented in [8]. Most recently, a five-dimensional smoking biomathematical model that accounts for distinct degrees of smoking under deterministic and stochastic constraints was presented in 2024 [9] and a smoking model that splits population into six groups, including: non-users, experimental, recreational, addicted, hospitalized, and prisoners' class was established and analyzed in [10].
In this paper, we are interested with the model which was suggested in [11] as a non-linear model that studies and analyzes smoking spread in a population. It assumes that the total population $\Phi(t)$ can be split to five categories: potential smokers $P(t)$, occasional smokers $O(t)$, smokers $S(t)$, temporary quitters $Q(t)$, and permanent quitters $L(t)$. That is, $\Phi(t) = P(t) + O(t) + S(t) + Q(t) + L(t)$. The dynamics of smoking in this model was formulated by the following system of ordinary differential equations (ODEs):

$$\begin{aligned}
P'(t) &= \vartheta - \delta PS - \kappa P \\
O'(t) &= \delta PS - \sigma O - \kappa O \\
S'(t) &= \sigma O + \alpha SQ - \kappa S - \zeta S \\
Q'(t) &= -\alpha SQ - \kappa Q + \zeta(1-\varsigma)S \\
L'(t) &= \varsigma\zeta S - \kappa L,
\end{aligned} \tag{1}$$

where ϑ indicates the recruitment rate in $P(t)$, δ represents the effective rate of contact between $P(t)$ and $S(t)$, while the natural rate of death is given by the parameter κ. The rate of change from occasional to regular smokers is represented by σ, and the contact rate between smokers and temporary smoking departures is α. Moreover, the rate of quitting smoking is symbolized by ζ and the fraction of smoking who permanently quit smoking is indicated by ς.
The smoking model in (1) was studied by many experts. For example, in [11], the stability of the system was investigated through theoretical and numerical discussion. In [12], [13] and [14], the classical integer order was generalized to Caputo fractional derivative and analytical approximate methodologies were carried out to discuss the solvability of the new fractional smoking model and how Caputo derivative affects this epidemic system. Its approximate solution was obtained via fractional differential transformation method in [12], by using the Laplace Adomian method in [13], and via the q-homotopy analysis transform method [14]. In fact, many biomathematical models ware generalized to fractional case due to the great success of fractional calculus in studying different real-life phenomena over the past few decades. Examples of recent applications of fractional operators can be found in describing viscoelastic phenomena in soft solids [15], in studying the kinetics of adsorption and extraction processes [16], in investigating economic growth model that models gross domestic product [17], in modeling electric circuits [18], and in the study of hybrid systems [19].
However, due to the evolution of fractional calculus, many definitions for fractional operators can be found. Some of which are Grünwald-Letnikov, Erdélyi-Kober, Riemann-Liouville, Riesz-Feller, Caputo, Caputo-Fabrizio,and many other formulations [20], [21] and [22]. Although Caputo and Riemann-Liouville operators enjoy some advantages [23], both have singular kernel functions, which diminish their effectiveness in modeling real-world issues. This limitation encouraged mathematicians to explore new fractional derivatives. One of the recent suggestions is a straightforward fractional derivative presented in 2014 and called "conformable fractional derivative" [24]. It is characterized by its resemblance with the classical integer order derivative in most features like product and chain rules which attracted the interest of many researchers who adopted it in their studies of mathematical models. See for examples [25], [26] and [27].
In this paper, we study a generalized form of the smoking model (1) with classical integer order is replaced be the conformable fractional derivative. So our aim is to investigate analytic and approximate solution to the conformable smoking model (CSM):

$$\begin{aligned}
\mathfrak{T}_t^{\gamma} P(t) &= \vartheta - \delta PS - \kappa P \\
\mathfrak{T}_t^{\gamma} O(t) &= \delta PS - \sigma O - \kappa O \\
\mathfrak{T}_t^{\gamma} S(t) &= \sigma O + \alpha SQ - \kappa S - \zeta S \\
\mathfrak{T}_t^{\gamma} Q(t) &= -\alpha SQ - \kappa Q + \zeta(1-\varsigma)S \\
\mathfrak{T}_t^{\gamma} L(t) &= \varsigma\zeta S - \kappa L,
\end{aligned} \tag{2}$$

subject to the initial conditions (ICs):

$$\begin{aligned}
P(0) &= p_0, \\
O(0) &= o_0, \\
S(0) &= s_0, \\
Q(0) &= q_0, \\
L(0) &= l_0.
\end{aligned} \tag{3}$$

In (2), the symbol $\mathfrak{T}_t^{\gamma}$ indicates the conformable derivative of order γ, $0 < \gamma \leq 1$, with respect to the time variable t.
Unfortunately, dealing with fractional differential equations (FDEs), especially the non-linear FDEs, and getting their exact or even analytic solutions is not an easy task. This difficulty led researchers to develop distinct methods to evaluate their approximate or analytic solutions. Among these methods are the Adomian decomposition method [28], the homotopy analysis method [29], variational iteration method [30], finite difference method [31], reproducing kernel Hilbert space method [32] and differential transform method [33].
In this paper we are going to solve this model by two analytical and numerical methods. The first technique is the conformable Laplace decomposition method (CLDM), which is an efficient combination of the

conformable Laplace transform and the Adomian decomposition method. The CLDM has been effectively applied to tackle a variety of nonlinear problems. For instance, it was employed to solve fractional Newell-Whitehead-Segel equation in [34], Fractional Burger's Type Equations in [35], and conformable fractional fokker-planck equation in [36]. The second technique is the conformable residual power series algorithm (CRPSA), which minimizes the residual error of the equation under study. The residual power series method (RPSM) was firstly developed and applied to fuzzy equations by Arqub in 2013 [37]. It have been proved its efficiency, accuracy and simplicity in many consequent researches when dealing with various problems. So, it was rapidly modified to FDEs [38], [39] and [40].

For more recent works that employ Laplace decomposition method and fractional RPSM to solve real-life models, the reader is suggested to see [41], [42], [43] and [44].

This paper is organized as follows: In section 2, we give some basic concepts related to conformable derivative, conformable Laplace operator, and fractional power series. In section 3, a summarized description for the CLDM when employed to solve the CSM. Similarly, in Section 4, the FRPSA is described through an organized algorithm and the algorithm is applied to the CSM. Some numerical results for both methods are given and discussed in Section 5. Finally, it is end by a conclusion.

2 Preliminaries related to conformable derivative and fractional power series

In this section, we present the basic concepts that we need in our paper. We start by the definition of conformable derivative, followed by conformable Laplace transform (CLT) and some properties, then we give the form of fractional power series with some necessary theorems.

Definition 1.[45] *Consider $\Psi : [\mathfrak{a},\infty) \to \Re$, and $\gamma \in (0,1]$. Then the conformable derivative (CD) of Ψ is*

$$\mathfrak{T}_t^{\gamma}\Psi(t) = \begin{cases} \lim_{\varepsilon\to 0} \frac{\Psi(t+\varepsilon(t-\mathfrak{a})^{1-\gamma})-\Psi(t)}{\varepsilon}, & t > \mathfrak{a}, \\ \lim_{t\to\mathfrak{a}^+} (\mathfrak{T}_t^{\gamma}\Psi)(t), & t = \mathfrak{a}. \end{cases} \quad (4)$$

If $\mathfrak{T}_t^{\gamma}\Psi$ exists over $[\mathfrak{a},\infty)$, then Ψ is said to be γ-conformable differentiable on $[\mathfrak{a},\infty.)$

A beautiful property of the γ-conformable differentiable function is its relation with classical integer order derivative which is:

$$(\mathfrak{T}_t^{\gamma}\Psi)(t) = t^{1-\gamma}\Psi'(t). \quad (5)$$

Now, we present the CLT which is a modification of classical well-known Laplace operator to fit the CD.

Definition 2.[45] *Let $\mathfrak{a} \in \Re, 0 < \gamma \le 1$ and $\Psi(t) : [\mathfrak{a},\infty) \to \Re$. Then the CLT of order γ starting from $\mathfrak{a}$ of $\Psi(t)$ is given by*

$$\mathfrak{L}_{\gamma}\{\Psi(t)\} = \Psi_{\gamma}(s) = \int_{\mathfrak{a}}^{\infty} e^{-s\frac{(t-\mathfrak{a})^{\gamma}}{\gamma}}\Psi(t)(t-\mathfrak{a})^{\gamma-1}dt. \quad (6)$$

Theorem 1.[45] *Let $\mathfrak{a} \in \Re, 0 < \gamma \le 1$ and $\Psi(t) : [\mathfrak{a},\infty) \to \Re$ be differentiable. Then*

$$\mathfrak{L}_{\gamma}\{\mathfrak{T}_t^{\gamma}\Psi)(t)\} = s\Psi_{\gamma}(s) - \Psi(\mathfrak{a}). \quad (7)$$

Some properties of CLT including the relationship between the classical Laplace transform and the CLT are listed below [45].

$*\mathfrak{L}_{\gamma}\{\Psi(t)\} = \mathfrak{L}\{\Psi((\gamma t)^{\frac{1}{\gamma}})\}.$
$*\mathfrak{L}_{\gamma}\{1\} = \frac{1}{s}, s > 0,$
$*\mathfrak{L}_{\gamma}\{\frac{t^{\gamma}}{\gamma}\} = \frac{1}{s^2}, s > 0,$
$*\mathfrak{L}_{\gamma}\{t^k\} = \frac{\gamma^{\frac{k}{\gamma}}}{s^{1+\frac{k}{\gamma}}}\Gamma(1+\frac{1}{\gamma}), s > 0,$
$*\mathfrak{L}_{\gamma}\{e^{\frac{t^{\gamma}}{\gamma}}\} = \frac{1}{s-1}, s > 1,$
$*\mathfrak{L}_{\gamma}\{sin(\tau\frac{t^{\gamma}}{\gamma})\} = \frac{\tau}{s^2+\tau^2},$
$*\mathfrak{L}_{\gamma}\{cos(\tau\frac{t^{\gamma}}{\gamma})\} = \frac{s}{s^2+\tau^2},$
$*\mathfrak{L}_{\gamma}\{e^{-k\frac{t^{\gamma}}{\gamma}}\Psi(t)\} = \mathfrak{L}\{e^{-kt}\Psi((\gamma t)^{\frac{1}{\gamma}})\}.$

Now, we give the definition of conformable power series followed by some related theorems.

