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Existence of Solution for a Dirichlet Problem with Tempered Fractional Derivatives

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Abstract: In this we deal with the existence of weak solution for a Dirichlet problem with tempered fractional derivatives. To this end, we use variational methods and critical point theory. Our results are new in the literature and as a consequence of our result we get an existence result to the Dirichlet problem with Riemann-Liouville fractional derivatives.

Keywords: Tempered fractional derivative, tempered fractional space of Sobolev type, variational methods, critical points.

1 Introduction

The fractional calculus has more than three centuries of history, and the development of fractional calculus theory is mainly focused on the pure mathematical field at the first stage. The earliest more or less systematic studies seem to have been made in the 19th century by Liouville, Riemann, Leibniz, Abel, etc. [1,2,3]. During the last four decades practical implementations emerged for the fractional calculus theory and it is now recognized to be an important tool to describe phenomena that classical integer-order calculus neglects [4,5,6,7,8,9,10]. We highlight in a special way, when it comes to applications in: medicine, engineering, physics, biology among other areas [1,2,11,12,13,14].

Tempered fractional calculus is a natural generalization of fractional calculus. The tempered fractional derivatives and integrals are obtained when the fractional derivatives and integrals are multiplied by an exponential factor [15,16,17]. Differential equations with tempered fractional derivatives have appeared in geophysics, statistical physics, plasma physics or in the context of astrophysics [18,19,20,21,22,17,23]. Moreover, tempered fractional calculus have been applied in finance [24].

In a recent work, Almeida and Morgado [25] introduced the tempered fractional calculus of variations. More precisely, they considered the functional

$$\begin{aligned} &\mathscr{M} : C^1[\varpi_1,\varpi_2] \to \mathbb{R} \\ &\eta \to \mathscr{M}(\eta) = \int_{\varpi_1}^{\varpi_2} \Theta(t,\eta(\xi),{}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\eta(\xi))d\xi, \end{aligned} \tag{1}$$

where $\nu \in (0,1)$, $\Xi \geq 0$ and $\Theta : [\varpi_1,\varpi_2]\times\mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable with respect to the second and third variables and showed that if η^* is a local minimizer of $\mathscr{M}$, subject to the boundary conditions

$$\eta(\varpi_1) = \beta_1 \quad \text{and} \quad \eta(\varpi_2) = \beta_2,$$

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then η^* is a solution of the Euler-Lagrange equation

$$\partial_2\Theta[\eta](\xi)+\mathbb{D}^{\nu,\Xi}_{\varpi_2^-}(\partial_3\Theta[\eta](\xi))=0, \quad \xi\in[\varpi_1,\varpi_2], \tag{2}$$

where $\Theta[\eta](\xi)=\Theta(\xi,\eta(\xi),\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\eta(\xi))$. In particular, if we choose

$$\Theta(\xi,\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\eta)=\frac{1}{2}[\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\eta]^2-\lambda\xi^2,$$

equation (2) can be rewritten as

$$\mathbb{D}^{\nu,\Xi}_{\varpi_2^-}(\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\eta(\xi))-\lambda\eta(\xi)=0,\ \ \xi\in[\varpi_1,\varpi_2].$$

Motivated by these previous works and the growing interest by the scientific community in considering tempered models, in this paper we would like to study variational structure for the tempered fractional derivative operator and we have justified some fundamental properties in the variational structure. Also, we investigate the following tempered fractional boundary value problem

$$\begin{cases} \mathbb{D}^{\nu,\Xi}_{\varpi_2^-}({}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u(\xi))=\daleth(\xi,u),\ \ x\in(\varpi_1,\varpi_2),\\ u(\varpi_1)=u(\varpi_2)=0, \end{cases} \tag{3}$$

where $\nu\in(\frac{1}{2},1)$ and $\Xi>0$ and $\daleth:(\varpi_1,\varpi_2)\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying some suitable conditions.

When $\Xi=0$, problem (3) reduces to the following problem:

$$\begin{cases} D^{\nu}_{\varpi_2^-}({}^C D^{\nu}_{\varpi_1^+}u(\xi))=\daleth(\xi,u),\ \ \xi\in(\varpi_1,\varpi_2),\\ u(\varpi_1)=u(\varpi_2)=0. \end{cases} \tag{4}$$

where $D^{\nu}_{\varpi_2^-}$ and ${}^C D^{\nu}_{\varpi_1^+}$ are the right Riemann-Liouville and left Caputo fractional derivatives of order ν, respectively. For some recent results on problem (4), we refer to for example [26,27] and the reference therein. Also, in the case $\daleth(x,u)=h(x,u)-V(\xi)u$, $\varpi_1=0$ and $\varpi_2=T$ and impulsive effects where V,h satisfies some general conditions, for example, see [28], [29], [30] for related discussions.

Now we make precise assumptions on $\daleth$, that is, we assume:

($\daleth_1$) $\liminf_{|\xi|\to+\infty}\frac{\widehat{\daleth}(\xi,x)}{x^{\vartheta}}>\delta_1$, for some $\delta_1>0$ and $\vartheta>2$, where $\widehat{\daleth}(\xi,u):=\int_0^u\daleth(\xi,s)ds$.

($\daleth_2$) There exist positive constants $\eta_1,M_1>0$ and $\zeta_1\geq 2$ such that for all $(\xi,u)\in[\varpi_1,\varpi_2]\times\mathbb{R}$, $|u|\geq M_1$

$$\widehat{\daleth}(\xi,u)\leq\eta_1|u|^{\zeta_1}.$$

($\daleth_3$) There exist positive constants $\eta_2,M_2>0$ and $\theta_1>\zeta_1-2$ such that for any $(\xi,u)\in[\varpi_1,\varpi_2]\times\mathbb{R}$, $|u|\geq M_2$

$$u\daleth(\xi,u)-2\widehat{\daleth}(\xi,u)\geq\eta_2|u|^{\theta_1}.$$

($\daleth_4$) $\limsup_{u\to 0}\frac{\max_{t\in[\varpi_1,\varpi_2]}\widehat{\daleth}(\xi,u)}{|u|^2}\leq\frac{1}{2(\varpi_2-\varpi_1)\left(\frac{[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}}{(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}\right)^2}$, where $\Psi(\cdot,\cdot)$ is defined as

$$\Psi(\nu,\xi)=\int_0^{\xi}t^{\nu-1}e^{-t}dt.$$

The first main result is as follows:

Theorem 1. *Let $\nu\in(\frac{1}{2},1)$ and $\Xi>0$. Moreover, assume that $\daleth$ satisfy $(\daleth_1)-(\daleth_4)$. Then (3) has at least one nontrivial weak solution.*

To state our second result, we suppose the following conditions:

($\daleth_5$) $|\daleth(\xi,u))|\leq C(|u|^{r-1}+1)$, for some $\xi\in[\varpi_1,\varpi_2]$, $r>2$, all $u\in\mathbb{R}$ and a positive constant $C>0$;

($\daleth_6$) $\lim_{|u|\to\infty}\left(\frac{1}{4}\daleth(\xi,u)u-\widehat{\daleth}(\xi,u)+\frac{\lambda_1}{2}u^2\right)=+\infty$ uniformly in $\xi\in[\varpi_1,\varpi_2]$;

($\daleth_7$) there exists $\mu>\lambda_1$ such that $\widehat{\daleth}(\xi,u)\geq\frac{\mu}{2}u^2$ for $|u|$ small;

$(\daleth_8)$ $\lim_{|u|\to\infty}\left(\frac{\lambda_1}{4}u^2-\widehat{\daleth}(\xi,u)\right)=+\infty$ uniformly in $\xi\in[\varpi_1,\varpi_2]$, where λ_1 is the first eigenvalue of the tempered fractional differential operator $\mathbb{D}^{\nu,\Xi}_{\varpi_2^-}({}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+})$.

The second main result is as follows:

Theorem 2. *Let $\nu\in(\frac{1}{2},1)$ and $\Xi>0$. Moreover, suppose that $(\daleth_5)-(\daleth_8)$ hold. Then (3) has at least one nontrivial solution.*

In the particular case $\Xi=0$, Theorem 1 and Theorem 2 give us an existence results to problem (4) which are new in the literature.

The rest of this article is arranged as follows In Section 2, we review the tempered fractional calculus theory. In Section 3, a tempered fractional space of Sobolev type and some principal properties is given. In Sections 4 and 5, we prove Theorem 1 and Theorem 2, respectively.

2 Preliminaries

For $\nu>0$ and $x\geq 0$, we define the incomplete Gamma function as

$$\Psi(\nu,\xi)=\int_0^{\xi}t^{\nu-1}e^{-t}dt,$$

which is convergen for all $\nu>0$. Moreover we have

$$e^{-\xi}\frac{\xi^{\nu}}{\nu}\leq\Psi(\nu,\xi)\leq\frac{\xi^{\nu}}{\nu}, \tag{5}$$

and integration by parts yields that

$$\Psi(\nu+1,\xi)=\nu\Psi(\nu,\xi)-\xi^{\nu}e^{-\xi}. \tag{6}$$

Also, if $\nu\in(-1,0)$ and $\xi>0$, then

$$\Psi(\nu,\xi)=\frac{1}{\nu}\Psi(\nu+1,\xi)+\frac{1}{\nu}\xi^{\nu}e^{-\xi}. \tag{7}$$

For more details, we can see [31].

Let $\nu\in(0,1)$ and $\Xi\geq 0$. According to [32], the left and right Riemann-Liouville tempered fractional integrals of order ν for a function u are defined as

$$\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u(\xi)=\frac{1}{\Gamma(\nu)}\int_{\varpi_1}^{\xi}(\xi-s)^{\nu-1}e^{-\Xi(\xi-s)}u(s)ds,\ \ \xi>\varpi_1, \tag{8}$$

and

$$\mathbb{I}^{\nu,\Xi}_{\varpi_2^-}u(\xi)=\frac{1}{\Gamma(\nu)}\int_{\xi}^{\varpi_1}(s-\xi)^{\nu-1}e^{-\Xi(s-\xi)}u(s)ds,\ \ \xi<\varpi_2, \tag{9}$$

respectively.

The following properties of these integrals hold.

1.For $\nu_1,\nu_2>0$, $\Xi\geq 0$ and for all $u\in L^p(\varpi_1,\varpi_2)$ with $p\in[1,\infty]$ we have

$$\mathbb{I}^{\nu_1,\Xi}_{a^+}\mathbb{I}^{\nu_2,\Xi}_{a^+}u(x)=\mathbb{I}^{\nu_1+\nu_2,\Xi}_{a^+}u(x)\quad\text{and}\quad\mathbb{I}^{\nu_1,\Xi}_{b^-}\mathbb{I}^{\nu_2,\Xi}_{b^-}u(x)=\mathbb{I}^{\nu_1+\nu_2,\Xi}_{b^-}u(x).$$

2.For any $\nu>0$, $\Xi\geq 0$ and $p\geq 1$, $\mathbb{I}^{\nu,\Xi}_{a^+}$ is bounded in $L^p(\varpi_1,\varpi_2)$. Moreover

$$\|\mathbb{I}^{\nu,\Xi}_{a^+}u\|_{L^p(\varpi_1,\varpi_2)}\leq\frac{(\varpi_2-\varpi_1)^{\nu}}{\Gamma(\nu+1)}\|e^{\Xi\cdot}u(\cdot)\|_{L^p(\varpi_1,\varpi_2)}.$$

For more details see [32].