Definition 3.[46] *A power series which has the form*

$$\sum_{\aleph=0}^{\infty} C_{\aleph}(t-\mathfrak{a})^{\aleph\gamma} = C_0 + C_1(t-\mathfrak{a})^{\gamma} + C_2(t-\mathfrak{a})^{2\gamma} + \cdots,$$

where $0 < \gamma \le 1, t \ge \mathfrak{a}$, is called conformable power series(CPS) expansion about $\mathfrak{a}$ and $C_0, C_1, C_2, \cdots$ are called the coefficients of the series.

Theorem 2.[46] *Suppose that the CPS $\Psi(t) = \sum_{\aleph=0}^{\infty} C_{\aleph} t^{\aleph\gamma}$ has radius of convergence $\wp > 0$ for all $0 \le t < \wp$. Then $\Psi(t)$ is infinitely γ-differentiable over the interval $[0,\wp)$ with $C_{\aleph} = \frac{(^{(\aleph)}\mathfrak{T}_t^{\gamma}\Psi)(0)}{\gamma^{\aleph}\aleph!}$, where $^{(\aleph)}\mathfrak{T}_t^{\gamma}\Psi = \mathfrak{T}_t^{\gamma}(\mathfrak{T}_t^{\gamma}(\mathfrak{T}_t^{\gamma}\cdots\mathfrak{T}_t^{\gamma}(\Psi)))$ ($\aleph$-times).*

3 Conformable Laplace Decomposition Algorithm for Solving CSM

In this section, we present a summarized description for the CLDM through a simple organized algorithm. After that, we employ this algorithm to produce recurrence formulas to solve the CSM in (2) and (3).

Algorithm 1: To get analytic-approximate solution for the system of conformable ordinary differential equations (CODEs) of the form:

$$\begin{aligned} &\mathfrak{T}_t^{\gamma}\Psi_1(t)=\mathfrak{H}_1(t,\Psi_1,\Psi_1,\cdots,\Psi_m), 0<\gamma\leq 1, t\geq 0,\\ &\mathfrak{T}_t^{\gamma}\Psi_1(t)=\mathfrak{H}_2(t,\Psi_1,\Psi_1,\cdots,\Psi_m),\\ &\quad\vdots\\ &\mathfrak{T}_t^{\gamma}\Psi_m(t)=\mathfrak{H}_m(t,\Psi_1,\Psi_1,\cdots,\Psi_m), \end{aligned} \tag{8}$$

subject to

$$\Psi_1(0)=\delta_1,\quad \Psi_2(0)=\delta_2,\cdots\quad \Psi_m(m)=\delta_m, \tag{9}$$

by the CLDM, we do the following steps for each $i=1,2,\cdots,m$:

Step 1: We start by separating each $\mathfrak{H}_i$ into linear and nonlinear functions of $\Psi_i(t)$. That is

$$\begin{aligned} \mathfrak{H}_i(t,\Psi_1,\Psi_2,\ldots,\Psi_m) =&\ell_i(\Psi_1,\Psi_2,\ldots,\Psi_m)+\\ &\mathcal{N}_i(\Psi_1,\Psi_2,\ldots,\Psi_m), \end{aligned} \tag{10}$$

where ℓ_i and $\mathcal{N}_i$ are the linear and nonlinear components of $\mathfrak{H}_i$, respectively.

Step 2: Take the CLT to both sides of each equation in (10). So, we get:

$$\begin{aligned} s\mathfrak{L}_\gamma\{\Psi_i(t)\} =&\delta_i+\mathfrak{L}_\gamma\{\ell_i(\Psi_1,\Psi_2,\ldots,\Psi_m)\}\\ &+\mathfrak{L}_\gamma\{\mathcal{N}_i(\Psi_1,\Psi_2,\ldots,\Psi_m)\}. \end{aligned} \tag{11}$$

Step 3: Make suitable manipulation and take the inverse CLT to both sides of each equation in system (11) to obtain:

$$\begin{aligned} \Psi_i(t) =&\delta_i+\mathfrak{L}_\gamma^{-1}\{\frac{1}{s}\mathfrak{L}_\gamma\{\ell_i(\Psi_1,\Psi_2,\ldots,\Psi_m)\}\}\\ &+\mathfrak{L}_\gamma^{-1}\{\frac{1}{s}\mathfrak{L}_\gamma\{\mathcal{N}_i(\Psi_1,\Psi_2,\ldots,\Psi_m)\}\}. \end{aligned} \tag{12}$$

Step 4: Replace each unknown function $\Psi_i(t)$ in the linear terms of (12) by the infinite series $\sum_{k=0}^{\infty}\psi_{ik}(t)$, and replace each nonlinear term $\mathcal{N}_i$ by the series of Adomian polynomials:

$$\mathcal{N}_i(\Psi_1,\Psi_2,\ldots,\Psi_m)=\sum_{k=0}^{\infty}\mathscr{A}_{ik}.$$

The Adomian polynomials A_{ik} can be calculated by

$$\begin{aligned} &\mathscr{A}_{i,k}\\ &=\frac{1}{k!}\frac{d^k\mathcal{N}_i}{d\eta^k}\left(\sum_{j=0}^{k}\eta^j\psi_{1j},\sum_{j=0}^{k}\eta^j\psi_{2j},\ldots,\sum_{j=0}^{k}\eta^j\psi_{mj}\right)\Bigg|_{\eta=0},\\ &\quad k=0,1,2,\ldots. \end{aligned} \tag{13}$$

Step 5: From the resulting system of equations:

$$\begin{aligned} \sum_{k=0}^{\infty}\psi_{ik}=\delta_i&\\ &+\mathfrak{L}_\gamma^{-1}\{\frac{1}{s}\mathfrak{L}_\gamma\{\ell_i\left(\sum_{k=0}^{\infty}\psi_{1k},\sum_{k=0}^{\infty}\psi_{2k},\ldots,\sum_{k=0}^{\infty}\psi_{mk}\right)\}\}\\ &+\mathfrak{L}_\gamma^{-1}\{\frac{1}{s}\mathfrak{L}_\gamma\{\sum_{k=0}^{\infty}\mathscr{A}_{ik}\}\}, \end{aligned} \tag{14}$$

we deduce the recurrence relations:

$$\begin{aligned} &\psi_{i0}(t)=\delta_i,\\ &\psi_{i(k+1)}(t)=\mathfrak{L}_\gamma^{-1}\{\frac{1}{s}\mathfrak{L}_\gamma\{\psi_{ik}(t)\}\}+\mathfrak{L}_\gamma^{-1}\{\frac{1}{s}\mathfrak{L}_\gamma\{\mathscr{A}_{ik}\}\},\\ &\qquad k=0,1,2,\ldots. \end{aligned} \tag{15}$$

Step 6: Choose a suitable number of iterations to get the approximate solution:

$$\Psi_{iN}(t)=\sum_{k=0}^{N}\psi_{ik}(t).$$

Surely, a larger number of iterations N leads to more accuracy. In fact,

$$\lim_{N\to\infty}\Psi_{iN}(t)=\sum_{k=0}^{\infty}\psi_{ik}(t)=\Psi_i(t). \tag{16}$$

Convergence conditions of this series have been examined by several authors, mainly in [47]. Additional references related to the use of the ADM, combined with the CLT, can be found in [48], [49], and [50].

Now, we implement Algorithm 1 to solve the CSM in (2) with the ICs in (3). We start by separating the linear and non-linear terms of each equation in (2) and then taking the CLT to get the desired results.

$$\begin{aligned} &s\mathfrak{L}_\gamma\{P(t)\}=p_0+\frac{\vartheta}{s}-\kappa\mathfrak{L}_\gamma\{P\}-\delta\mathfrak{L}_\gamma\{P(t)S(t)\},\\ &s\mathfrak{L}_\gamma\{O(t)\}=o_0-(\sigma+\kappa)\mathfrak{L}_\gamma\{O\}+\delta\mathfrak{L}_\gamma\{P(t)S(t)\},\\ &s\mathfrak{L}_\gamma\{S(t)\}=s_0+\mathfrak{L}_\gamma\{\sigma O-(\kappa+\zeta)S\}+\alpha\mathfrak{L}_\gamma\{S(t)Q(t)\},\\ &s\mathfrak{L}_\gamma\{Q(t)\}=q_0-\mathfrak{L}_\gamma\{\kappa Q-\zeta(1-\varsigma)S\}-\alpha\mathfrak{L}_\gamma\{S(t)Q(t)\},\\ &s\mathfrak{L}_\gamma\{L(t)\}=l_0+\mathfrak{L}_\gamma\{\varsigma\zeta S-\kappa L\}. \end{aligned} \tag{17}$$

Rearranging the terms of (17), and taking the inverse CLT, we obtain

$$
\begin{aligned}
P(t) &= p_0 + \frac{\vartheta t^{\gamma}}{\gamma} - \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\kappa}{s}\mathfrak{L}_{\gamma}\{P\}\right\} \\
&\quad - \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\delta}{s}\mathfrak{L}_{\gamma}\{P(t)S(t)\}\right\}, \\
O(t) &= o_0 - \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\sigma+\kappa}{s}\mathfrak{L}_{\gamma}\{O\}\right\} \\
&\quad + \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\delta}{s}\mathfrak{L}_{\gamma}\{P(t)S(t)\}\right\}, \\
S(t) &= s_0 + \mathfrak{L}_{\gamma}^{-1}\left\{\frac{1}{s}\mathfrak{L}_{\gamma}\{\sigma O - (\kappa+\zeta)S\}\right\} \\
&\quad + \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\alpha}{s}\mathfrak{L}_{\gamma}\{S(t)Q(t)\}\right\}, \\
Q(t) &= q_0 - \mathfrak{L}_{\gamma}^{-1}\left\{\frac{1}{s}\mathfrak{L}_{\gamma}\{\kappa Q - \zeta(1-\varsigma)S\}\right\} \\
&\quad - \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\alpha}{s}\mathfrak{L}_{\gamma}\{S(t)Q(t)\}\right\}, \\
L(t) &= l_0 + \mathfrak{L}_{\gamma}^{-1}\left\{\frac{1}{s}\mathfrak{L}_{\gamma}\{\varsigma\zeta S - \kappa L\}\right\}.
\end{aligned} \tag{18}
$$

The CLDM presents the solution of system (2) in the form of infinite series as

$$
\begin{aligned}
P(t) &= \sum_{k=0}^{\infty} p_k(t), \\
O(t) &= \sum_{k=0}^{\infty} o_k(t), \\
S(t) &= \sum_{k=0}^{\infty} s_k(t), \\
Q(t) &= \sum_{k=0}^{\infty} q_k(t), \\
L(t) &= \sum_{k=0}^{\infty} l_k(t),
\end{aligned} \tag{19}
$$

while it expresses the nonlinear terms $P(t)S(t)$ and $S(t)Q(t)$ in terms of Adomian polynomials as follows:

$$
\begin{aligned}
P(t)S(t) &= \sum_{k=0}^{\infty} \mathscr{A}_k(t), \\
\mathscr{A}_k &= \frac{1}{k!}\frac{d^k}{d\eta^k}\left((\sum_{\iota=0}^{k}\eta^{\iota}p_{\iota})(\sum_{\iota=0}^{k}\eta^{\iota}s_{\iota}) \right)\Bigg|_{\eta=0}, \\
S(t)Q(t) &= \sum_{k=0}^{\infty} \mathscr{B}_k(t), \\
\mathscr{B}_k &= \frac{1}{k!}\frac{d^k}{d\eta^k}\left((\sum_{\iota=0}^{k}\eta^{\iota}s_{\iota})(\sum_{\iota=0}^{k}\eta^{\iota}q_{\iota}) \right)\Bigg|_{\eta=0}, \quad k = 0, 1, \cdots.
\end{aligned} \tag{20}
$$

As an illustration, the first 6 Adomian polynomials are:

$$
\begin{aligned}
\mathscr{A}_0 &= p_0 s_0, \\
\mathscr{A}_1 &= p_1 s_0 + p_0 s_1, \\
\mathscr{A}_2 &= p_2 s_0 + p_1 s_1 + p_0 s_2, \\
\mathscr{A}_3 &= p_3 s_0 + p_2 s_1 + p_1 s_2 + p_0 s_3, \\
\mathscr{A}_4 &= p_4 s_0 + p_3 s_1 + p_2 s_2 + p_1 s_3 + p_0 s_4, \\
\mathscr{A}_5 &= p_5 s_0 + p_4 s_1 + p_3 s_2 + p_2 s_3 + p_1 s_4 + p_0 s_5,
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{B}_0 &= q_0 s_0, \\
\mathscr{B}_1 &= q_1 s_0 + q_0 s_1, \\
\mathscr{B}_2 &= q_2 s_0 + q_1 s_1 + q_0 s_2, \\
\mathscr{B}_3 &= q_3 s_0 + q_2 s_1 + q_1 s_2 + q_0 s_3, \\
\mathscr{B}_4 &= q_4 s_0 + q_3 s_1 + q_2 s_2 + q_1 s_3 + q_0 s_4, \\
\mathscr{B}_5 &= q_5 s_0 + q_4 s_1 + q_3 s_2 + q_2 s_3 + q_1 s_4 + q_0 s_5.
\end{aligned}
$$

Substituting (19) and (20) into (18) implies:

$$
\begin{aligned}
\sum_{k=0}^{\infty} p_k &= p_0 + \frac{\vartheta t^{\gamma}}{\gamma} - \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\kappa}{s}\mathfrak{L}_{\gamma}\left\{\sum_{k=0}^{\infty} p_k\right\}\right\} \\
&\quad - \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\delta}{s}\mathfrak{L}_{\gamma}\left\{\sum_{k=0}^{\infty} \mathscr{A}_k\right\}\right\},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{\infty} o_k &= o_0 - \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\sigma+\kappa}{s}\mathfrak{L}_{\gamma}\left\{\sum_{k=0}^{\infty} o_k\right\}\right\} \\
&\quad + \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\delta}{s}\mathfrak{L}_{\gamma}\left\{\sum_{k=0}^{\infty} \mathscr{A}_k\right\}\right\}, \\
\sum_{k=0}^{\infty} s_k &= s_0 + \mathfrak{L}_{\gamma}^{-1}\left\{\frac{1}{s}\mathfrak{L}_{\gamma}\left\{\sigma\sum_{k=0}^{\infty} o_k - (\kappa+\zeta)\sum_{k=0}^{\infty} s_k\right\}\right\} \\
&\quad + \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\alpha}{s}\mathfrak{L}_{\gamma}\left\{\sum_{k=0}^{\infty} \mathscr{B}_k\right\}\right\}, \\
\sum_{k=0}^{\infty} q_k &= q_0 - \mathfrak{L}_{\gamma}^{-1}\left\{\frac{1}{s}\mathfrak{L}_{\gamma}\left\{\kappa\sum_{k=0}^{\infty} q_k - \zeta(1-\varsigma)\sum_{k=0}^{\infty} s_k\right\}\right\} \\
&\quad - \mathfrak{L}_{\gamma}^{-1}\left\{\frac{\alpha}{s}\mathfrak{L}_{\gamma}\left\{\sum_{k=0}^{\infty} \mathscr{B}_k\right\}\right\}, \\
\sum_{k=0}^{\infty} l_k &= l_0 + \mathfrak{L}_{\gamma}^{-1}\left\{\frac{1}{s}\mathfrak{L}_{\gamma}\left\{\varsigma\zeta\sum_{k=0}^{\infty} s_k - \kappa\sum_{k=0}^{\infty} l_k\right\}\right\}.
\end{aligned} \tag{21}
$$

which suggests the recurrence relations:

$$\begin{aligned}
&p_0(t)=p_0+\frac{\vartheta t^{\gamma}}{\gamma},\quad o_0(t)=o_0,\quad s_0(t)=s_0,\\
&q_0(t)=q_0,\quad l_0(t)=l_0,\\
&p_{k+1}=-\mathfrak{L}_{\gamma}^{-1}\left\{\frac{\kappa}{s}\mathfrak{L}_{\gamma}\{p_k\}\right\}-\mathfrak{L}_{\gamma}^{-1}\left\{\frac{\delta}{s}\mathfrak{L}_{\gamma}\{\mathscr{A}_k\}\right\},\\
&o_{k+1}=-\mathfrak{L}_{\gamma}^{-1}\left\{\frac{\sigma+\kappa}{s}\mathfrak{L}_{\gamma}\{o_k\}\right\}+\mathfrak{L}_{\gamma}^{-1}\left\{\frac{\delta}{s}\mathfrak{L}_{\gamma}\{\mathscr{A}_k\}\right\},\\
&s_{k+1}=\mathfrak{L}_{\gamma}^{-1}\left\{\frac{1}{s}\mathfrak{L}_{\gamma}\{\sigma o_k-(\kappa+\zeta)s_k\}\right\}\\
&\qquad+\mathfrak{L}_{\gamma}^{-1}\left\{\frac{\alpha}{s}\mathfrak{L}_{\gamma}\{\mathscr{B}_k\}\right\},\\
&q_{k+1}=-\mathfrak{L}_{\gamma}^{-1}\left\{\frac{1}{s}\mathfrak{L}_{\gamma}\{\kappa q_k-\zeta(1-\varsigma)s_k\}\right\}\\
&\qquad-\mathfrak{L}_{\gamma}^{-1}\left\{\frac{\alpha}{s}\mathfrak{L}_{\gamma}\{\mathscr{A}_k\}\right\},\\
&l_{k+1}=\mathfrak{L}_{\gamma}^{-1}\left\{\frac{1}{s}\mathfrak{L}_{\gamma}\{\varsigma\zeta s_k-\kappa l_k\}\right\},\quad k=0,1,2,\ldots
\end{aligned}\tag{22}$$

As a final step in applying the CLDM to solve CSM in (2), we choose a suitable number of iterations for (22) to get a good approximation.

4 Description of Conformable Residual Power Series Method to Solve CSM

Our target in this section is to employ CRPSA to solve the CSM in (2) and (3). We first summarize this methodology by an algorithm, then employ it to the CSM.

Algorithm 2: To get a conformable fractional series solution for the system of CODEs in (8) and (9), we follow the following successive steps for each $i=1,2,\ldots,m$:

Step 1: Start by assuming that the solution of (8) and (9) has a CPS of the form

$$\Psi_i(t)=\sum_{k=0}^{\infty}\psi_{ik}t^{k\gamma}.\tag{23}$$

Step 2: Use the ICs in (9) to get $\psi_{i0}=\delta_i$. So, the series solution can be rewritten as:

$$\Psi_i(t)=\delta_i+\sum_{k=1}^{\infty}\psi_{ik}t^{k\gamma}.\tag{24}$$

Step 3: Use the R-th truncated series as an approximation of each unknown function in (8) as:

$$\Psi_{iR}(t)=\delta_i+\sum_{k=1}^{R}\psi_{ik}t^{k\gamma}.\tag{25}$$

Step 4: Define the R-th residual function as:

$$Resid\Psi_{iR}(t)=\mathfrak{T}_t^{\gamma}\Psi_{iR}(t)-\mathfrak{H}_i(t,\Psi_{1R},\Psi_{2R},\ldots,\Psi_{mR}),\tag{26}$$

and the residual function by:

$$\begin{aligned}Resid\Psi_i(t)&=\lim_{R\to\infty}Resid\Psi_{iR}(t)\\&=\mathfrak{T}_t^{\gamma}\Psi_i(t)-\mathfrak{H}_i(t,\Psi_1,\Psi_2,\ldots,\Psi_m).\end{aligned}\tag{27}$$

Obviously, $\mathfrak{T}_t^{\gamma}\mathrm{Resid}_q(t)=0$ for $t\geq 0$. And more generally,

$$\begin{aligned}&({}^{(\aleph)}\mathfrak{T}_t^{\gamma}Resid\Psi_i)(0)=({}^{(\aleph)}\mathfrak{T}_t^{\gamma}Resid\Psi_{iR})(0)=0\\&\qquad\text{for each }\aleph=0,1,2,\ldots,R-1.\end{aligned}$$

Step 5: For $\aleph=0,1,2,\ldots,R-1$, solve:

$$({}^{(\aleph)}\mathfrak{T}_t^{\gamma}\mathrm{Resid}\Psi_{iR})(0)=0\tag{28}$$

to get the coefficients ψ_{iR}.