Moreover, for $\nu \in (0,1)$ and $\Xi > 0$, the left and right Riemann-Liouville tempered fractional derivatives of order ν for a function u are defined as

$$\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u(\xi) = (e^{-\Xi\xi}{}_{\varpi_1}D^{\nu}_{\xi}e^{\Xi\xi})u(\xi) = \frac{e^{-\Xi\xi}}{\Gamma(1-\nu)}\frac{d}{d\xi}\int_{\varpi_1}^{\xi}(\xi-s)^{-\nu}e^{\Xi s}u(s)ds, \ \ \xi > \varpi_1, \tag{10}$$

and

$$\mathbb{D}^{\nu;\Xi}_{\varpi_2^-}u(\xi) = (e^{\Xi\xi}{}_{x}D^{\nu}_{\varpi_2}e^{-\Xi\xi})u(\xi) = \frac{-e^{\Xi x}}{\Gamma(1-\nu)}\frac{d}{d\xi}\int_{\xi}^{\varpi_2}(s-\xi)^{-\nu}e^{-\Xi s}u(s)ds, \ \ \xi < \varpi_2, \tag{11}$$

respectively. Also, the left-sided and right-sided tempered Caputo fractional derivatives of order ν for a function u are defined as

$$^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u(\xi) = \left(e^{-\Xi\xi}\,{}^{C}_{\varpi_1}D^{\nu}_{\xi}e^{\Xi\xi}\right)u(\xi) = \frac{e^{-\Xi\xi}}{\Gamma(1-\nu)}\int_{\varpi_1}^{\xi}(\xi-s)^{-\nu}[e^{\Xi s}u(s)]'ds, \ \ \xi > \varpi_1, \tag{12}$$

and

$$^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_2^-}u(\xi) = \left(e^{\Xi x}\,{}^{C}_{\xi}D^{\nu}_{\varpi_2}e^{-\Xi\xi}\right)u(\xi) = \frac{-e^{\Xi\xi}}{\Gamma(1-\nu)}\int_{\xi}^{\varpi_2}(s-\xi)^{-\nu}[e^{-\Xi s}u(s)]'ds, \ \ \xi < \varpi_2, \tag{13}$$

respectively.

Lemma 1. *[33] Let $\nu > 0$, $\Xi > 0$ and $u \in AC[\varpi_1,\varpi_2]$. Then $\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u, \mathbb{I}^{\nu,\Xi}_{\varpi_2^-}u$ are well defined. Furthermore,*

$$\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u(\xi) = \frac{u(a)}{\Xi^{\nu}\Gamma(\nu)}\Psi(\nu,\Xi(\xi-\varpi_1)) + \frac{1}{\Xi^{\nu}\Gamma(\nu)}\int_{\varpi_1}^{\xi}\Psi(\nu,\Xi(\xi-s))u'(s)ds, \tag{14}$$

and

$$\mathbb{I}^{\nu,\Xi}_{\varpi_2^-}u(\xi) = \frac{u(\varpi_2)}{\Xi^{\nu}\Gamma(\nu)}\Psi(\nu,\Xi(\varpi_2-\xi)) - \frac{1}{\Xi^{\nu}\Gamma(\nu)}\int_{\xi}^{\varpi_2}\Psi(\nu,\Xi(s-\xi))u'(s)ds. \tag{15}$$

Moreover we have

Theorem 3. *[33] Let $\nu \in (0,1)$, $\Xi > 0$, $p \in [1,\infty]$. Then $\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}, \mathbb{I}^{\nu,\Xi}_{\varpi_2^-} : L^p(\varpi_1,\varpi_2) \to L^p(\varpi_1,\varpi_2)$ are bounded. Furthermore,*

$$\|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u\|_{L^p(\varpi_1,\varpi_2)} \leq \frac{\Psi(\nu,\Xi(\varpi_2-\varpi_1))}{\Xi^{\nu}\Gamma(\nu)}\|u\|_{L^p(\varpi_1,\varpi_2)}, \tag{16}$$

and

$$\|\mathbb{I}^{\nu,\Xi}_{\varpi_2^-}u\|_{L^p(\varpi_1,\varpi_2)} \leq \frac{\Psi(\nu,\Xi(\varpi_2-\varpi_1))}{\Xi^{\nu}\Gamma(\nu)}\|u\|_{L^p(\varpi_1,\varpi_2)}. \tag{17}$$

Theorem 4. *[33] Let $\nu \in \left(0,\frac{1}{p}\right)$, $p > 1$, $\Xi > 0$ and $p^*_\nu = \frac{p}{1-\nu p}$. Then $\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}, \mathbb{I}^{\nu,\Xi}_{\varpi_2^-}$ are bounded from $L^p(\varpi_1,\varpi_2)$ into $L^{p^*_\nu}(\varpi_1,\varpi_2)$.*

Using ideas of [34], we have the following integration by parts.

Theorem 5. *Let $\nu \in (0,1)$, $\Xi > 0$, $1 < p < \infty$, $1 < q < \infty$ and*

$$\frac{1}{p} + \frac{1}{q} \leq 1 + \nu.$$

If $\digamma_1 \in L^p(\varpi_1,\varpi_2)$ and $\digamma_2 \in L^q(\varpi_1,\varpi_2)$, then

$$\int_{\varpi_1}^{\varpi_2}\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}\digamma_1(\xi)\digamma_2(\xi)d\xi = \int_{\varpi_1}^{\varpi_2}\digamma_1(\xi)\mathbb{I}^{\nu,\Xi}_{\varpi_2^-}\digamma_2(\xi)d\xi. \tag{18}$$

Theorem 6. *[33] Let $\nu \in (0,1)$, $\Xi > 0$ and $u \in C[\varpi_1,\varpi_2]$. Then $\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u, \mathbb{I}^{\nu,\Xi}_{\varpi_2^-}$ are continuos on $[\varpi_1,\varpi_2]$ and*

$$\lim_{\xi\to\varpi_1^+} \mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u(\xi) = 0 \quad \text{and} \quad \lim_{\xi\to\varpi_2^-} \mathbb{I}^{\nu,\Xi}_{\varpi_2^-}u(\xi) = 0. \tag{19}$$

Moreover,

$$\|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u\|_\infty \le \frac{1}{\Xi^\nu\Gamma(\nu)}\Psi(\nu,\Xi(\varpi_2-\varpi_1))\|u\|_\infty,$$

and

$$\|\mathbb{I}^{\nu,\Xi}_{\varpi_2^-}u\|_\infty \le \frac{1}{\Xi^\nu\Gamma(\nu)}\gamma(\nu,\Xi(\varpi_2-\varpi_1))\|u\|_\infty.$$

The following result were considered by Torres et. all. [33, Theorem 3.9]. More precisely, assuming that $\nu \in (\frac{1}{2},1)$, $\Xi > 0$ and $u \in L^2(\varpi_1,\varpi_2)$, then Torres et. all. proved that $\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u, \mathbb{I}^{\nu,\Xi}_{\varpi_2^-}u \in C[\varpi_1,\varpi_2]$. We note that under a carefully analysis we are able to prove that $\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u, \mathbb{I}^{\nu,\Xi}_{\varpi_2^-}u$ are Hölder continuous with order $\nu-\frac{1}{2}$. We need the following inequality To state our result: For any $x_1 \ge x_2 \ge 0$ and $q \ge 1$

$$(x_1-x_2)^q \le x_1^q - x_2^q. \tag{20}$$

Theorem 7. *Let $\nu \in (\frac{1}{2},1)$ and $\Xi > 0$. Then, for any $u \in L^2(\varpi_1,\varpi_2)$, $\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u \in H^{\nu-\frac{1}{2}}(\varpi_1,\varpi_2)$ and*

$$\lim_{\xi\to\varpi_1^+} \mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u(\xi) = 0,$$

where $H^{\nu-\frac{1}{2}}(\varpi_1,\varpi_2)$ denotes the Hölder space of order $\nu-\frac{1}{2} > 0$.

Proof. Let $\varpi_1 < x_1 < x_1 \le \varpi_2$ and $u \in L^2(\varpi_1,\varpi_2)$, then by Hölder inequality

$$\begin{aligned}
&|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u(x_1) - \mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u(x_2)| \\
&\le \frac{1}{\Gamma(\nu)}\left[\int_{\varpi_1}^{x_1}\left|(x_1-s)^{\nu-1}e^{-\Xi(x_1-s)} - (x_2-s)^{\nu-1}e^{-\Xi(x_2-s)}\right||u(s)|ds\right. \\
&\left. + \int_{x_1}^{x_2}(x_2-s)^{\nu-1}e^{-\Xi(x_2-s)}|u(s)|ds\right] \\
&\le \frac{1}{\Gamma(\nu)}\left(\int_{\varpi_1}^{x_1}\left|(x_1-s)^{\nu-1}e^{-\Xi(x_1-s)} - (x_2-s)^{\nu-1}e^{-\Xi(x_2-s)}\right|^2 ds\right)^{1/2}\left(\int_{\varpi_1}^{x_1}|u(s)|^2ds\right)^{1/2} \\
&+ \frac{1}{\Gamma(\nu)}\left(\int_{x_1}^{x_2}(x_2-s)^{2\nu-2}e^{-2\Xi(x_2-s)}ds\right)^{1/2}\left(\int_{x_1}^{x_2}|u(s)|^2ds\right)^{1/2}.
\end{aligned} \tag{21}$$

Using (5) and the change of variable $t = 2\Xi(x_2-s)$, we get

$$\int_{x_1}^{x_2}(x_2-s)^{2\nu-2}e^{-2\Xi(x_2-s)}ds = \frac{1}{(2\Xi)^{2\nu-1}}\Psi(2\nu-1,2\Xi(x_2-x_1)) \le \frac{(x_2-x_1)^{2\nu-1}}{2\nu-1}.$$

Hence

$$\left(\int_{x_1}^{x_2}(x_2-s)^{2\nu-2}e^{-2\Xi(x_2-s)}ds\right)^{1/2} \le \frac{1}{(2\nu-1)^{1/2}}(x_2-x_1)^{\nu-\frac{1}{2}}. \tag{22}$$

Also, the change of variable $t = \frac{x_1-s}{x_2-x_1}$ yields that

$$\begin{aligned}
&\int_{\varpi_1}^{x_1}\left|(x_1-s)^{\nu-1}e^{-\Xi(x_1-s)} - (x_2-s)^{\nu-1}e^{-\Xi(x_2-s)}\right|^2 ds \\
&= (x_2-x_1)^{2\nu-1}\int_0^{\frac{x_1-a}{x_2-x_1}}\left|t^{\nu-1}e^{-\Xi t(x_2-x_1)} - (1+t)^{\nu-1}e^{-\Xi(1+t)(x_2-x_1)}\right|^2 dt.
\end{aligned} \tag{23}$$