Step 6: Repeat Steps 3-6 so that you can find more coefficients of the CPS and achieve the required accuracy.

Now, we implement the CRPSA to solve the CSM in (2) and (3) as follows. Assume that the solution of (2) and (3) has a CPS of the form:

$$\begin{aligned}
P(t)&=\sum_{k=0}^{\infty}p_kt^{k\gamma},\\
O(t)&=\sum_{k=0}^{\infty}o_kt^{k\gamma},\\
S(t)&=\sum_{k=0}^{\infty}s_kt^{k\gamma},\\
Q(t)&=\sum_{k=0}^{\infty}q_kt^{k\gamma},\\
L(t)&=\sum_{k=0}^{\infty}l_kt^{k\gamma}.
\end{aligned}\tag{29}$$

Using the ICs in (3), we can approximate the solution by the R-th truncated series of the forms:

$$\begin{aligned}
P_R(t)&=p_0+\sum_{k=1}^{R}p_kt^{k\gamma},\\
O_R(t)&=o_0+\sum_{k=1}^{R}o_kt^{k\gamma},\\
S_R(t)&=s_0+\sum_{k=1}^{R}s_kt^{k\gamma},\\
Q_R(t)&=q_0+\sum_{k=1}^{R}q_kt^{k\gamma},\\
L_R(t)&=l_0+\sum_{k=1}^{R}l_kt^{k\gamma}.
\end{aligned}\tag{30}$$

Now, we define the Rth-residual functions as

$$\begin{aligned}
\text{Resid}P_R(t) &= \mathfrak{T}_t^{\gamma} P_R - \vartheta + \delta P_R S_R + \kappa P_R,\\
\text{Resid}O_R(t) &= \mathfrak{T}_t^{\gamma} O_R - \delta P_R S_R + (\sigma + \kappa) O_R,\\
\text{Resid}S_R(t) &= \mathfrak{T}_t^{\gamma} S_R - \sigma O_R - \alpha S_R Q_R + (\kappa + \zeta) S_R,\\
\text{Resid}Q_R(t) &= \mathfrak{T}_t^{\gamma} Q_R + \alpha S_R Q_R + \kappa Q_R - \zeta(1-\varsigma) S_R,\\
\text{Resid}L_R(t) &= \mathfrak{T}_t^{\gamma} L_R - \varsigma\zeta S_R + \kappa L_R.
\end{aligned} \tag{31}$$

The main step is to solve the system of algebraic equations with $\aleph = 0, 1, 2, \ldots, R-1$.

$$\begin{aligned}
({}^{(\aleph)}\mathfrak{T}^{\gamma}\text{Resid}P_R)(0) &= 0,\\
({}^{(\aleph)}\mathfrak{T}_t^{\gamma}\text{Resid}O_R)(0) &= 0,\\
({}^{(\aleph)}\mathfrak{T}_t^{\gamma}\text{Resid}S_R)(0) &= 0,\\
({}^{(\aleph)}\mathfrak{T}_t^{\gamma}\text{Resid}Q_R)(0) &= 0,\\
({}^{(\aleph)}\mathfrak{T}_t^{\gamma}\text{Resid}L_R)(0) &= 0.
\end{aligned} \tag{32}$$

So, we compute the coefficients of each truncated CRPS in (30). To illustrate this, let $R = 1$ in systems (30) and (31). Then:

$$\begin{aligned}
P_1(t) &= p_0 + p_1 t^{\gamma},\\
O_1(t) &= o_0 + o_1 t^{\gamma},\\
S_1(t) &= s_0 + s_1 t^{\gamma},\\
Q_1(t) &= q_0 + q_1 t^{\gamma},\\
L_1(t) &= l_0 + l_1 t^{\gamma}.
\end{aligned}$$

$$\begin{aligned}
ResidP_1(t) &= \gamma p_1 + \kappa p_0 + \delta p_0 s_0 + \delta p_1 s_1 t^{2\gamma} + \delta p_1 s_0 t^{\gamma}\\
&\quad + \delta p_0 s_1 t^{\gamma} + \kappa p_1 t^{\gamma} - \vartheta,\\
ResidO_1(t) &= \alpha o_0 + \gamma o_1 + \kappa o_0 + \alpha o_1 t^{\gamma} + \kappa o_1 t^{\gamma} - \delta p_0 s_0\\
&\quad - \delta p_1 s_1 t^{2\gamma} - \delta p_1 s_0 t^{\gamma} - \delta p_0 s_1 t^{\gamma},\\
ResidS_1(t) &= -o_0\sigma + o_1 - \sigma t^{\gamma} - \alpha q_0 s_0 - \alpha q_1 s_1 t^{2\gamma}\\
&\quad - \alpha q_1 s_0 t^{\gamma} - \alpha q_0 s_1 t^{\gamma} + \gamma s_1 + \zeta s_0 + \kappa s_0\\
&\quad + \zeta s_1 t^{\gamma} + \kappa s_1 t^{\gamma},\\
ResidQ_1(t) &= \gamma q_1 + \kappa q_0 + \alpha q_0 s_0 + \alpha q_1 s_1 t^{2\gamma} + \alpha q_1 s_0 t^{\gamma}\\
&\quad + \alpha q_0 s_1 t^{\gamma} + \kappa q_1 t^{\gamma} + \zeta s_0 \varsigma - \zeta s_0 + \zeta s_1 \varsigma t^{\gamma}\\
&\quad - \zeta s_1 t^{\gamma},\\
ResidL_1(t) &= \gamma l_1 + \kappa l_0 + \kappa l_1 t^{\gamma} - \zeta s_0 \varsigma - \zeta s_1 \varsigma t^{\gamma}.
\end{aligned}$$

From (32), we have to solve the system of algebraic equations $(ResidP_1)(0) = 0$, $(ResidO_1)(0) = 0$, $(ResidS_1)(0) = 0$, $(ResidQ_1)(0) = 0$, and $(ResidL_1)(0) = 0$, which implies

$$\begin{aligned}
p_1 &= \frac{\vartheta - p_0(\kappa + \delta s_0)}{\gamma},\\
o_1 &= \frac{\delta p_0 s_0 - o_0(\alpha + \kappa)}{\gamma},\\
s_1 &= \frac{o_0\sigma - s_0(\zeta + \kappa - \alpha q_0)}{\gamma},\\
q_1 &= -\frac{q_0(\kappa + \alpha s_0) + \zeta s_0(\varsigma - 1)}{\gamma},\\
l_1 &= \frac{\zeta s_0 \varsigma - \kappa l_0}{\gamma}.
\end{aligned}$$

To find the second set of coefficients, we put R=2 in systems (30) and (31). Then

$$\begin{aligned}
P_2(t) &= p_0 + p_1 t^{\gamma} + p_2 t^{2\gamma},\\
O_2(t) &= o_0 + o_1 t^{\gamma} + o_2 t^{2\gamma},\\
S_2(t) &= s_0 + s_1 t^{\gamma} + s_2 t^{2\gamma},\\
Q_2(t) &= q_0 + q_1 t^{\gamma} + q_2 t^{2\gamma},\\
L_2(t) &= l_0 + l_1 t^{\gamma} + l_2 t^{2\gamma},
\end{aligned}$$

$$\begin{aligned}
ResidP_2(t) &= \frac{-\gamma\vartheta - \kappa p_0^2(\gamma + \kappa t^{\gamma}) + p_0\vartheta(\gamma + \kappa t^{\gamma})}{\gamma}\\
&\quad + \delta p_0 s_0 \left(\frac{o_0\sigma\vartheta t^{2\gamma}}{\gamma^2} - p_0\right) + p_2 t^{\gamma}(2\gamma + \kappa t^{\gamma})\\
&\quad + \frac{\delta s_0 t^{\gamma}\left(\gamma o_0 p_2 \sigma t^{2\gamma} - \kappa p_0^2(\gamma + o_0\sigma t^{\gamma})\right)}{\gamma^2}\\
&\quad - \frac{\delta p_0 s_0^2 t^{2\gamma}(\vartheta(\zeta + \kappa) - p_0(\kappa(\zeta + \kappa) - \delta o_0\sigma))}{\gamma^2}\\
&\quad - \frac{\delta s_0^2 t^{2\gamma}(\alpha p_0 q_0(\kappa p_0 - \vartheta) + \gamma p_2(\zeta + \kappa) t^{\gamma})}{\gamma^2}\\
&\quad + \frac{\delta s_0^2 t^{2\gamma}\left(\delta p_0^2 s_0(\zeta + \kappa - \alpha q_0) + \alpha\gamma p_2 q_0 t^{\gamma}\right)}{\gamma^2}\\
&\quad + \frac{\delta s_2 t^{3\gamma}\left(-p_0^2(\kappa + \delta s_0) + \gamma p_2 t^{\gamma} + p_0\vartheta\right)}{\gamma},
\end{aligned}$$

$$
\begin{aligned}
ResidO_2(t) =& \frac{\alpha\gamma o_2 t^{2\gamma} - o_0^2(\alpha+\kappa)\left(\gamma+(\alpha+\kappa)t^{\gamma}\right)}{\gamma} \\
&+ \frac{\delta o_0 p_0 s_0\left(\gamma+(\alpha+\kappa)t^{\gamma}\right)+\gamma o_2 t^{\gamma}\left(2\gamma+\kappa t^{\gamma}\right)}{\gamma} \\
&- \frac{\delta s_0 t^{2\gamma}\left(o_0\sigma\left(-\kappa p_0^2+\gamma p_2 t^{\gamma}+p_0\vartheta\right)-\zeta p_0 s_0\vartheta\right)}{\gamma^2} \\
&+ \frac{\delta p_0 s_0^2 t^{2\gamma}\left(\kappa\vartheta - p_0\left(\kappa(\zeta+\kappa)-\delta o_0\sigma\right)\right)}{\gamma^2} \\
&+ \frac{\delta s_0^2 t^{2\gamma}\left(\alpha p_0 q_0\left(\kappa p_0-\vartheta\right)+\gamma p_2(\zeta+\kappa)t^{\gamma}\right)}{\gamma^2} \\
&- \frac{\delta s_0^2 t^{2\gamma}\left(\delta p_0^2 s_0\left(\zeta+\kappa-\alpha q_0\right)+\alpha\gamma p_2 q_0 t^{\gamma}\right)}{\gamma^2} \\
&- \frac{\delta s_2 t^{3\gamma}\left(-p_0^2\left(\kappa+\delta s_0\right)+\gamma p_2 t^{\gamma}+p_0\vartheta\right)}{\gamma},
\end{aligned}
$$