So, if $\frac{x_1-\varpi_1}{x_2-x_1} \leq 1$, by (20) we derive

$$\begin{aligned}
&\int_0^{\frac{x_1-\varpi_1}{x_2-x_1}} \left|t^{\nu-1}e^{-\Xi t(x_2-x_1)} - (1+t)^{\nu-1}e^{-\Xi(1+t)(x_2-x_1)}\right|^2 dt \\
&\leq \int_0^1 \left|t^{\nu-1}e^{-\Xi t(x_2-x_1)} - (1+t)^{\nu-1}e^{-\Xi(1+t)(x_2-x_1)}\right|^2 dt \\
&\leq \int_0^1 \left(t^{2\nu-2}e^{-2\Xi t(x_2-x_1)} - (1+t)^{2\nu-2}e^{-2\Xi(1+t)(x_2-x_1)}\right) dt \\
&= \frac{1}{[2\Xi(x_2-x_1)]^{2\nu-1}}\Big(2\Psi(2\nu-1, 2\Xi(x_2-x_1)) - \Psi(2\nu-1, 4\Xi(x_2-x_1))\Big).
\end{aligned}$$

Now, note that by (5) we obtain

$$\Psi(2\nu-1, 2\Xi(x_2-x_1)) \leq \frac{[2\Xi(x_2-x_1)]^{2\nu-1}}{2\nu-1},$$

and

$$e^{-4\Xi(x_2-x_1)}\frac{[4\Xi(x_2-x_1)]^{2\nu-1}}{2\nu-1} \leq \Psi(2\nu-1, 4\Xi(x_2-x_1)),$$

consequently,

$$2\Psi(2\nu-1, 2\Xi(x_2-x_1)) - \Psi(2\nu-1, 4\Xi(x_2-x_1)) \leq \frac{[2\Xi(x_2-x_1)]^{2\nu-1}}{2\nu-1}\left(2 - 2^{2\nu-1}e^{-4\Xi(x_2-x_1)}\right).$$

Therefore, replacing in (21) we obtain

$$\begin{aligned}
&\int_{\varpi_1}^{x_1} \left|(x_1-s)^{\nu-1}e^{-\Xi(x_1-s)} - (x_2-s)^{\nu-1}e^{-\Xi(x_2-s)}\right|^2 ds \\
&\leq \frac{(x_2-x_1)^{2\nu-1}}{2\nu-1}\left(2 - 2^{2\nu-1}e^{-4\Xi(x_2-x_1)}\right) \leq \frac{2}{2\nu-1}(x_2-x_1)^{2\nu-1}.
\end{aligned} \tag{24}$$

Also, if $\frac{x_1-\varpi_1}{x_2-x_1} > 1$, then

$$\begin{aligned}
&\int_0^{\frac{x_1-\varpi_1}{x_2-x_1}} \left|t^{\nu-1}e^{-\Xi t(x_2-x_1)} - (1+t)^{\nu-1}e^{-\Xi(x_2-x_1)(1+t)}\right|^2 dt \\
&= \int_0^1 \left|t^{\nu-1}e^{-\Xi t(x_2-x_1)} - (1+t)^{\nu-1}e^{-\Xi(x_2-x_1)(1+t)}\right|^2 dt \\
&+ \int_1^{\frac{x_1-a}{x_2-x_1}} \left|t^{\nu-1}e^{-\Xi t(x_2-x_1)} - (1+t)^{\nu-1}e^{-\Xi(x_2-x_1)(1+t)}\right|^2 dt \\
&\leq \frac{2}{2\nu-1} + \int_1^{\frac{x_1-a}{x_2-x_1}} \left|t^{\nu-1}e^{-\Xi t(x_2-x_1)} - (1+t)^{\nu-1}e^{-\Xi(x_2-x_1)}(1+t)\right|^2 dt.
\end{aligned} \tag{25}$$

In view of he change of variable $\lambda = 2\Xi t(x_2-x_1)$ and the mean value theorem, we get

$$\begin{aligned}
&\int_1^{\frac{x_1-\varpi_1}{x_2-x_1}} \left|t^{\nu-1}e^{-\Xi t(x_2-x_1)} - (1+t)^{\nu-1}e^{-\Xi(1+t)(x_2-x_1)}\right|^2 dt \\
&= \int_1^{\frac{x_1-\varpi_1}{x_2-x_1}} \left((1-\nu)r^{\nu-2}e^{-\Xi t(x_2-x_1)} + \Xi(x_2-x_1)t^{\nu-1}e^{-\Xi t(x_2-x_1)}\right)^2 dt \\
&\leq 2(1-\nu)^2\int_1^{\frac{x_1-\varpi_1}{x_2-x_1}} t^{2\nu-4}e^{-2\Xi t(x_2-x_1)}dt + 2\Xi^2(x_2-x_1)^2\int_1^{\frac{x_1-\varpi_1}{x_2-x_1}} t^{2\nu-2}e^{-2\Xi t(x_2-x_1)}dt \\
&= \frac{2(1-\nu)^2}{(2\Xi)^{2\nu-3}}(x_2-x_1)^{3-2\nu}\int_{2\Xi(x_2-x_1)}^{2\Xi(x_1-\varpi_1)} \lambda^{2\nu-4}e^{-\lambda}d\lambda + \frac{2\Xi^2}{(2\Xi)^{2\nu-1}}(x_2-x_3)^{3-2\nu}\int_{2\Xi(x_2-x_1)}^{2\Xi(x_1-\varpi_1)} \lambda^{2\nu-2}e^{-\lambda}d\lambda.
\end{aligned}$$

Note that, integrating by parts the first integral of the last expression we get

$$\int_{2\Xi(x_2-x_1)}^{2\Xi(x_1-\varpi_1)}\lambda^{2\nu-4}e^{-\lambda}d\lambda=\frac{(2\Xi)^{2\nu-3}}{2\nu-3}\left((x_1-\varpi_1)^{2\nu-3}e^{-2\Xi(x_1-\varpi_1)}-(x_2-x_1)^{2\nu-3}e^{-2\Xi(x_2-x_1)}\right)$$
$$+\frac{(2\Xi)^{2\nu-2}}{(2\nu-3)(2\nu-2)}\left((x_1-\varpi_1)^{2\nu-2}e^{-2\Xi(x_1-\varpi_1)}-(x_2-x_1)^{2\nu-2}e^{-2\Xi(x_2-x_1)}\right)$$
$$+\frac{1}{(2\nu-2)(2\nu-3)}\int_{2\Xi(x_2-x_1)}^{2\Xi(x_1-\varpi_1)}\lambda^{2\nu-2}e^{-\lambda}d\lambda.$$

Consequently, replacing in the last inequality we derive

$$\int_1^{\frac{x_1-\varpi_1}{x_2-x_1}}\left|t^{\nu-1}e^{-\Xi t(x_2-x_1)}-(1+t)^{\nu-1}e^{-\Xi(1+t)(x_2-x_1)}\right|^2dt$$
$$\leq\frac{2(1-\nu)^2}{(2\Xi)^{2\nu-3}}(x_2-x_1)^{3-2\nu}\int_{2\Xi(x_2-x_1)}^{2\Xi(x_1-\varpi_1)}\lambda^{2\nu-4}e^{-\lambda}d\lambda+\frac{2\Xi^2}{(2\Xi)^{2\nu-1}}(x_2-x_3)^{3-2\nu}\int_{2\Xi(x_2-x_1)}^{2\Xi(x_1-\varpi_1)}\lambda^{2\nu-2}e^{-\lambda}d\lambda$$
$$=\frac{2(1-\nu)^2}{(2\Xi)^{2\nu-3}}(x_2-x_1)^{3-2\nu}\left[\frac{(2\Xi)^{2\nu-3}}{2\nu-3}\left((x_1-\varpi_1)^{2\nu-3}e^{-2\Xi(x_1-\varpi_1)}-(x_2-x_1)^{2\nu-3}e^{-2\Xi(x_2-x_1)}\right)\right.$$
$$\left.+\frac{(2\Xi)^{2\nu-2}}{(2\nu-3)(2\nu-2)}\left((x_1-\varpi_1)^{2\nu-2}e^{-2\Xi(x_1-\varpi_1)}-(x_2-x_1)^{2\nu-2}e^{-2\Xi(x_2-x_1)}\right)\right]$$
$$+\left(\frac{2\Xi^2}{(2\Xi)^{2\nu-1}}+\frac{2(1-\nu)^2}{(2\Xi)^{2\nu-3}(2\nu-2)(2\nu-3)}\right)(x_2-x_3)^{3-2\nu}\int_{2\Xi(x_2-x_1)}^{2\Xi(x_1-\varpi_1)}\lambda^{2\nu-2}e^{-\lambda}d\lambda$$
$$\leq\frac{2(1-\nu)^2}{2\nu-3}(x_2-x_1)^{3-2\nu}\left((x_1-\varpi_1)^{2\nu-3}e^{-2\Xi(x_1-\varpi_1)}-(x_2-x_1)^{2\nu-3}e^{-2\Xi(x_2-x_1)}\right)$$
$$+\left(\frac{2\Xi^2}{(2\Xi)^{2\nu-1}}+\frac{2(1-\nu)^2}{(2\Xi)^{2\nu-3}(2\nu-2)(2\nu-3)}\right)(x_2-x_3)^{3-2\nu}\int_{2\Xi(x_2-x_1)}^{2\Xi(x_1-\varpi_1)}\lambda^{2\nu-2}e^{-\lambda}d\lambda.$$

By other side, (5) yields that

$$\int_{2\Xi(x_2-x_1)}^{2\Xi(x_1-\varpi_1)}\lambda^{2\nu-2}e^{-\lambda}d\lambda=\Psi(2\nu-1,2\Xi(x_1-\varpi_1))-\Psi(2\nu-1,2\Xi(x_2-x_1))$$
$$\leq\frac{(2\Xi)^{2\nu-1}}{2\nu-1}(x_2-x_1)^{2\nu-1}\left(\left(\frac{x_1-\varpi_1}{x_2-x_1}\right)^{2\nu-1}-e^{-2\Xi(x_2-x_1)}\right).$$

Hence

$$\int_1^{\frac{x_1-\varpi_1}{x_2-x_1}}\left|t^{\nu-1}e^{-\Xi t(x_2-x_1)}-(1+t)^{\nu-1}e^{-\Xi(1+t)(x_2-x_1)}\right|^2dt$$
$$\leq\frac{2(1-\nu)^2}{2\nu-3}\left(\left(\frac{x_1-a}{x_2-x_1}\right)^{2\nu-3}e^{-2\Xi(x_1-a)}-e^{-2\Xi(x_2-x_1)}\right)\tag{26}$$
$$+\left(\frac{2\Xi^2}{2\nu-1}+\frac{8(1-\nu)^2\Xi^2}{(2\nu-1)(2\nu-2)(2\nu-3)}\right)(x_2-x_1)^2\left(\left(\frac{x_1-\varpi_1}{x_2-x_1}\right)^{2\nu-1}-e^{-2\Xi(x_2-x_1)}\right).$$

Finally, combining (23) with (25) and (26) we derive

$$\int_{\varpi_1}^{x_1}\left|(x_1-s)^{\nu-1}e^{-\Xi(x_1-s)}-(x_2-s)^{\nu-1}e^{-\Xi(x_2-s)}\right|^2ds\leq\mathfrak{M}(x_2-x_1)^{2\nu-1},\tag{27}$$

where

$$\mathfrak{M}=\frac{2}{2\nu-1}+\frac{2(1-\nu)^2}{2\nu-3}\left(\left(\frac{x_1-\varpi_1}{x_2-x_1}\right)^{2\nu-3}e^{-2\Xi(x_1-a)}-e^{-2\Xi(x_2-x_1)}\right)$$
$$+\left(\frac{2\Xi^2}{2\nu-1}+\frac{8(1-\nu)^2\Xi^2}{(2\nu-1)(2\nu-2)(2\nu-3)}\right)(x_2-x_1)^2\left(\left(\frac{x_1-\varpi_1}{x_2-x_1}\right)^{2\nu-1}-e^{-2\Xi(x_2-x_1)}\right).$$

Therefore, by (21), (22), (24) and (27), we get

$$
\begin{aligned}
&|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u(x_1)-\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u(x_2)|\\
&\leq \frac{1}{\Gamma(\nu)}\left(\int_{\varpi_1}^{x_1}\left|(x_1-s)^{\nu-1}e^{-\Xi(x_1-s)}-(x_2-s)^{\nu-1}e^{-\Xi(x_2-s)}\right|^2ds\right)^{1/2}\left(\int_{\varpi_1}^{x_1}|u(s)|^2ds\right)^{1/2}\\
&+\frac{1}{\Gamma(\nu)}\left(\int_{x_1}^{x_2}(x_2-s)^{2\nu-2}e^{-2\Xi(x_2-s)}ds\right)^{1/2}\left(\int_{x_1}^{x_2}|u(s)|^2ds\right)^{1/2}\\
&\leq \frac{\mathfrak{M}^{1/2}}{\Gamma(\nu)}(x_2-x_1)^{\nu-\frac{1}{2}}\|u\|_{L^2(\varpi_1,\varpi_2)}+\frac{1}{(2\nu-1)^{1/2}\Gamma(\nu)}(x_2-x_1)^{\nu-\frac{1}{2}}\|u\|_{L^2(\varpi_1,\varpi_2)}\\
&=\frac{1}{\Gamma(\nu)}\left(\mathfrak{M}^{1/2}+\frac{1}{(2\nu-1)^{1/2}}\right)\|u\|_{L^2(\varpi_1,\varpi_2)}(x_2-x_1)^{\nu-\frac{1}{2}},
\end{aligned}
$$

which implies that $\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u\in H^{\nu-\frac{1}{2}}(\varpi_1,\varpi_2)$.

Finally, by $u\in L^2(\varpi_1,\varpi_2)$ and Hölder inequality, we obtain

$$
\begin{aligned}
|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u(\xi)| &\leq \frac{1}{\Gamma(\nu)}\int_{\varpi_1}^{\xi}(\xi-s)^{\nu-1}e^{-\Xi(\xi-s)}|u(s)|ds\\
&\leq \frac{1}{\Gamma(\nu)}\left(\int_{\varpi_1}^{\xi}(\xi-s)^{2(\nu-1)}e^{-\Xi 2(\xi-s)}ds\right)^{1/2}\left(\int_{\varpi_1}^{\xi}|u(s)|^2ds\right)^{1/2}\\
&\leq \frac{1}{2^{\nu-\frac{1}{2}}\Gamma(\nu)}\frac{[\Psi(2\nu-1,2\Xi(\xi-\varpi_1))]^{1/2}}{\Xi^{\nu-\frac{1}{2}}}\|u\|_{L^2(\varpi_1,\varpi_2)}.
\end{aligned}
$$

Furthermore, (5) yields that

$$
e^{-\Xi(\xi-\varpi_1)}\frac{2^{\nu-\frac{1}{2}}}{(2\nu-1)^{1/2}}(\xi-\varpi_1)^{\nu-\frac{1}{2}}\leq\frac{(\Psi(2\nu-1,2\Xi(x-a)))^{1/2}}{\Xi^{\nu-\frac{1}{2}}}\leq\frac{2^{\nu-\frac{1}{2}}}{(2\nu-1)^{1/2}}(\xi-\varpi_1)^{\nu-\frac{1}{2}},
$$

which implies

$$
\lim_{\xi\to\varpi_1^+}\frac{(\Psi(2\nu-1,2\Xi(\xi-\varpi_1)))^{1/2}}{\Xi^{\nu-\frac{1}{2}}}=0.
$$

So, combining this limit with the last inequality we get

$$
\lim_{\xi\to\varpi_1^+}\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}u(\xi)=0.
$$

We also have the following result:

Theorem 8. *Let $\nu\in(0,1)$, $\Xi>0$ and $u\in AC[\varpi_1,\varpi_2]$. Then*

$$
\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u(\xi)=\frac{u(\varpi_1)}{\Gamma(1-\nu)}(\xi-\varpi_1)^{-\nu}e^{-\Xi(\xi-\varpi_1)}+{}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u(\xi), \tag{28}
$$

and

$$
\mathbb{D}^{\nu,\Xi}_{\varpi_2^-}u(\xi)=\frac{u(\varpi_2)}{\Gamma(1-\nu)}(\varpi_2-\xi)^{-\nu}e^{-\Xi(\varpi_2-\xi)}+{}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_2^-}u(\xi). \tag{29}
$$

Proof. Using the definition, we get

$$
\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u(\xi)=e^{-\Xi\xi}\frac{d}{d\xi}{}_{\varpi_1}I_{\xi}^{1-\nu}(e^{\Xi\xi}u(\xi)).
$$

Note that, for any function $v\in AC[\varpi_1,\varpi_2]$, the Riemann-Liouville fractional integral has the following representation

$$
{}_{\varpi_1}I_{\xi}^{\nu}v(\xi)=\frac{v(\varpi_1)}{\nu\Gamma(\nu)}(\xi-\varpi_1)^{\nu}+\frac{1}{\nu\Gamma(\nu)}\int_{\varpi_1}^{\xi}(\xi-s)^{\nu}v'(s)ds.
$$

Combining this equality with the fact that $e^{\Xi \cdot}u \in AC[\varpi_1, \varpi_2]$ we derive

$$_{\varpi_1}I_{\xi}^{1-\nu}e^{\Xi\xi}u(\xi) = \frac{e^{\Xi\varpi_1}u(\varpi_1)}{(1-\nu)\Gamma(1-\nu)}(\xi-\varpi_1)^{1-\nu} + \frac{1}{(1-\nu)\Gamma(1-\nu)}\int_{\varpi_1}^{\xi}(\xi-s)^{1-\nu}[e^{\Xi s}u(s)]'ds.$$

Hence

$$\mathbb{D}_{\varpi_1^+}^{\nu,\Xi}u(\xi) = \frac{e^{-\Xi(\xi-\varpi_1)}u(\xi)}{\Gamma(1-\nu)}(\xi-\varpi_1)^{-\nu} + \frac{e^{-\Xi\xi}}{\Gamma(1-\nu)}\int_{\varpi_1}^{\xi}(\xi-s)^{-\nu}[e^{\Xi s}u(s)]'ds,$$

consequently, we get (28).

Since for any $v \in AC[\varpi_1, \varpi_2]$ we have

$$_{\xi}I_{\varpi_2}^{\nu}v(x) = \frac{u(\varpi_2)}{\nu\Gamma(\nu)}(\varpi_2-\xi)^{\nu} - \frac{1}{\nu\Gamma(\nu)}\int_{\xi}^{\varpi_2}(s-\xi)^{\nu}v'(s)ds.$$

Following the same lines as above we get (29).

*Remark.*Note that (28) and (29) were considered without any proof and with a misprint by [17].

Theorem 9. *For $\nu \in (0,1)$, $\Xi > 0$ and $u \in AC[\varpi_1, \varpi_2]$, we have*

1.

$$^{C}\mathbb{D}_{\varpi_1^+}^{\nu,\Xi}\mathbb{I}_{\varpi_1^+}^{\nu,\Xi}u(\xi) = u(\xi),$$
$$^{C}\mathbb{D}_{\varpi_2^-}^{\nu,\Xi}\mathbb{I}_{\varpi_2^-}^{\nu,\Xi}u(\xi) = u(\xi).$$

2.

$$\mathbb{I}_{\varpi_1^+}^{\nu,\Xi}{}^{C}\mathbb{D}_{\varpi_1^+}^{\nu,\Xi}u(\xi) = u(\xi) - e^{-\Xi(\xi-\varpi_1)}u(\varpi_1),$$
$$\mathbb{I}_{\varpi_2^-}^{\nu,\Xi}{}^{C}\mathbb{D}_{\varpi_2^-}^{\nu,\Xi}u(\xi) = u(\xi) - e^{-\Xi(\xi-\varpi_2)}u(\varpi_2).$$

The integration by parts theorem for Riemann-Liouville tempered fractional derivative is as follows:

Theorem 10. *Let $\nu \in (0,1)$, $\Xi > 0$ and $u, v \in AC[\varpi_1, \varpi_2]$, then*

$$\int_{\varpi_1}^{\varpi_2}u(\xi)\mathbb{D}_{\varpi_1^-}^{\nu,\Xi}v(x)dx = \lim_{x\to a^+}u(x)\mathbb{I}_{\varpi_2^-}^{1-\nu,\Xi}v(x) - \lim_{x\to\varpi_2^-}u(x)\mathbb{I}_{\varpi_2^-}^{1-\nu,\Xi}v(x) + \int_{\varpi_1}^{\varpi_2}{}^{C}\mathbb{D}_{a^+}^{\nu,\Xi}u(x)v(x)dx. \tag{30}$$

Proof. Note that, as in Lemma 1 we can show that, if $\varphi \in AC[\varpi_1, \varpi_2]$, then $_{\varpi_1}I_x^{\nu}\varphi, {}_xI_{\varpi_2}^{\nu}\varphi \in AC[\varpi_1, \varpi_2]$. Hence, $_xI_{\varpi_2}^{1-\nu}e^{-\Xi\cdot}v \in AC[\varpi_1, \varpi_2]$ and then

$$_xD_{\varpi_2}^{\nu}e^{-\Xi x}v(x) = -\frac{d}{dx}{}_xI_{\varpi_2}^{1-\nu}e^{-\Xi x}v(x) \in L^1[\varpi_1, \varpi_2].$$

Consequently,

$$\int_{\varpi_1}^{\varpi_2}|\mathbb{D}_{\varpi_2^-}^{\nu,\Xi}v(\xi)|d\xi = \int_{\varpi_1}^{\varpi_2}|e^{\Xi x}{}_{\xi}D_{\varpi_2}^{\nu}e^{-\Xi\xi}v(\xi)|d\xi \le e^{\Xi b}\int_{\varpi_1}^{\varpi_2}|_{\xi}D_{\varpi_2}^{\nu}e^{-\Xi\xi}v(\xi)|d\xi < \infty.$$