$$
\begin{aligned}
ResidS_2(t) =& \frac{\sigma\left(o_0 s_0\left(\gamma+\zeta t^{\gamma}\right)-\gamma o_2 t^{2\gamma}+o_0^2(\alpha+\kappa)t^{\gamma}\right)}{\gamma} \\
&- \frac{o_0\sigma s_0 t^{\gamma}\left(\gamma\delta p_0+\alpha t^{\gamma}\left(\gamma q_2 t^{\gamma}-\kappa q_0^2\right)\right)}{\gamma^2} \\
&- \frac{s_0^2\left((\zeta+\kappa)\left(\gamma+(\zeta+\kappa)t^{\gamma}\right)-\alpha\gamma q_0\right)}{\gamma} \\
&+ \frac{\alpha q_0 s_0^2 t^{\gamma}\left(\gamma(\zeta+\kappa)+\zeta o_0\sigma(\varsigma-1)t^{\gamma}\right)}{\gamma^2} \\
&+ \frac{\alpha q_0^2 s_0^2 t^{2\gamma}\left(\alpha o_0\sigma-\kappa\left(\zeta+\kappa-\alpha q_0\right)\right)}{\gamma^2} \\
&- \frac{\alpha t^{3\gamma}\left(-q_2 s_0^2\left(\zeta+\kappa-\alpha q_0\right)-\kappa q_0^2 s_2\right)}{\gamma} \\
&- \frac{\alpha\zeta q_0 s_0^3 t^{2\gamma}\left(\zeta\varsigma+\kappa(\varsigma-1)+2\alpha q_0\right)}{\gamma^2} \\
&+ \frac{\alpha q_0 s_0^3 t^{2\gamma}\left(\zeta^2+\alpha^2 q_0^2+\alpha q_0(\zeta\varsigma-\kappa)\right)}{\gamma^2} \\
&- s_2 t^{\gamma}\left(-2\gamma+\alpha q_2 t^{3\gamma}-(\zeta+\kappa)t^{\gamma}\right) \\
&+ \frac{\alpha q_0 s_0 s_2 t^{3\gamma}\left(\gamma\zeta(\varsigma-1)+\alpha\kappa o_0 q_0\sigma s_0 t^{\gamma}\right)}{\gamma^2},
\end{aligned}
$$

$$
\begin{aligned}
ResidQ_2(t) =& \frac{\gamma q_2 t^{\gamma}\left(2\gamma+\kappa t^{\gamma}\right)-\kappa q_0^2\left(\gamma+\alpha s_2 t^{3\gamma}+\kappa t^{\gamma}\right)}{\gamma} \\
&+ \frac{\zeta s_0(\varsigma-1)\left(o_0\sigma t^{\gamma}-q_0\left(\gamma+\kappa t^{\gamma}\right)\right)}{\gamma} \\
&- \frac{\alpha s_0\left(o_0 q_2(-\sigma)t^{3\gamma}+\gamma q_0^2+\zeta q_0 s_2(\varsigma-1)t^{3\gamma}\right)}{\gamma} \\
&- \frac{s_0 t^{\gamma}\left(\zeta s_0\left((\varsigma-1)(\zeta+\kappa)+\alpha q_0\right)+\alpha^2 q_0^2 s_2 t^{2\gamma}\right)}{\gamma} \\
&+ \frac{\alpha q_0 s_0 t^{2\gamma}\left(\zeta\kappa q_0 s_0-o_0\sigma\left(\kappa q_0+\zeta s_0(\varsigma-1)\right)\right)}{\gamma^2} \\
&- \frac{\alpha s_0^2 t^{2\gamma}\left(q_0^2\left(\alpha o_0\sigma-\kappa^2\right)+\alpha\kappa q_0^3+\gamma\zeta q_2 t^{\gamma}\right)}{\gamma^2} \\
&+ \frac{\alpha s_0^2 t^{2\gamma}\left(\gamma q_2\left(-t^{\gamma}\right)\left(\kappa-\alpha q_0\right)-\zeta q_0 s_0(\zeta+\kappa)\right)}{\gamma^2} \\
&- \frac{\alpha q_0 s_0^3 t^{2\gamma}\left(\alpha q_0(\zeta(\varsigma-2)-\kappa)-\zeta\varsigma(\zeta+\kappa)\right)}{\gamma^2} \\
&+ \frac{t^{2\gamma}\left(\gamma^2 s_2\left(\zeta(\varsigma-1)+\alpha q_2 t^{2\gamma}\right)-\alpha^3 q_0^3 s_0^3\right)}{\gamma^2} \\
&- \frac{\alpha\kappa q_0^2 s_0 t^{\gamma}}{\gamma},
\end{aligned}
$$

$$
\begin{aligned}
ResidL_2(t) =& -\kappa l_0^2 - \frac{\kappa^2 l_0^2 t^{\gamma}}{\gamma} + \kappa l_2 t^{2\gamma} + 2\gamma l_2 t^{\gamma} - \zeta s_2\varsigma t^{2\gamma} \\
&+ \zeta l_0 s_0\varsigma + \frac{\zeta\kappa l_0 s_0\varsigma t^{\gamma}}{\gamma} - \frac{\zeta o_0\sigma s_0\varsigma t^{\gamma}}{\gamma} \\
&+ \frac{\zeta^2 s_0^2\varsigma t^{\gamma}}{\gamma} + \frac{\zeta\kappa s_0^2\varsigma t^{\gamma}}{\gamma} - \frac{\alpha\zeta q_0 s_0^2\varsigma t^{\gamma}}{\gamma}.
\end{aligned}
$$

From (2), we have to solve the system of algebraic equations
$(\mathfrak{T}_t^{\gamma} ResidP_2)(0) = 0$,
$(\mathfrak{T}_t^{\gamma} ResidO_2)(0) = 0$,
$(\mathfrak{T}_t^{\gamma} ResidS_2)(0) = 0$,
$(\mathfrak{T}_t^{\gamma} ResidQ_2)(0) = 0$, and
$(\mathfrak{T}_t^{\gamma} ResidL_2)(0) = 0$.
So, we get the system

$$
\begin{aligned}
&2\gamma^2 p_2 - \kappa p_0^2\left(\kappa+\delta s_0\right)+\kappa p_0\vartheta = 0, \\
&-o_0^2(\alpha+\kappa)^2 + 2\gamma^2 o_2 + \delta o_0 p_0 s_0(\alpha+\kappa) = 0, \\
&o_0^2\sigma(\alpha+\kappa) + o_0\sigma s_0\left(\zeta+\kappa-\delta p_0\right) \\
&-s_0^2(\zeta+\kappa)\left(\zeta+\kappa-\alpha q_0\right)+2\gamma^2 s_2 = 0, \\
&-\zeta s_0(\varsigma-1)\left(s_0(\zeta+\kappa)-o_0\sigma\right)+2\gamma^2 q_2 \\
&+\zeta q_0 s_0(\varsigma-1)\left(\alpha s_0-\kappa\right)-\kappa q_0^2\left(\kappa+\alpha s_0\right) = 0, \\
&2\gamma^2 l_2 - \kappa^2 l_0^2 + \zeta s_0\varsigma\left(\kappa l_0 - o_0\sigma\right)+\zeta s_0^2\varsigma\left(\zeta+\kappa-\alpha q_0\right) = 0.
\end{aligned}
$$

Solving the resulting system, we get

$$p_2 = -\frac{\kappa p_0\left(-\vartheta + p_0\left(\kappa+\delta s_0\right)\right)}{2\gamma^2},$$
$$o_2 = \frac{o_0(\alpha+\kappa)\left(o_0(\alpha+\kappa)-\delta p_0 s_0\right)}{2\gamma^2},$$
$$s_2 = \frac{1}{2\gamma^2}\big(o_0^2\sigma(\alpha+\kappa)+o_0\sigma s_0\left(\zeta+\kappa-\delta p_0\right) - s_0^2(\zeta+\kappa)\left(\zeta+\kappa-\alpha q_0\right)\big),$$
$$q_2 = \frac{-1}{2\gamma^2}\big(\sigma(\kappa+\sigma)o_0^2+\sigma o_0(\zeta+\kappa-\delta p_0)s_0 - (\zeta+\kappa)(\zeta+\kappa-\alpha q_0)s_0^2\big),$$
$$l_2 = \frac{\kappa^2 l_0^2 - \zeta\kappa l_0 s_0 \varsigma + \zeta s_0 \varsigma\left(o_0\sigma - s_0\left(\zeta+\kappa-\alpha q_0\right)\right)}{2\gamma^2}.$$

This procedure can be repeated many times to get the required number of coefficients for the approximate solution of the CSM in (2) and (3), and consequently the required accuracy. Of course, higher accuracy can be achieved by evaluating more components.