By other side, if $u \in AC[\varpi_1, \varpi_2]$, then $u \in C[\varpi_1, \varpi_2]$. Hence,

$$\int_{\varpi_1}^{\varpi_2}u(\xi)\mathbb{D}_{\varpi_2^-}^{\nu,\Xi}v(\xi)d\xi \le \left|\int_{\varpi_1}^{\varpi_2}u(\xi)\mathbb{D}_{\varpi_2^-}^{\nu,\Xi}v(\xi)d\xi\right| \le \|u\|_{\infty}\int_{\varpi_1}^{\varpi_2}|\mathbb{D}_{\varpi_2^-}^{\nu,\Xi}v(\xi)|d\xi < \infty.$$

Now, we are going to show (30). So, by Theorem 5 and integration by parts, we obtain

$$\begin{aligned}\int_{\varpi_1}^{\varpi_2}u(\xi)\mathbb{D}_{\varpi_2^-}^{\nu,\Xi}v(\xi)d\xi &= \int_{\varpi_1}^{\varpi_2}u(\xi)e^{\Xi\xi}{}_{\xi}D_{\varpi_2}^{\nu}e^{-\Xi\xi}v(\xi)d\xi = -\int_{\varpi_1}^{\varpi_2}u(\xi)e^{\Xi\xi}\frac{d}{d\xi}{}_{\xi}I_{\varpi_2}^{1-\nu}e^{-\Xi\xi}v(\xi)d\xi \\ &= -\left[u(\xi)\mathbb{I}_{\varpi_2^-}^{1-\nu,\Xi}v(\xi)\Big|_{\varpi_1}^{\varpi_2} - \int_{\varpi_1}^{\varpi_2}[e^{\Xi x}u(\xi)]'{}_{\xi}I_{\varpi_2}^{1-\nu}e^{-\Xi\xi}v(\xi)d\xi\right] \\ &= \lim_{\xi\to\varpi_1^+}u(\xi)\mathbb{I}_{\varpi_2^-}^{1-\nu,\Xi}v(\xi) - \lim_{\xi\to\varpi_2^-}u(\xi)\mathbb{I}_{\varpi_2^-}^{1-\nu,\Xi}v(\xi) + \int_{\varpi_1}^{\varpi_2}{}^{C}\mathbb{D}_{\varpi_1^+}^{\nu,\Xi}u(\xi)v(\xi)d\xi.\end{aligned}$$

3 Tempered fractional Sobolev type space

We begin this section by considering the following definition.

Definition 1. *Let $\nu \in (0,1)$ and $\Xi > 0$. A function $\phi \in C^1(\varpi_1, \varpi_2)$ is called a solution of (3) if: ${}_{\xi}I^{\nu}_{\varpi_2}\left(e^{-\Xi\xi}{}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi(\xi)\right)$ is continuously differentiable for any $\xi \in (\varpi_1, \varpi_2)$, ${}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi$ is continuous for every $\xi \in (\varpi_1, \varpi_2)$, and ϕ satisfies (3).*

Now, let $\phi \in C_0^\infty(\varpi_1, \varpi_2)$ be a solution of (3) according to Definition 1 and choose $\psi \in C_0^\infty(\varpi_1, \varpi_2)$, then

$$\int_{\varpi_1}^{\varpi_2} \mathbb{D}^{\nu,\Xi}_{\varpi_2^-}({}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi(\xi))\psi(\xi)d\xi = \int_{\varpi_1}^{\varpi_2} \gimel(\xi,\phi(\xi))\psi(\xi)d\xi. \tag{31}$$

Theorem 10 yields that

$$\int_{\varpi_1}^{\varpi_2} \mathbb{D}^{\nu,\Xi}_{\varpi_2^-}({}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi(\xi))\psi(\xi)d\xi = \lim_{\xi\to\varpi_1^+}\psi(\xi)\mathbb{I}^{1-\nu,\Xi}_{\varpi_2^-}({}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi(\xi)) - \lim_{\xi\to\varpi_2^-}\psi(\xi)\mathbb{I}^{1-\nu,\Xi}_{\varpi_2^-}({}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi(\xi))$$
$$+\int_{\varpi_1}^{\varpi_2} {}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi(\xi){}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\psi(\xi)d\xi.$$

By definition ${}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi$ is continuous, hence by Theorem 6 we get

$$\lim_{\xi\to\varpi_2^-}\mathbb{I}^{\nu,\Xi}_{\varpi_2^-}({}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi(\xi)) = 0.$$

So, as $\psi \in C_0^\infty(\varpi_1, \varpi_2)$, next

$$\int_{\varpi_1}^{\varpi_2} \mathbb{D}^{\nu,\Xi}_{\varpi_2^-}({}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi(\xi))\psi(\xi)d\xi = \int_{\varpi_1}^{\varpi_2} {}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi(\xi){}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\psi(\xi)d\xi.$$

Hence, (31) can be written as

$$\int_{\varpi_1}^{\varpi_2} {}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\phi(\xi){}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\psi(\xi)d\xi = \int_{\varpi_1}^{\varpi_2} \gimel(\xi,\phi(\xi))\psi(\xi)d\xi. \tag{32}$$

Motivated by this expression, we define the tempered fractional Sobolev type space $\mathbb{H}_0^{\nu,\Xi}(\varpi_1, \varpi_2)$ as

$$\mathbb{H}_0^{\nu,\Xi}(\varpi_1, \varpi_2) = \overline{C_0^\infty(\varpi_1, \varpi_2)}^{\|\cdot\|_{\nu,\Xi}},$$

where

$$\|\daleth\|_{\nu,\Xi} = \left(\int_{\varpi_1}^{\varpi_2} |\daleth(\xi)|^2 dx + \int_{\varpi_1}^{\varpi_2} |{}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\daleth(\xi)|^2 d\xi\right)^{1/2}, \tag{33}$$

with the inner product

$$\langle \daleth, \beth\rangle_{\nu,\Xi} = \int_{\varpi_1}^{\varpi_2} \daleth(\xi)\,\beth(\xi)d\xi + \int_{\varpi_1}^{\varpi_2} {}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\daleth(\xi)\;{}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\beth(\xi)d\xi. \tag{34}$$

As in [26, Theorem 3.7] we can show that $(\mathbb{H}_0^{\nu,\Xi}(\varpi_1, \varpi_2), \langle\cdot,\cdot,\rangle_{\nu,\Xi})$ is a Hilbert space.
Now we study some properties of this function space.

Lemma 2. *For any $u \in \mathbb{H}_0^{\nu,\Xi}(\varpi_1, \varpi_2)$, we have*

$$\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}{}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u(\xi) = u(\xi), \;\; a.e. \text{ in } (\varpi_1, \varpi_2).$$

Proof. By the definition of $\mathbb{H}_0^{\nu,\Xi}(\varpi_1, \varpi_2)$, there exists $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\varpi_1, \varpi_2)$ such that

$$\lim_{n\to\infty}\|u-\varphi_n\|_{\nu,\Xi} = 0.$$

Hence

$$\lim_{n\to\infty}\|u-\varphi_n\|_{L^2(\varpi_1,\varpi_2)}=0 \quad \text{and} \quad \lim_{n\to\infty}\|{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}(u-\varphi_n)\|_{L^2(\varpi_1,\varpi_2)}=0. \tag{35}$$

Fatou's Lemma yields that

$$\begin{aligned}&\int_{\varpi_1}^{\varpi_2}|u(\xi)|^2dx\le\liminf_{n\to\infty}\int_{\varpi_1}^{\varpi_2}|\varphi_n(\xi)|^2dx<\infty \quad \text{and}\\ &\int_{\varpi_1}^{\varpi_2}|{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u(\xi)|^2dx\le\liminf_{n\to\infty}\int_{\varpi_1}^{\varpi_2}|{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\varphi_n(\xi)|^2dx<+\infty.\end{aligned} \tag{36}$$

By other side

$$\|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u-u\|_{L^2(\varpi_1,\varpi_2)}\le\|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}(u-\varphi_n)\|_{L^2(\varpi_1,\varpi_2)}+\|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\varphi_n-\varphi_n\|_{L^2(\varpi_1,\varpi_2)}+\|\varphi_n-u\|_{L^2(\varpi_1,\varpi_2)}. \tag{37}$$

Since $\varphi_n(a)=0$, Theorem 9 implies that

$$\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\varphi_n(\xi)=\varphi_n(\xi),$$

next

$$\|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\varphi_n-\varphi_n\|_{L^2(\varpi_1,\varpi_2)}=0 \ \ \forall n\in\mathbb{N}. \tag{38}$$

By other side, Theorem 3 yields that

$$\|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}(u-\varphi_n)\|_{L^2(\varpi_1,\varpi_2)}\le\frac{\Psi(\nu,\Xi(\varpi_2-\varpi_1))}{\Xi^\nu\Gamma(\nu)}\|{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}(u-\varphi_n)\|_{L^2(\varpi_1,\varpi_2)}. \tag{39}$$

Therefore, by (35), (38), (39) and (37) we obtain that

$$\|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u-u\|_{L^2(\varpi_1,\varpi_2)}=0,$$

which we have the conclusion.

Corollary 1. *Let* $\nu\in(0,1)$, $\Xi>0$. *Then,*

$$\|u\|_{L^2(\varpi_1,\varpi_2)}\le\frac{\Psi(\nu,\Xi(\varpi_2-\varpi_1))}{\Xi^\nu\Gamma(\nu)}\|{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u\|_{L^2(\varpi_1,\varpi_2)}, \tag{40}$$

for any $u\in\mathbb{H}^{\nu,\Xi}_0(\varpi_1,\varpi_2)$.

Proof. Since $u\in\mathbb{H}^{\nu,\Xi}_0(\varpi_1,\varpi_2)$, Theorem 3 and Lemma 2 imply that

$$\|u\|_{L^2(\varpi_1,\varpi_2)}=\|\mathbb{I}^{\nu,\Xi}_{\varpi_1^+}{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u\|_{L^2(\varpi_1,\varpi_2)}\le\frac{\Psi(\nu,\Xi(\varpi_2-\varpi_1))}{\Xi^\nu\Gamma(\nu)}\|{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u\|_{L^2(\varpi_1,\varpi_2)}.$$

Remark. By Corollary 1 we can endowed $\mathbb{H}^{\nu,\Xi}_0(\varpi_1,\varpi_2)$ with the norm

$$\|u\|=\left(\int_{\varpi_1}^{\varpi_2}|{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}u(\xi)|^2d\xi\right)^{1/2},$$

which is equivalent with $\|\cdot\|_{\nu,\Xi}$.

In the following result we are able to show that $\mathbb{H}^{\nu,\Xi}_0(\varpi_1,\varpi_2)$ is continuously embedded into $C(\varpi_1,\varpi_2)$, more precisely we get:

Theorem 11. *Let* $\nu\in(\frac{1}{2},1)$ *and* $\Xi>0$, *then* $\mathbb{H}^{\nu,\Xi}_0(\varpi_1,\varpi_2)$ *is continuously embedded into* $C(\varpi_1,\varpi_2)$.

Proof. Let $u \in \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$, Lemma 2 implies that $u, {}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi} u \in L^2(\varpi_1,\varpi_2)$ and

$$u(\xi) = \mathbb{I}_{\varpi_1^+}^{\nu,\Xi} {}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi} u(\xi) \text{ a.e. } \xi \in (\varpi_1,\varpi_2).$$

Hence, Theorem 7 yields that

$$\begin{aligned}
\|u\|_\infty &= \|\mathbb{I}_{\varpi_1^+}^{\nu,\Xi} {}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi} u\|_\infty \\
&\leq \frac{1}{(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}\|{}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi} u\|_{L^2(\varpi_1,\varpi_2)} \\
&= \frac{1}{(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}\|u\|,
\end{aligned}$$

which implies the desired result.

The following compactness result will be crucial for our purpose.

Theorem 12. *Let $\nu \in (\frac{1}{2},1)$ and $\Xi > 0$. Then the embedding*

$$\mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2) \hookrightarrow C(\overline{\varpi_1,\varpi_2})$$

is compact.

Proof. Assume that Σ be a bounded subset of $\mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$, so we must show that Σ is relative compact in $C(\overline{\varpi_1,\varpi_2})$. So, by virtue of the Arzelá-Ascoli theorem, we will prove that Σ is equibounded and equicontinuous in $C(\overline{\varpi_1,\varpi_2})$. Theorem 11 implies that $\mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$ is continuously embedded in $C(\overline{\varpi_1,\varpi_2})$, and

$$\|u\|_\infty \leq \frac{[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}}{(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}\|u\|, \text{ for every } u \in \Sigma. \tag{41}$$

So, Σ is equibounded in $C(\overline{\varpi_1,\varpi_2})$. Furthermore, in view of Lemma 2 and Theorem 7, there is a constant $\mathscr{H} > 0$ such that

$$\begin{aligned}
|u(\xi)-u(\tau)| &= |\mathbb{I}_{\varpi_1^+}^{\nu,\Xi} {}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi} u(\xi) - \mathbb{I}_{\varpi_1^+}^{\nu,\Xi} {}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi} u(\tau)| \\
&\leq \mathscr{H}\|{}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi} u\|_{L^2(\varpi_1,\varpi_2)}|\xi-\tau|^{\nu-\frac{1}{2}},
\end{aligned}$$

so Σ is equicontinuous. Therefore, we have the conclusion.

Remark. If $\nu \in (\frac{1}{2},1)$ and $\Xi > 0$, then for any $u \in \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$, there exists $(\phi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\varpi_1,\varpi_2)$ such that

$$\lim_{n\to\infty}\|u-\phi_n\| = 0.$$

Then, by Theorem 11, we obtain

$$0 \leq |u(\varpi_1)| = |u(\varpi_1)-\phi_n(\varpi_1)| \leq \frac{1}{(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}\|u-\phi_n\| \to 0 \text{ as } n\to\infty.$$

Therefore, $u(\varpi_1) = 0$. By Similar argument we have that $u(\varpi_2) = 0$.
Also, Lemma 2 implies that

$${}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi} u \in L^2(\varpi_1,\varpi_2).$$

Hence, $\mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$ can be rewritten as

$$\mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2) = \{u \in L^2(\varpi_1,\varpi_2) : {}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi} u \in L^2(\varpi_1,\varpi_2) \text{ and } u(\varpi_1) = u(\varpi_2) = 0\}.$$

4 Proof of Theorem 1

We first present the following important theorem and remark to prove of Theorem 1.

Theorem 13. *[35, Theorem 4.10] Assume that $\mathscr{E}$ be a Banach space and $\mathscr{J} \in C^1(\mathscr{E},\mathbb{R})$ satisfy the Palais-Smale condition ((PS)-condition). Let there exist $\widetilde{\eta}_0, \widetilde{\eta}_1 \in \mathscr{E}$ and a bounded open neighborhood Ω of $\widetilde{\eta}_0$ such that $\widetilde{\eta}_1 \in \mathscr{E} \setminus \overline{\Omega}$ and*

$$\max\{\mathscr{J}(\widetilde{\eta}_0), \mathscr{J}(\widetilde{\eta}_1)\} < \inf_{z\in\partial\Omega} \mathscr{J}(z).$$

Set

$$\Gamma = \{\omega \in C([0,1],\mathscr{E}) : \ \omega(0) = \widetilde{\eta}_0, \omega(1) = \widetilde{\eta}_1\},$$
$$\lceil = \inf_{\omega\in\Gamma} \max_{\tau\in[0,1]} \mathscr{J}(\omega(\tau)).$$

Then $\lceil$ is a critical point of $\mathscr{J}$, that is, there exists η^* such that $\mathscr{J}'(\eta^*) = \Theta$ and $\mathscr{J}(\eta^*) = \tau$, where $\lceil > \max\{\mathscr{J}(\widetilde{\eta}_0), \mathscr{J}(\widetilde{\eta}_1)\}$.

If for any $\{\eta_n\} \subset \mathscr{E}$, $\{\eta_n\}$ has a convergent subsequence if $\mathscr{J}(\eta_n)$ is bounded and $(1+\|\eta_n\|)\|\mathscr{J}'(\eta_n)\| \to 0$ as $n \to \infty$, then we say that $\mathscr{J}$ satisfies condition (C).

Remark. In [36] be proved that Theorem 13 holds true when we replace the (PS)-condition with the condition (C).

Proof of Theorem 1. Define

$$\mathscr{J}(\daleth) = \frac{1}{2}\int_{\varpi_1}^{\varpi_2} |{}^C\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\daleth(\xi)|^2 d\xi - \int_{\varpi_1}^{\varpi_2} \widehat{\beth}(\xi,\daleth(\xi))d\xi.$$

First, we prove that $\mathscr{J}$ satisfies the condition (C). Let $\{\eta_n\} \subset \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$ is a sequence such that $\mathscr{J}(\eta_n)$ is bounded and

$$(1+\|\eta_n\|)\|\mathscr{J}'(\eta_n)\| \to 0 \quad \text{as } n \to \infty,$$

so,

$$|\mathscr{J}(\eta_n)| \le D, \quad (1+\|\eta_n\|)\|\mathscr{J}'(\eta_n)\| \le D, \quad \forall\, n \in \mathbb{N}, \tag{42}$$

positive constant D. By $(\beth_3)$, there exists $\zeta_1 > 0$ such that

$$\eta\beth(\xi,u) - 2\widehat{\beth}(\xi,\eta) \ge \eta_2|\eta|^{\theta_1} - \zeta_1 \tag{43}$$

for all $(\xi,u) \in [\varpi_1,\varpi_2] \times \mathbb{R}$. (42) and (43) yield that

$$\begin{aligned} 3D &\ge 2\mathscr{J}(\eta_n) - \mathscr{J}'(\eta_n)\eta_n \\ &= \int_{\varpi_1}^{\varpi_2} \Big(\beth(\xi,\eta_n(\xi))\eta_n(\xi) - \widehat{\beth}(\xi,\eta_n(\xi))\Big)d\xi \\ &\ge \eta_2 \int_{\varpi_1}^{\varpi_2} |\eta_n(\xi)|^{\theta_1} d\xi - \zeta_1 M(\varpi_2 - \varpi_1). \end{aligned}$$

So, $\int_{\varpi_1}^{\varpi_2} |\eta_n(\xi)|^{\theta_1} d\xi$ is bounded. In view of $(\beth_4)$, we can fix ϑ satisfying

$$\limsup_{\eta\to 0} \frac{\max_{\xi\in[\varpi_1,\varpi_2]} \widehat{\beth}(\xi,\eta)}{|\eta|^2} < \chi < \frac{1}{2(\varpi_2-\varpi_1)\left(\frac{[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}}{(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}\right)^2}. \tag{44}$$

By $(\beth_2)$ and (44), there exists $\chi_1 > 0$ such that

$$\widehat{\beth}(\xi,\eta) \le \chi|\eta|^2 + \chi_1|\eta|^{\zeta_1}, \tag{45}$$

for all $(\xi,\eta) \in [\varpi_1,\varpi_2] \times \mathbb{R}$. In view of (41), (42), (44) and (45), we get

$$
\begin{aligned}
D &\geq \mathscr{J}(\eta_n)\\
&= \frac{1}{2}\int_{\varpi_1}^{\varpi_2} |{}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\eta_n(\xi)|^2 dx - \int_{\varpi_1}^{\varpi_2} \widehat{\beth}(\xi,\eta_n(\xi))d\xi\\
&\geq \frac{1}{2}\|\eta_n\|^2 - \int_{\varpi_1}^{\varpi_2} (\chi|\eta_n(\xi))|^2 + \chi_1|\eta_n(\xi))|^{\zeta_1})d\xi\\
&\geq \left[\frac{1}{2} - \chi(\varpi_2-\varpi_1)\left(\frac{[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}}{(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}\right)^2\right]\|\eta_n\|^2 - \chi_1\int_{\varpi_1}^{\varpi_2}|\eta_n(\xi))|^{\zeta_1}d\xi.
\end{aligned}
$$

If $\chi_1 > \zeta_1$, Hölder's inequality yields that

$$
\int_{\varpi_1}^{\varpi_2}|\eta_n(\xi))|^{\zeta_1}d\xi \leq (\varpi_2-\varpi_1)^{\frac{\chi_1-\zeta_1}{\chi_1}}\left(\int_{\varpi_1}^{\varpi_2}|\eta_n(\xi))|^{\theta_1}d\eta\right)^{\frac{\zeta_1}{\chi_1}}.
$$

(46) and the above inequality yields that $\{\eta_n\}$ is bounded in $\mathbb{H}^{\nu,\Xi}_0(\varpi_1,\varpi_2)$. If $\chi_1 \leq \zeta_1$, (41) implies that

$$
\begin{aligned}
\int_{\varpi_1}^{\varpi_2}|\eta_n(\xi))|^{\zeta_1}dx &= \int_{\varpi_1}^{\varpi_2}|\eta_n(\xi))|^{\zeta_1-\chi_1}|\eta_n(\xi))|^{\chi_1}d\xi\\
&\leq \|\eta_n\|_\infty^{\zeta_1-\chi_1}\int_{\varpi_1}^{\varpi_2}|\eta_n(\xi))|^{\theta_1}d\xi\\
&\leq \left(\frac{[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}}{(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}\right)^{\zeta_1-\chi_1}\|\eta_n\|^{\zeta_1-\chi_1}\int_{\varpi_1}^{\varpi_2}|\eta_n(\xi))|^{\theta_1}d\xi.
\end{aligned}
$$

By (46) and $\zeta_1-\chi_1<2$ we get that $\{\eta_n\}$ is bounded in $\mathbb{H}^{\nu,\Xi}_0(\varpi_1,\varpi_2)$. From compactly embedded $\mathbb{H}^{\nu,\Xi}_0(\varpi_1,\varpi_2)$ into $C([\varpi_1,\varpi_2])$ and a standard argument, we obtain $\{\eta_n\}$ has a strong convergent subsequence, and so, $\mathscr{J}$ satisfies the condition (C).