5 Numerical Results

In this section, we apply both proposed techniques that are described in Algorithms 1 and 2 to investigate approximate numerical solutions for the CSM with specified values of the parameters. In fact, we assume the same values of the parameters as given in the classical smoking model of integer order derivative [11]. So, we assign values for each parameters as follows:

$$\vartheta = 1, \quad \delta = 0.14, \quad \kappa = 0.05, \quad \sigma = 0.002,$$

$$\alpha = 0.0025, \quad \zeta = 0.8, \quad \varsigma = 0.1,$$

and the initial populations of potential smokers, occasional smokers, smokers, temporary quitters, and permanent quitters are assumed to be:

$$p_0 = 40, \quad o_0 = 1, \quad s_0 = 20, \quad q_0 = 10, \quad l_0 = 5,$$

successively.
To perform our calculations, we use Mathematica 10 software and compute 25 iterations for the CSM solutions with arbitrary conformable order. In fact, the solution terms that result from both methods are identical. A sample of the results is the tenth truncated series solution that is obtained by both techniques as

$$P_{10}(t) = 40 + \frac{203.109t^{10\gamma}}{\gamma^{10}} - \frac{257.646t^{9\gamma}}{\gamma^9} + \frac{312.847t^{8\gamma}}{\gamma^8} - \frac{360.864t^{7\gamma}}{\gamma^7} + \frac{391.628t^{6\gamma}}{\gamma^6} - \frac{394.829t^{5\gamma}}{\gamma^5} + \frac{363.319t^{4\gamma}}{\gamma^4} - \frac{297.238t^{3\gamma}}{\gamma^3} + \frac{207.169t^{2\gamma}}{\gamma^2} - \frac{113t^{\gamma}}{\gamma},$$

$$O_{10}(t) = 10 - \frac{203.161t^{10\gamma}}{\gamma^{10}} + \frac{257.716t^{9\gamma}}{\gamma^9} - \frac{312.938t^{8\gamma}}{\gamma^8} + \frac{360.977t^{7\gamma}}{\gamma^7} - \frac{391.7605t^{6\gamma}}{\gamma^6} + \frac{394.976t^{5\gamma}}{\gamma^5} - \frac{363.469t^{4\gamma}}{\gamma^4} + \frac{297.377t^{3\gamma}}{\gamma^3} - \frac{207.242t^{2\gamma}}{\gamma^2} + \frac{t^{\gamma}}{\gamma},$$

$$S_{10}(t) = 20 + \frac{0.0592332t^{10\gamma}}{\gamma^{10}} - \frac{0.0811802t^{9\gamma}}{\gamma^9} + \frac{0.10849899t^{8\gamma}}{\gamma^8} - \frac{0.143958t^{7\gamma}}{\gamma^7} + \frac{0.201203t^{6\gamma}}{\gamma^6} - \frac{0.339961t^{5\gamma}}{\gamma^5} + \frac{0.800531t^{4\gamma}}{\gamma^4} - \frac{2.421034t^{3\gamma}}{\gamma^3} + \frac{7.24448t^{2\gamma}}{\gamma^2} - \frac{16.48t^{\gamma}}{\gamma},$$

$$Q_{10}(t) = 10 - \frac{0.00668552t^{10\gamma}}{\gamma^{10}} + \frac{0.0101606t^{9\gamma}}{\gamma^9} - \frac{0.0160952t^{8\gamma}}{\gamma^8} + \frac{0.0287447t^{7\gamma}}{\gamma^7} - \frac{0.0636761t^{6\gamma}}{\gamma^6} + \frac{0.179750t^{5\gamma}}{\gamma^5} - \frac{0.599103t^{4\gamma}}{\gamma^4} + \frac{2.07556t^{3\gamma}}{\gamma^3} - \frac{6.3968t^{2\gamma}}{\gamma^2} + \frac{13.4t^{\gamma}}{\gamma},$$

$$L_{10}(t) = 5 - \frac{0.000654304t^{10\gamma}}{\gamma^{10}} + \frac{0.000972514t^{9\gamma}}{\gamma^9} - \frac{0.001454t^{8\gamma}}{\gamma^8} + \frac{0.002333t^{7\gamma}}{\gamma^7} - \frac{0.004644t^{6\gamma}}{\gamma^6} + \frac{0.0133183t^{5\gamma}}{\gamma^5} - \frac{0.0509799t^{4\gamma}}{\gamma^4} + \frac{0.204735t^{3\gamma}}{\gamma^3} - \frac{0.69295t^{2\gamma}}{\gamma^2} + \frac{1.35t^{\gamma}}{\gamma}.$$

Comparisons between the 25th approximate solutions for the epidemic smoking model for $\gamma \in \{1, 0.95, 0.85\}$ are presented in Tables 1-5. The solution natures for $\gamma \in \{1, 0.95, 0.85, 0.75\}$ are explored with time change for each category of smokers in the CSM in Figures 1-5.
It is clear form these graphs that the numbers of both potential smokers and smokers decrease over time, while the numbers of smokers in the remaining three categories increase with time. Moreover, we notice that the use of conformable operator instead of the classical integer order generalizes the epidemic model and gives us the opportunity of obtaining variety of results and expectations. In our proposed smoking model, at fixed time, the number of potential smokers and smokers decrease when conformable derivative order become less than 1 which is clear since the curves in fractional case appear below the curves of $\gamma = 1$ as shown in Figures 1 and 3. Conversely, the solution curves of fractional orders for O, Q and L are above the curves of $\gamma = 1$ in Figures 2, 4 and 5.
To check the validity of our results, and due to the absence of the exact solution of this CSM, we calculate some values for the 25th residual errors depending on the formulas (25) with $R = 25$. These values are given in Tables 6-10.

Table 1: Approximate number of potential smokers $P_{25}(t)$.

t_i	$\gamma = 1$	$\gamma = 0.95$	$\gamma = 0.85$
0.	40.	40.	40.
0.1	30.5072	29.1166	25.8128
0.2	23.785	22.2577	18.9452
0.3	18.9262	17.5369	14.6692
0.4	15.3484	14.1636	11.7893
0.5	12.6687	11.6867	9.75302
0.6	10.6306	9.8277	8.26193
0.7	9.05873	8.40668	7.13469
0.8	7.83055	7.29999	6.16484
0.9	6.84653	6.36993	4.11757
1.	5.85895	4.8901	-9.63683

Table 2: Approximate number of potential smokers $O_{25}(t)$.

t_i	$\gamma = 1$	$\gamma = 0.95$	$\gamma = 0.85$
0.	10.	10.	10.
0.1	19.3402	20.703	23.9331
0.2	25.909	27.393	30.5957
0.3	30.614	31.9489	34.6821
0.4	34.038	35.1601	37.3822
0.5	36.5641	37.4773	39.2459
0.6	38.4489	39.1785	40.5701
0.7	39.8679	40.4436	41.5352
0.8	40.9438	41.396	42.347
0.9	41.776	42.1734	44.24
1.	42.6125	43.5021	57.8448

Table 3: Approximate number of potential smokers $S_{25}(t)$.

0.	20.	20.	20.
0.1	18.4221	18.1508	17.4509
0.2	16.9756	16.5902	15.6542
0.3	15.6484	15.2053	14.1662
0.4	14.4298	13.9617	12.8889
0.5	13.3102	12.8377	11.7724
0.6	12.281	11.8175	10.785
0.7	11.3344	10.8889	9.90465
0.8	10.4633	10.0417	9.11493
0.9	9.66146	9.26717	8.40295
1.	8.92304	8.55802	7.75713

Table 4: Approximate number of potential smokers $Q_{25}(t)$.

t_i	$\gamma = 1$	$\gamma = 0.95$	$\gamma = 0.85$
0.	10.	10.	10.
0.1	11.278	11.4967	12.0593
0.2	12.4398	12.7476	13.4913
0.3	13.4959	13.846	14.6619
0.4	14.4556	14.8215	15.6529
0.5	15.3276	15.6924	16.5068
0.6	16.1194	16.4725	17.2503
0.7	16.8381	17.1726	17.9022
0.8	17.4898	17.8015	18.4766
0.9	18.0802	18.3667	18.9845
1.	18.6145	18.8748	19.4348

6 Conclusion

In this paper, we investigated the solution of a biomathematical model that is related to the population of smoker. We replaced classical first derivative by the conformable fractional derivative. The CRPSA and CLDM are implemented to obtain analytic and approximate solution and some numerical results are produced by the help of Mathematica 10 software. From the tabulated and graphical results that are shown in Tables 1-10 and in Figures 1-5, we can conclude the following:

*Regardless of the values of conformable orders, the numbers of both potential smokers and smokers decrease over time, while the numbers of smokers in the remaining three categories increase with time.
*The use of conformable operator instead of the classical integer order generalizes the epidemic model and gives us the opportunity of obtaining variety of results and expectations. In CSM, at fixed time, the number of potential smokers and smokers decrease when conformable derivative order become less than 1 which is clear since the curves in fractional case appear below the curves of $\gamma = 1$ as shown in Figures

Table 5: Approximate number of potential smokers $L_{25}(t)$.

t_i	$\gamma = 1$	$\gamma = 0.95$	$\gamma = 0.85$
0.	5.	5.	5.
0.1	5.12827	5.15011	5.20611
0.2	5.24384	5.27425	5.34733
0.3	5.34778	5.38195	5.46091
0.4	5.44105	5.47623	5.5553
0.5	5.52453	5.55902	5.63491
0.6	5.59902	5.63175	5.70255
0.7	5.66527	5.69557	5.76021
0.8	5.72393	5.75142	5.8094
0.9	5.77564	5.80013	5.85129
1.	5.82094	5.84239	5.88684

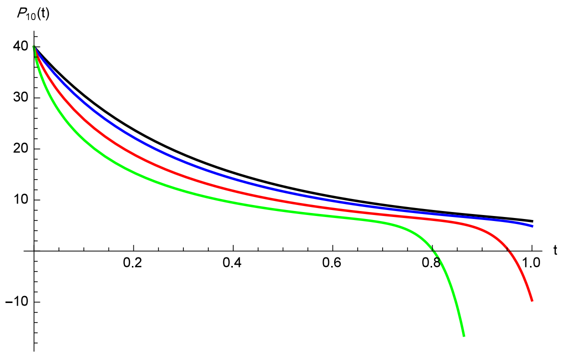

Fig. 1: Plots of the 25th approximate solution for potential smoker with (Black: $\gamma = 1$, Blue: $\gamma = 0.95$, Red: $\gamma = 0.85$, Green: $\gamma = 0.75$)

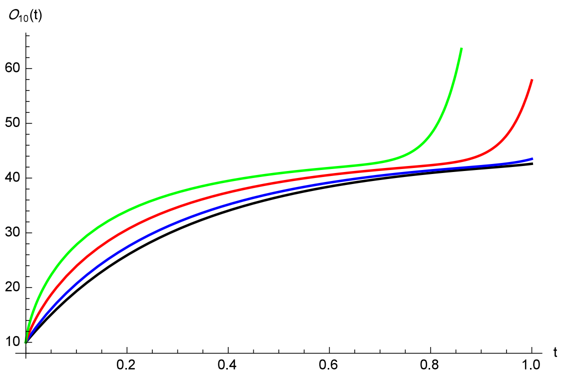

Fig. 2: Plots of the 25th approximate solution for occasional smoker with (Black: $\gamma = 1$, Blue: $\gamma = 0.95$, Red: $\gamma = 0.85$, Green: $\gamma = 0.75$)