(44) yields that there exists $\delta>0$ such that for every $\eta\in\mathbb{R}$ with $|\eta|\leq\delta$ and $\xi\in[\varpi_1,\varpi_2]$, one get

$$
\widehat{\beth}(\xi,\eta) < \chi|\eta|^2. \tag{46}
$$

Hence, for any $\eta\in\mathbb{H}^{\nu,\Xi}_0(\varpi_1,\varpi_2)$ with $\|\eta\|\leq\frac{\delta(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}{[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}}$, from (41) and (46) we have

$$
\begin{aligned}
\mathscr{J}(\eta) &= \frac{1}{2}\int_{\varpi_1}^{\varpi_2}|{}^{C}\mathbb{D}^{\nu,\Xi}_{\varpi_1^+}\eta(\xi)|^2dx - \int_{\varpi_1}^{\varpi_2}\widehat{\beth}(\xi,\eta(\xi))d\xi\\
&\geq \frac{1}{2}\|\eta\|^2 - \int_{\varpi_1}^{\varpi_2}\chi|\eta(\xi))|^2d\xi\\
&\geq \left[\frac{1}{2}-\chi(\varpi_2-\varpi_1)\left(\frac{[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}}{(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}\right)^2\right]\|\eta\|^2.
\end{aligned} \tag{47}
$$

Choose

$$
\beta_1 = \left[\frac{1}{2}-\chi(\varpi_2-\varpi_1)\left(\frac{[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}}{(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}\right)^2\right]\frac{\delta^2}{\left(\frac{[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}}{(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}\right)^2},
$$

$$
\rho_1 = \frac{\delta(2\Xi)^{\nu-\frac{1}{2}}\Gamma(\nu)}{[\Psi(2\nu-1,2\Xi(\varpi_2-\varpi_1))]^{1/2}},
$$

then $\mathscr{J}(\eta)\geq\beta_1>0$ for any $\eta\in\partial\Sigma_{\rho_1}$, where $\Sigma_{\rho_1}=\{\eta\in\mathbb{H}^{\nu,\Xi}_0(\varpi_1,\varpi_2):\ \|\eta\|<\rho_1\}$.

For this ρ_1, from $(\beth_1)$, we take a sufficiently large ρ_2 such that for any $\|\eta\|\geq\rho_2>\rho_1$ satisfying

$$
\widehat{\beth}(\xi,\eta)\geq\delta_1|\eta|^{\vartheta}. \tag{48}
$$

So, for every finite dimensional subspace $X \subseteq \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$, for every $\eta \in X$ by $\|\eta\| \geq \rho_2 > \rho_1$ and $\tau > 0$, in view of (41), (48) and ($\beth_1$), we get

$$\begin{aligned}\mathscr{J}(t\eta) &= \frac{1}{2}\int_{\varpi_1}^{\varpi_2} |{}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi} t\xi(\xi)|^2 d\xi - \int_{\varpi_1}^{\varpi_2} \widehat{\beth}(\xi, tu(\xi))d\xi \\ &\leq \frac{t^2}{2}\|\eta\|^2 - \delta_1 t^{\vartheta}\int_{\varpi_1}^{\varpi_2}|\eta(\xi)|^{\vartheta}d\xi \\ &\leq \frac{t^2}{2}\|\eta\|^2 - \rho_1\tau^{\vartheta}C_0\|\eta\|^{\vartheta}, \end{aligned} \tag{49}$$

for come constant $C_0 > 0$ and so $\mathscr{J}(t\eta) \to -\infty$ as $t \to +\infty$, since $\vartheta > 2$, then there exists a sufficiently large k_0 such that $\mathscr{J}(k_0\eta) < 0$. Hence, we choose $\widetilde{\eta}_1 = t_0\eta$ with $\|\widetilde{\eta}_1\| \geq \rho_2$ large enough such that $\mathscr{J}(\widetilde{\eta}_1) < 0$. Let $\widetilde{\eta}_0 = 0$, so $\mathscr{J}(\widetilde{\eta}_0) = 0$, then Theorem 13 yields that $\mathscr{J}$ possesses a critical value $\widetilde{d}_1 > 0$ given by

$$\widetilde{d}_1 = \inf_{\omega\in\Gamma}\max_{\tau\in[0,1]}\mathscr{J}(\omega(\tau)),$$

where

$$\Gamma = \{\omega \in C([0,1], \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)) :\ \omega(0) = \widetilde{\eta}_0, \omega(1) = \widetilde{\eta}_1\}.$$

Hence, there exists $\widehat{\eta} \in \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$ such that $\mathscr{J}(\widehat{\eta}) = \widetilde{d}_1$ and $\mathscr{J}'(\widehat{\eta}) = 0$. Therefore, $\widehat{\eta}$ is a weak solution of (3). Since $\widetilde{d}_1 > 0$ then $\widehat{\eta}$ is a nontrivial weak solution. □

5 Proof of Theorem 2

Before proving our main second results, we recall the eigenvalues of the following eigenvalue problem

$$\begin{cases} \mathbb{D}_{\varpi_2^-}^{\nu,\Xi}({}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi}u(\xi)) = \lambda u(\xi),\ \ \xi \in (\varpi_1,\varpi_2), \\ u(\varpi_1) = u(\varpi_2) = 0. \end{cases} \tag{50}$$

Its weak solution $\phi \in \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$ satisfies

$$\int_{\varpi_1}^{\varpi_2} {}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi}\phi(\xi)\,{}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi}\psi(\xi)d\xi = \mu\int_{\varpi_1}^{\varpi_2}(\phi(\xi),\psi(\xi))d\xi \tag{51}$$

for every $\psi \in \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$. Note that,

$$\lambda_1 = \min_{u\in\mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)\setminus\{0\}} \frac{\int_{\varpi_1}^{\varpi_2}|{}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi}u(\xi)|^2d\xi}{\int_{\varpi_1}^{\varpi_2}|u(\xi)|^2d\xi} = \inf_{u\in\mathscr{M}}\|u\|^2,$$

where $\mathscr{M} = \{u \in \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2) :\ \int_{\varpi_1}^{\varpi_2}|u|^2dx = 1\}$.

By similar methods in [37,38], we can define

$$E_i := \oplus_{j\leq i}\ker(\mathbb{D}_{\varpi_2^-}^{\nu,\Xi}({}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi}u(\xi)) - \lambda_j),$$

where $0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots$, are the eigenvalue of $(\mathbb{D}_{\varpi_2^-}^{\nu,\Xi}({}^C\mathbb{D}_{\varpi_1^+}^{\nu,\Xi}u(\xi)), \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2))$.

Theorem 14(Mountain Pass Theorem [39,40]). *Assume that $\mathscr{E}$ is a real Banach space and $\mathscr{J} \in C^1(\mathscr{E},\mathbb{R})$ satisfying the* (C) *condition. Let $\mathscr{J}(0) = 0$,*

(i)there are positive constants $\varsigma, \alpha > 0$ such that $\mathscr{J}|_{\partial\Sigma_\varsigma} \geq \alpha$ where

$$\Sigma_\varsigma = \{\eta \in \mathscr{E} :\ \|\eta\| \leq \varsigma\};$$

(ii)there is $\widetilde{\eta}_1 \in \mathscr{E}$ and $\|\widetilde{\eta}_1\| > \varsigma$ with $\mathscr{J}(\widetilde{\eta}_1) < 0$.
Then $\mathscr{J}$ possesses a critical value $\sigma \geq \alpha$. Furthermore, σ be characterized as

$$\sigma = \inf_{\omega \in \Gamma} \max_{\eta \in \omega([0,1])} \mathscr{J}(\eta), \quad \Gamma = \{\omega \in C([0,1]) : \ \omega(0) = 0, \ \omega(1) = \widetilde{\eta}_1\}.$$

Lemma 3. *Suppose that $(\daleth_5)$ and $(\daleth_6)$ hold. Then $\mathscr{J} : \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2) \to \mathbb{R}$ satisfies the (C) condition.*

Proof. Assume that $\{\eta_n\} \subseteq \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$ be a $(\mathrm{C})_\sigma$ sequence for $\sigma \in \mathbb{R}$,

$$\mathscr{J}(\eta_n) \to \sigma, \quad (1 + \|\eta_n\|)\mathscr{J}'(\eta_n) \to 0 \quad \text{as } n \to \infty. \tag{52}$$

We claim that $\{\eta_n\}$ is a bounded. By (52), we get

$$\begin{aligned} 1 + \sigma &\geq \mathscr{J}(\eta_n) - \frac{1}{4}\mathscr{J}'(\eta_n)\eta_n \\ &= \frac{1}{4}\|\eta_n\|^2 + \int_{\varpi_1}^{\varpi_2} \left(\frac{1}{4}\daleth(\xi,\eta_n(\xi))\eta_n(\xi) - \widehat{\daleth}(\xi,\eta_n(\xi))\right) d\xi. \end{aligned} \tag{53}$$

By $(\daleth_6)$, there exists $\varsigma_0 > 0$ with

$$-\varsigma_0 \leq \frac{1}{4}\daleth(\xi,\eta)\eta - \widehat{\daleth}(\xi,\eta) + \frac{\lambda_1}{4}|\eta|^2, \quad \forall \xi \in [\varpi_1,\varpi_2], \ \eta \in \mathbb{R}. \tag{54}$$

We now define $\eta_n = u_n + \psi_n$, where $u_n \in E_1$ and $\psi_n \in E_1^\perp$. In view of (53) and (54), we get

$$\begin{aligned} 1 + \sigma &\geq \frac{1}{4}\|\eta_n\|^2 - \frac{\lambda_1}{4}\|\eta_n\|_{L^2}^2 \\ &\quad + \int_{\varpi_1}^{\varpi_2} \left(\frac{1}{4}\daleth(\xi,\eta_n(\xi))\eta_n(\xi) - \widehat{\daleth}(\xi,\eta_n(\xi) + \frac{\lambda_1}{4}|\eta_n(\xi)|^2)\right) d\xi \\ &\geq \frac{1}{4}(1 - \frac{\lambda_1}{\lambda_2})\|\psi_n\|^2 - \varsigma_0(\varpi_2 - \varpi_1). \end{aligned} \tag{55}$$

So, $\|\psi_n\|$ is bounded. Let $\{\eta_n\}$ is unbounded sequence, so there exists a subsequence $\{\eta_n\}$ of $\{\eta_n\}$ satisfying $\|\eta_n\| \to +\infty$ as $n \to +\infty$. So, we get $\frac{\psi_n}{\|\eta_n\|} \to 0 \in \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$. Since $\frac{u_n}{\|\eta_n\|}$ is bounded in finite dimensional E_1, we have $\frac{u_n}{\|\eta_n\|} \to w$ in E_1. Using

$$k_n := \frac{\eta_n}{\|\eta_n\|} = \frac{u_n + \psi_n}{\|\eta_n\|} = \frac{u_n}{\|\eta_n\|} + \frac{\psi_n}{\|\eta_n\|} \to k,$$

in E_1, yields

$$\frac{\eta_n(\xi)}{\|\eta_n\|} \to k(\xi) \quad \text{a.e. in } [\varpi_1,\varpi_2]. \tag{56}$$

So, by this fact $\|k\| = 1$ (because $\|k_n\| = 1$), $k \in E_1$ and (56), we obtain

$$|\eta_n(\xi)| \to +\infty \quad \text{as } n \to +\infty. \tag{57}$$

From the Fatou's lemma, $(\daleth_6)$, (55) and (57), we get

$$\begin{aligned} 1 + \sigma &\geq \mathscr{J}(\eta_n) - \frac{1}{4}\mathscr{J}'(\eta_n)\eta_n \\ &= \frac{1}{4}\|\eta_n\|^2 + \int_{\varpi_1}^{\varpi_2} \left(\frac{1}{4}\daleth(\xi,\eta_n(\xi))\eta_n(\xi) - \widehat{\daleth}(x,\eta_n(\xi))\right) d\xi \\ &\geq \int_{\varpi_1}^{\varpi_2} \left(\frac{1}{4}\daleth(x,\eta_n(\xi))\eta_n(\xi) - \widehat{\daleth}(\xi,\eta_n(\xi) + \frac{\lambda_1}{4}|\eta_n(\xi)|^2)\right) d\xi \\ &\to +\infty \quad \text{as } n \to +\infty, \end{aligned} \tag{58}$$

that is a contradiction. Then, $\{\eta_n\}$ is bounded in $\mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$. By $(\daleth_5)$, $\{\eta_n\}$ has a convergence subsequence. Hence, $\mathscr{J}$ satisfies the (C) condition.

Proof of Theorem 2. From Lemma 3, $\mathscr{J}: \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2) \to \mathbb{R}$ satisfies the (C) condition. So, we must show that (i) and (ii) of Theorem 14 hold.

Now, we show that there are positive constant $\varsigma_0, \alpha > 0$ such that $\mathscr{J}(\eta) \geq \alpha$ for all $\|\eta\| = \varsigma_0$. From $(\beth_5)$ and $(\beth_8)$, we obtain

$$\widehat{\beth}(\xi,\eta) \leq \frac{\lambda_1}{4}|\eta|^2 + C|\eta|^r, \tag{59}$$

for any $\eta \in \mathbb{R}$ and $\xi \in [\varpi_1,\varpi_2]$. Hence, the definition of λ_1 and (59) yields that

$$\begin{aligned}\mathscr{J}(\eta) &= \frac{1}{2}\|\eta\|^2 - \int_{\varpi_1}^{\varpi_2} \widehat{\beth}(\xi,\eta(\xi))\,d\xi \\ &\geq \frac{1}{4}\|\eta\|^2 + \frac{1}{4}\|\eta\|^2 - \frac{\lambda_1}{4}\|\eta\|_{L^2}^2 - C\int_{\varpi_1}^{\varpi_2} |\eta(\xi)|^r\,dx \\ &\geq \frac{1}{4}\|\eta\|^2 - CC_r\|\eta\|^r.\end{aligned}$$

Since $r > 2$, so for $\varsigma_0 > 0$ small enough, there exists $\alpha > 0$ such that $\mathscr{J}(\eta) \geq \alpha$ for all $\|\eta\| = \varsigma_0$.

Now, we prove that there exists $\widetilde{\eta}_1 \in \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$ and $\|\widetilde{\eta}_1\| > \varsigma_0$ such that $\mathscr{J}(\widetilde{\eta}_1) < 0$. From the definition of λ_1, for small enough $\varepsilon > 0$, we can choose $\eta \in \mathscr{M}$ satisfying

$$\lambda_1 + \frac{\varepsilon}{2} \geq \|\eta\|^2. \tag{60}$$

Also, by $(\beth_5)$ and $(\beth_7)$, one has

$$\beth(\xi,\eta) > \frac{\lambda_1 + 2\varepsilon}{2}\eta^2 - C. \tag{61}$$

Hence, In view of (60) and (61), we obtain

$$\begin{aligned}\mathscr{J}(\kappa\eta) &= \frac{1}{2}\kappa^2\|\eta\|^2 - \int_{\varpi_1}^{\varpi_2} \widehat{\beth}(\xi,\kappa\eta(\xi))\,d\xi \\ &\leq \frac{1}{2}\kappa^2\|\eta\|^2 - \frac{\lambda_1 + 2\varepsilon}{2}\kappa^2\|\eta\|_{L^2}^2 + C(\varpi_2 - \varpi_1) \\ &= -\frac{\varepsilon}{2}\kappa^2 + C(\varpi_2 - \varpi_1).\end{aligned}$$

Then, $\mathscr{J}(\kappa\eta) \to -\infty$ as $\kappa \to \infty$. Hence, there exists $\widetilde{\eta}_1 \in \mathbb{H}_0^{\nu,\Xi}(\varpi_1,\varpi_2)$ and $\|\widetilde{\eta}_1\| > \varsigma_0$ such that $\mathscr{J}(\widetilde{\eta}_1) < 0$. Therefore, we have the conclusion. □

References

[1] K. Diethelm, *The analysis of fractional differential equations*, Springer-Verlag, Berlin, Heidelberg, 2010.
[2] A. Kilbas, O. Marichev and J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Vol. 204, Amsterdam, 2006.
[3] S. Samko, A. Kilbas and O. Marichev, *Fractional integral and derivatives: theory and applications*, Gordon and Breach Science, Yverdon, Switzerland, 1993.
[4] R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlin. Sci. Numer. Simul.* **44**, 460-481 (2017).
[5] F. Jarad and T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Disc. Cont. Dyn. Sys.- S* **13**(3), 709-722 (2020).
[6] F. Jarad, T. Abdeljawad and D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.* **2012**(1), 1-8 (2012).
[7] Y. Luchko and J. Trujillo, Caputo-type modification of the Erdélyi-Kober fractional derivative., *Frac. Calc. Appl. Anal.* **10**(3), 249-267 (2007).
[8] D. Oliveira and E. Capelas De Oliveira, Hilfer-Katugampola fractional derivatives, *Comput. Appl. Math.* **37**(3), 3672-3690 (2018).
[9] D. Tavares, A. Ricardo and D. Torres, Caputo derivatives of fractional variable order: numerical approximations, *Commun. Nonlin. Sci. Numer. Simul.*, *35*, 69-87 (2016).
[10] J. Vanterler da C. Sousa and E. Capelas De Oliveira, Leibniz type rule: ψ-Hilfer fractional operator, *Commun. Nonlin. Sci. Numer. Simul.* **77**, 305-311 (2019).

[11] D. Kumar and J. Singh, *Fractional calculus in medical and health science*, CRC Press, 2020.
[12] R. Magin, Fractional calculus in bioengineering, part3, *Critical Reviews^{TM} Biomed. Engin.* **32**(3), (2004).
[13] J. Machado, F. Mainardi, and V. Kiryakova, Fractional calculus: Quo vadimus?(Where are we going?), *Frac. Cal. Appl. Anal.* **18**(2), 495-526 (2015).
[14] V. Tarasov, *Handbook of fractional calculus with applications.* Vol. 5. Berlin, de Gruyter, 2019.
[15] A. Liemert and A. Kienle, Fundamental solution of the tempered fractional diffusion equation, *J. Math. Phys.* **56**(11), 113504 (2015).
[16] S. Saifullad, A. Ali, A. Khan, K. Shah and T. Abdeljawad, A novel tempered fractional transform: theory, properties and applications to differential equations, *Fract.*, (2023).
[17] M. Zayernouri, M. Ainsworth and G. Em Karniadakis, Tempered fractional Sturm-Liouville eigenproblems, *SIAM J. Sci. Comput.* **37**(4), F5777-A1800 (2015).
[18] D. del-Castillo-Negrete, Truncation effects in superdiffusive front propagation with Lévy flights, *Phys. Rev. E* **79**, 031120 (2009).
[19] J. Gajda and M. Magdziarz, Fractional Fokker-Planck equation with tempered α-stable waiting times: Langevin picture and computer simulation, *Phys. Rev. E* **82**, 011117 (2010).
[20] A. Kullberg and D. del-Castillo-Negrete, Transport in the spatially tempered, fractional Fokker-Planck equation, *J. Phys. A: Math. Theor.* **45**, 255101 (2012).
[21] M. Meerschaert, Y. Zhang and B. Baeumer, Tempered anomalous diffusion in heterogeneous systems, *Geophys. Res. Lett.* **35**, L17403 (2008).
[22] A. Stanislavsky, K. Weron and A. Weron, Diffusion and relaxation controlled by tempered α-stable processes, *Phys. Rev. E* **78**, 051106 (2008).
[23] Y. Zhang, Moments for tempered fractional advection-diffusion equations, *J. Stat. Phys.* **139**, 915-939 (2010).
[24] F. Sabzikar, M. Meerschaert and J. Chen, Tempered fractional calculus, *J. Comput. Phy.* **293**, 14-28 (2015).
[25] R. Almeida and M. Luísa Morgado, Analysis and numerical approximation of tempered fractional calculus of variations problems, *J. Comput. Appl. Math.* **361**, 1-12 (2019).
[26] C. Torres Ledesma and M. Montalvo Bonilla, Fractional Sobolev space with Riemann-Liouville fractional derivative and application to a fractional concave-convex problem, *Adv. Oper. Theor.* **6**, 65 (2021).
[27] C. Torres Ledesma, Boundary value problem with fractional p-Laplacian operator, *Adv. Nonlin. Anal.* **5** (2), 133-146 (2016).
[28] N. Nyamoradi and R. Rodríguez-López, On boundary value problems for impulsive fractional differential equations, *App. Math. Comput.* **271**, 874-892 (2015).
[29] N. Nyamoradi and R. Rodríguez-López, Multiplicity of solutions to fractional Hamiltonian systems with impulsive effects, *Chaos Solit. Fract.* **102**, 254-263 (2017).
[30] R. Rodríguez-López and S. Tersian, Multiple solutions to boundary value problem for impulsive fractional differential equations, *Fract. Calc. Appl. Anal.* **17**(4), 1016-1038 (2014).
[31] B. Fisher, B. Jolevsaka-Tuneska and Et. Adem KiliÇman, On defining the incomplete gamma function, *Int. Transf. Spec. Func.* **14**(4), 293-299 (2003).
[32] P. Pandey, R. Pandey, S. Yadav and O. Agrawal, Variational approach for tempered fractional Sturm-Liouville problem, *Int. J. Appl. Comput. Math.* **7**, 51 (2021).
[33] C. Torres Ledesma, H. Cuti Gutierrez, J. Ávalos Rodríguez and W. Zubiaga Vera, Some boundedness results for Riemann-Liouville tempered fractional integrals, *Fract. Calc. Appl. Anal.* **27**, 818-847 (2024).
[34] C. Torres Ledesma and J. Vanterler da C. Sousa, Fractional integration by parts and Sobolev-type inequalities for ψ-fractional operators, *Math. Meth. Appl. Sci.*, 1-22 (2022).
[35] J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Berlin, Springer-Verlag, 1989.
[36] P. Bartolo, V. Benci and D. Fortunato, Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity, *Nonlin. Anal.* **7**, 981-1012 (1983).
[37] A. Iannizzotto and M. Squassina, Weyl-type laws for fractional p-eigenvalue problems, *Asympt. Anal.* **88**(4), (2013).
[38] E. Lindgren and P. Lindqvist, Fractional eigenvalues, *Calc. Var.* **49**, 795-826 (2014).
[39] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14**(4), 349-381 (1973).
[40] I. Ekeland, *Convexity methods in Hamiltonian mechanics. Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* [Results in Mathematics and Related Areas (3)], Vol. 19, Springer-Verlag, Berlin, p.p. 247, 1990.