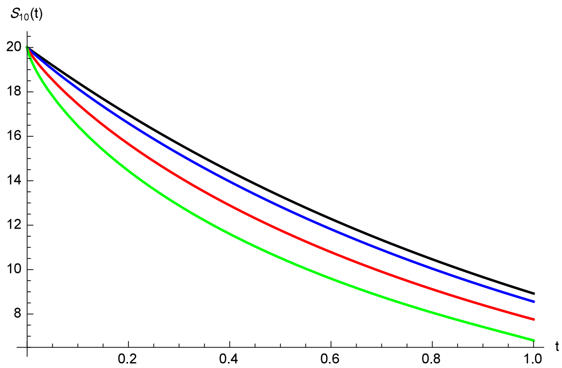

Fig. 3: Plots of the 25th approximate solution for smoker with (Black: $\gamma = 1$, Blue: $\gamma = 0.95$, Red: $\gamma = 0.85$, Green: $\gamma = 0.75$)

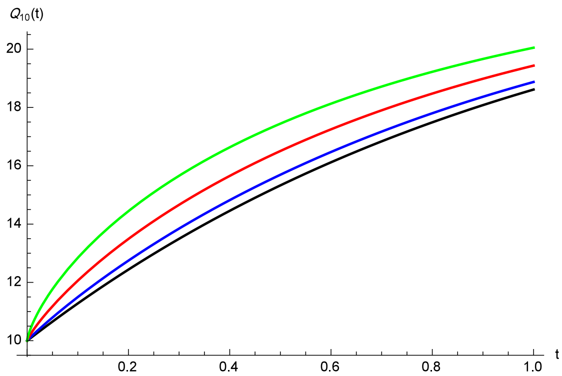

Fig. 4: Plots of the 25th approximate solution for temporary quitters with (Black: $\gamma = 1$, Blue: $\gamma = 0.95$, Red: $\gamma = 0.85$, Green: $\gamma = 0.75$)

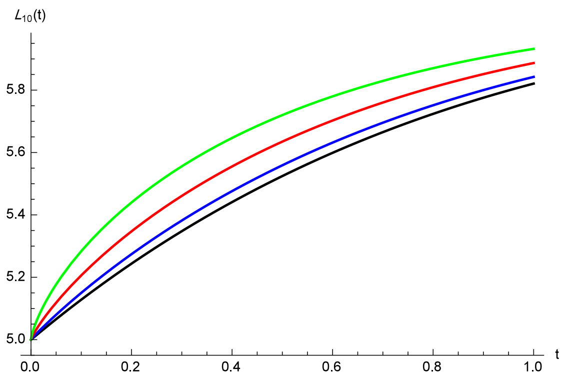

Fig. 5: Plots of the 25th approximate solution for permanent quitters with (Black: $\gamma = 1$, Blue: $\gamma = 0.95$, Red: $\gamma = 0.85$, Green: $\gamma = 0.75$)

1 and 3. Conversely, the solution curves of fractional orders for O, Q and L are above the curves of $\gamma = 1$ in Figures 2, 4 and 5.

*Matching results for the same number of iteration proves the validity and efficiency of the CRPSA and CLDM in solving epidemic model. In fact, small residual error values support the the accuracy of our approximations.

Table 6: The 25th residual errors $|ResidP_{25}|$ of approximating the number of potential smokers by $P_{25}(t)$.

t_i	$\gamma = 1$	$\gamma = 0.95$	$\gamma = 0.85$
0.	0	0	0
0.1	1.4211×10^{-14}	2.8422×10^{-14}	0.
0.2	7.1054×10^{-15}	7.1054×10^{-15}	7.3186×10^{-13}
0.3	7.3186×10^{-13}	1.1404×10^{-11}	3.5720×10^{-9}
0.4	9.0326×10^{-10}	1.0039×10^{-8}	1.5283×10^{-6}
0.5	2.2809×10^{-7}	1.9159×10^{-6}	0.000166597
0.6	0.0000207933	0.000138975	0.00765294
0.7	0.000938747	0.00517346	0.193733
0.8	0.0253469	0.118224	3.17187

Table 7: The 25th residual errors $|ResidO_{25}|$ of approximating the number of occasional smokers by $O_{25}(t)$.

t_i	$\gamma = 1$	$\gamma = 0.95$	$\gamma = 0.85$
0.	0	0	0
0.1	0.	0.	1.4211×10^{-14}
0.2	0.	7.1054×10^{-15}	6.9633×10^{-13}
0.3	7.2475×10^{-13}	1.1397×10^{-11}	3.5720×10^{-9}
0.4	9.0337×10^{-10}	1.0041×10^{-8}	1.5285×10^{-6}
0.5	2.2812×10^{-7}	1.9162×10^{-6}	0.000166624
0.6	0.0000207966	0.000138998	0.00765426
0.7	0.000938904	0.00517434	0.193768
0.8	0.0253513	0.118245	3.17246

Table 8: The 25th residual errors $|ResidS_{25}|$ of approximating the number of smokers by $S_{25}(t)$.

t_i	$\gamma = 1$	$\gamma = 0.95$	$\gamma = 0.85$
0.	0	0	0
0.1	0.	1.7764×10^{-15}	0.
0.2	3.5527×10^{-15}	1.7764×10^{-15}	1.7764×10^{-15}
0.3	3.5527×10^{-15}	1.7764×10^{-15}	5.5778×10^{-13}
0.4	1.4033×10^{-13}	1.5810×10^{-12}	2.5184×10^{-14}
0.5	3.6907×10^{-11}	3.1642×10^{-10}	2.8858×10^{-8}
0.6	3.5196×10^{-9}	2.4023×10^{-8}	1.3889×10^{-6}
0.7	1.6596×10^{-7}	9.3423×10^{-7}	0.0000367405
0.8	4.6734×10^{-6}	0.0000222642	0.000627088
0.9	0.0000887776	0.000365037	0.00765924
1.	0.00123624	0.004456	0.0718472

Acknowledgement

The authors are grateful to the referee for the great efforts that have been done to improve this work.

References

[1] C. Castillo-Garsow,G. Jordan-Salivia and A. R. Herrera, Mathematical models for the dynamics of tobacco use, recovery, and relapse. *(Technical Report No. BU-1505-M)*, Cornell University, (1997).

Table 9: The 25th residual errors $|ResidQ_{25}|$ of approximating the number of temporary quitters by $Q_{25}(t)$.

t_i	$\gamma = 1$	$\gamma = 0.95$	$\gamma = 0.85$
0.	0	0	0
0.1	0.	1.7764×10^{-15}	1.7764×10^{-15}
0.2	0.	3.5527×10^{-15}	0.
0.3	1.7764×10^{-15}	5.3291×10^{-15}	3.0198×10^{-14}
0.4	8.8818×10^{-15}	7.9936×10^{-14}	1.2730×10^{-11}
0.5	1.8741×10^{-12}	1.5984×10^{-11}	1.4459×10^{-9}
0.6	1.7704×10^{-10}	1.2041×10^{-9}	6.9054×10^{-8}
0.7	8.2867×10^{-9}	4.6485×10^{-8}	1.8137×10^{-6}
0.8	2.3175×10^{-7}	1.1003×10^{-6}	0.0000307567
0.9	4.3739×10^{-6}	0.0000179265	0.000373418
1.	0.0000605367	0.000217533	0.00348347

Table 10: The 25th residual errors $|ResidL_{25}|$ of approximating the number of permanent quitters by $L_{25}(t)$.

t_i	$\gamma = 1$	$\gamma = 0.95$	$\gamma = 0.85$
0.0	0.0	0.0	0.0
0.1	0.0	2.2204×10^{-16}	0.0
0.2	0.0	2.2204×10^{-16}	1.1102×10^{-16}
0.3	0.0	2.2204×10^{-16}	2.7756×10^{-15}
0.4	7.7716×10^{-16}	7.6605×10^{-15}	1.2479×10^{-12}
0.5	1.8274×10^{-13}	1.5676×10^{-12}	1.4304×10^{-10}
0.6	1.7442×10^{-11}	1.1907×10^{-10}	6.8873×10^{-9}
0.7	8.2273×10^{-10}	4.6323×10^{-9}	1.8225×10^{-7}
0.8	2.3177×10^{-8}	1.1043×10^{-7}	3.1117×10^{-6}
0.9	4.4042×10^{-7}	1.8112×10^{-6}	0.0000380177
1.	6.1349×10^{-6}	0.0000221164	0.000356724

[2] H. Ritchie and M. Roser , Smoking, *Published online at OurWorldInData.org. Retrieved from: 'https://ourworldindata.org/smoking'* [Online Resource], (2023).

[3] PK. Roy, X. Cao, X-Z. Li, P. Das, and S. Deo, editors, *Mathematical analysis and applications in modeling*, Springer Singapore, (2020).

[4] A. Sharma and AK. Misra, Backward bifurcation in a smoking cessation model with media campaigns. *Appl Math Model*, 39(3-4),1087-98 (2015).

[5] IR. Sofia and M. Ghosh, Mathematical modeling of smoking habits in the society. *Stoch Anal Appl* **2022**,1-20(2022).

[6] G. Zaman, Y. H. Kang and I. H. Jung, Dynamics of a smoking model with smoking death rate. *Applicationes Mathematicae*, **44**, 281-295(2017).

[7] V. Verma, Optimal control analysis of a mathematical model on smoking. *Model Earth Syst Environ* 6(4), 2535-42(2020).

[8] A. Sharma, Quantifying the effect of demographic stochasticity on the smoking epidemic in the presence of economic stimulus. *Physica A: Statistical Mechanics and its Applications,* **549,** 124412(2020).

[9] S. Zhang, Y. Meng, A. K. Chakraborty and H. Wang, Controlling smoking: A smoking epidemic model with different smoking degrees in deterministic and stochastic

environments. *Mathematical Biosciences,* **368**, 109132 (2024).
[10] A. Ullah, H. Sakidin, K. Shah, Y. Hamed and T Abdeljawad, A mathematical model with control strategies for marijuana smoking prevention. *Electronic Research Archive,* 32(**4**), 2342-2362(2024).
[11] R. Ullah, M. Khan, G. Zaman, S. Islam, M. A. Khan, S. Jan and T. Gul, Dynamical features of a mathematical model on smoking. *J. Appl. Environ. Biol. Sci,* 6(**1**), 92-96(2016).
[12] H. Gunerhan, H. Rezazadeh, W. Adel, M. Hatami, K. M. Sagayam, H. Emadifar, M.I. Asjad, F. Hamasalh and A. A. Hamoud, Analytical approximate solution of fractional order smoking epidemic model. *Adv. Mech. Eng.*, 14(**9**), 1-11 (2022).
[13] M. Abdullah, A. Ahmad, N. Raza, M. Farman and M. Ahmad, Approximate solution and analysis of smoking epidemic model with Caputo fractional derivatives. *International Journal of Applied and Computational Mathematics,* 4(**5**),112(2018).
[14] P. Veeresha, D. G. Prakasha and H. M. Baskonus, Solving smoking epidemic model of fractional order using a modified homotopy analysis transform method. *Mathematical Sciences,* **13**, 115-128(2019).
[15] I. Y. Miranda-Valdez, J. G. Puente-Cordova, F. Y. Renteria-Baltierrez, L. Fliri, M. Hummel, A. Puisto, A. Puisto and M. J. Alava, Viscoelastic phenomena in methylcellulose aqueous systems: Application of fractional calculus. *Food Hydrocolloids,* 147, 109334(2024).
[16] A. L.Ferrari, M. C. S. Gomes, A. C. R. Aranha,,S. M. Paschoal, G. de Souza Matias, L. M. de Matos Jorge and R. O. Defendi, Mathematical modeling by fractional calculus applied to separation processes. *Separation and Purification Technology,* 337, 126310(2024).
[17] Y. H. Cheow, K. H. Ng, C. Phang and K. H. Ng, The application of fractional calculus in economic growth modelling: An approach based on regression analysis. *Heliyon*, 10 (**15**) (2024).
[18] S. Hasan, M. Al-Smadi, H. Dutta, S. Momani and S. Hadid, Multi-step reproducing kernel algorithm for solving Caputo-Fabrizio fractional stiff models arising in electric circuits. *Soft Computing*, 26(**8**), 3713-3727(2022).
[19] S. Hasan, M. Al-Smadi, A. El-Ajou, S. Momani, S. Hadid and Z. Al-Zhour, Numerical approach in the Hilbert space to solve a fuzzy Atangana-Baleanu fractional hybrid system. *Chaos, Solitons and Fractals,* 143, 110506 (2021).
[20] K. Oldham and J. Spanier, *The fractional calculus theory and applications of differentiation and integration to arbitrary order.* Elsevier (1974).
[21] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations.* Wiley,(1993).
[22] M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* 1(**2**), 1-13 (2015).
[23] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations,* (**Vol. 204**). Elsevier(2006).
[24] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative. *Journal of computational and applied mathematics,* **264**, 65-70(2014).
[25] M. Yavari and A. Nazemi, On fractional infinite-horizon optimal control problems with a combination of conformable and Caputo-Fabrizio fractional derivatives. *ISA transactions, 101*, 78-90(2020).
[26] T. Chiranjeevi and R. K. Biswas, Closed-form solution of optimal control problem of a fractional order system. *Journal of King Saud University-Science,* 31(**4**), 1042-1047(2019).
[27] Z. Gong, C. Liu, K. L. Teo and Y. Wu, A gradient-based optimization algorithm to solve optimal control problems with conformable fractional-order derivatives. *Journal of Computational and Applied Mathematics*, **454**, 116169(2025).
[28] S. Momani and N. Shawagfeh, Decomposition method for solving fractional Riccati differential equations. *Applied Mathematics and Computation,* 182(**2**), 1083-1092(2006).
[29] A. F. Jameel, N. R. Anakira,A.K. Alomari, M. Al-Mahameed, and A. Saaban, A new approximate solution of the fuzzy delay differential equations. *International Journal of Mathematical Modelling and Numerical Optimisation,* 9(**3**), 221-240(2019).
[30] O. Acan, O. Firat and Y. Keskin, Conformable variational iteration method, conformable fractional reduced differential transform method and conformable homotopy analysis method for non-linear fractional partial differential equations. *Waves in Random and Complex Media,* 30(**2**), 250-268(2020).
[31] Y. Zhang, A finite difference method for fractional partial differential equation. *Applied Mathematics and Computation,* 215(**2**), 524-529(2009).
[32] S. Hasan, B. Maayah, S. Bushnaq and S. Momani, A modified reproducing kernel Hilbert space method for solving fuzzy fractional integro-differential equations. *Boletim da Sociedade Paranaense de Matematica*, **41**, 1-16(2023).
[33] A. F. Jameel, N. R. Anakira, M. M. Rashidi, A. K. Alomari, A. Saaban, and M. A. Shakhatreh, Differential Transformation Method For Solving High Order Fuzzy Initial Value Problems. *Italian Journal of Pure and Applied Mathematics,* **39**, 194-208(2018).
[34] M. Ayata and O. Ozkan, A new application of conformable Laplace decomposition method for fractional Newell-Whitehead-Segel equation. *AIMS mathematics*, 5(**6**), 7402-7412(2020).
[35] K. Ilhem, M. Al Horani and R. R. Khalil, Solution of Non-linear Fractional Burger's Type Equations Using The Laplace Transform Decomposition Method. *Results in Nonlinear Analysis*, 5(**2**), 131-150(2022).
[36] M. Ayata and O. Ozkan, An analytical solution to conformable fractional fokker-planck equation. *Karaelmas Fen ve Muhendislik Dergisi,* 12(**1**), 9-14(2022).
[37] O. A. Arqub, Series solution of fuzzy differential equations under strongly generalized differentiability. J. *Adv. Res. Appl. Math*, 5(**1**), 31-52(2013).
[38] M. I. Liaqat, A. Khan and A. Akgul, Adaptation on power series method with conformable operator for solving fractional order systems of nonlinear partial differential equations. Chaos, Solitons and Fractals, **157**, 111984(2022).
[39] M. Al-Smadi, O. A. Arqub and S. Momani, Numerical computations of coupled fractional resonant Schrodinger equations arising in quantum mechanics under conformable fractional derivative sense. *Physica Scripta*, 95(**7**), 075218(2020).
[40] S. Hasan, M. Al-Smadi, S. Momani and O. A. Arqub, Residual power series approach for solving linear fractional

swift-hohenberg problems. *In Mathematical Methods and Modelling in Applied Sciences* (pp. 33-43). Springer International Publishing(2020).

[41] A. Shaiab, Laplace Adomian Decomposition Fractional Order SIS Epidemic Model. *AlQalam Journal of Medical and Applied Sciences,* 740-747 (2024).

[42] P. Dunnimit, W. Sawangtong and P. Sawangtong, An approximate analytical solution of the time-fractional Navier-Stokes equations by the generalized Laplace residual power series method. *Partial Differential Equations in Applied Mathematics*, 9, 100629(2024).

[43] N. Aguegboh, P. Roy Kiogora, M. Felix, W. Okongo and B. Diallo, Analytic solution of a fractional order hepatitis model using Laplace Adomian decomposition method and optimal control analysis. Computational and Mathematical Biophysics, 12(1). 20230114 (2024).

[44] A. O. Yunus and M. O. Olayiwola, Mathematical Modeling of Malaria Epidemic Dynamics with Enlightenment and Therapy Intervention Using the Laplace-Adomian Decomposition Method and Caputo Fractional Order. *Franklin Open*, 100147(2024).

[45] T. Abdeljawad, On conformable fractional calculus. *Journal of computational and Applied Mathematics*, **279**, 57-66(2015).

[46] F. Martinez, I. Martinez, M. K. Kaabar and S. Paredes, Some new results on conformable fractional power series. *Asia Pacific Journal of Mathematics*, 7(**31**), 1-14(2020).

[47] K. Abbaoui and Y. Cherruault, Convergence of Adomian's method applied to differential equations. *Computers and Mathematics with Applications*, 28(**5**), 103-109(1994).

[48] N. A. Khan, O. A. Razzaq and M. Ayaz, Some properties and applications of conformable fractional Laplace transform (CFLT). J. Fract. Calc. Appl, 9(1), 72-81(2018).

[49] M. Vivas-Cortez, J. N. Valdes, J. E. H. Hernandez, J. V. Velasco and O.Larreal, On non conformable fractional Laplace transform. *Appl. Math. Inf. Sci,* 15(**4**), 403-409(2021).

[50] J.Younis, B. Ahmed, M. AlJazzazi, R. Al Hejaj and H. Aydi, Existence and uniqueness study of the conformable Laplace transform. *Journal of Mathematics,* **2022**(**1**), 4554065 (2022).

Shatha Hasan graduated from the University of Jordan in 2006 with a B.S. in Mathematics; and an M.S. degree in Mathematics in 2009. In 2016, she received her Ph.D. from the University of Jordan in applied mathematics.She then began work at the Al-balqa applied university in 2017 as an assistant professor of applied mathematics. Recently, she joined Jadara Research Center with the aim of collaborating with new research groups. Her research interests are in the areas of applied mathematics specially in mathematical analytical methods, numerical analysis, dynamical systems, and fuzzy and fractional differential equations.

Mohammad Al Zurayqat is mathematical researcher who earned Master's degree from Al-Balqa Applied University in 2023 and preparing for the Phd in applied mathematics. His research interests encompass a diverse range of fields, including dynamical systems, economics, applied economics, geometric dynamics, and their real-world applications.

Mohammad Dradka is a researcher in applied fields of mathematics. He got the Master degree in Mathematics from Al-Balqa Applied University, granted in 2024. He is interested in discovering the action between mathematical theory and real-world phenomena in areas like dynamical systems, economics, and geometric dynamics. His main research interests are dynamical systems, patterns of growth and sustainable development, mathematical economics, game theory, optimization theory, applied economics, differential geometry and applications, geometric dynamics and applications.

Shaher Momani received his Ph.D from the university of Wales (UK) in 1991. He then began work at Mutah university in 1991 as assistant professor of applied mathematics and promoted to full Professor in 2006. He left Mutah university to the university of Jordan in 2009 until now. His research interests focus on the numerical solution of fractional differential equations in fluid mechanics, non-Newtonian fluid mechanics and numerical analysis. Prof. Momani has written over 500 research papers and warded several national and international prizes. Also, he was classified as one of the top ten scientists in the world in fractional differential equations according to ISI web of knowledge.