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Integral Inequalities Involving New Conformable Derivatives

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Abstract: Here, we first describe the new conformable derivatives and integrals ([1]). Then, we establish Opial, Ostrowski, Poincaré, Sobolev, Polya and Hilbert-Pachpatte type integral inequalities.

Keywords: New conformable derivative and integral, integral inequalities.

1 Background

All in this section come from [1].

The motivation though comes from control theory [2]. A proportional derivative (PD) controller for controller output u at time t with two tuning parameters is given by

$$u(t) = k_p E(t) + k_d \frac{d}{dt} E(t), \tag{1}$$

where k_p is the proportional gain, k_d is the derivative gain, and E is the error between the state variable and the process variable.

The above motivate the following:

Definition 1. *(A class of New Conformable Derivatives) ([1]) Let $\alpha \in [0,1]$, and let the functions $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous such that*

$$\begin{gathered} \lim_{\alpha \to 0+} k_1(\alpha,t) = 1, \quad \lim_{\alpha \to 0+} k_0(\alpha,t) = 0, \ \forall\, t \in \mathbb{R}, \\ \lim_{\alpha \to 1-} k_1(\alpha,t) = 0, \quad \lim_{\alpha \to 1-} k_0(\alpha,t) = 1, \ \forall\, t \in \mathbb{R}, \\ k_1(\alpha,t) \neq 0,\ \alpha \in [0,1),\ k_0(\alpha,t) \neq 0, \ \ \alpha \in (0,1], \ \forall\, t \in \mathbb{R}. \end{gathered} \tag{2}$$

Then, the following differentiable operator D^α, defined via

$$D^\alpha f(t) = k_1(\alpha,t) f(t) + k_0(\alpha,t) f'(t) \tag{3}$$

is the New Conformable derivative given that $f'(t)$ exists for $t \in \mathbb{R}$.

Here, k_1 is a type of proportional gain k_p, k_0 is a type of derivative gain k_d, f is the error, and $u = D^\alpha f$ is the controller output. For example, one may choose $k_1 := (1-\alpha)\omega^\alpha$ and $k_0 := \alpha\omega^{1-\alpha}$ for any $\omega \in (0,\infty)$; or $k_1 := (1-\alpha)|t|^\alpha$ and $k_0 := \alpha|t|^{1-\alpha}$ on $\mathbb{R} - \{0\}$, so that

$$D^\alpha f(t) = (1-\alpha)|t|^\alpha f(t) + \alpha|t|^{1-\alpha} f'(t). \tag{4}$$

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Another class of conformable derivatives is

$$D^{\alpha}f(t)=\cos\left(\alpha\frac{\pi}{2}\right)|t|^{\alpha}f(t)+\sin\left(\alpha\frac{\pi}{2}\right)|t|^{1-\alpha}f'(t). \tag{5}$$

In general we have that $D^{\beta}D^{\alpha}\neq D^{\alpha}D^{\beta}$ *for* $\alpha,\beta\in[0,1]$.

Definition 2. *([1]) (Conformable Exponential Function). Let* $\alpha\in(0,1]$*, the points* $s,t\in\mathbb{R}$ *with* $s\leq t$*, and let the function* $p:[s,t]\to\mathbb{R}$ *be continuous. Let* $k_0,k_1:[0,1]\times\mathbb{R}\to[0,\infty)$ *be continuous and satisfy (2) with* $\frac{p}{k_0}$ *and* $\frac{k_1}{k_0}$ *Riemann integrable on* $[s,t]$*. Then the exponential function with respect to* D^{α} *in (3) is defined to be*

$$e_p(t,s):=e^{\int_s^t\frac{p(\tau)-k_1(\alpha,\tau)}{k_0(\alpha,\tau)}d\tau},\quad e_0(t,s):=e^{-\int_s^t\frac{k_1(\alpha,\tau)}{k_0(\alpha,\tau)}d\tau}. \tag{6}$$

Using (3) and (6) we have the following basic results.

Lemma 1. *([1]) (Basic Derivatives). Let the conformable differential operator* D^{α} *be given as in (3), where* $\alpha\in[0,1]$*. Let the function* $p:[s,t]\to\mathbb{R}$ *be continuous. Let* $k_0,k_1:[0,1]\times\mathbb{R}\to[0,\infty)$ *be continuous and satisfy (2), with* $\frac{p}{k_0}$ *and* $\frac{k_1}{k_0}$ *Riemann integrable on* $[s,t]$*. Assume the functions* f *and* g *are differentiable as needed. Then*

(i) $D^{\alpha}[af+bg]=aD^{\alpha}[f]+bD^{\alpha}[g]$ *for all* $a,b\in\mathbb{R}$*;*

(ii) $D^{\alpha}c=ck_1(\alpha,\cdot)$ *for all constants* $c\in\mathbb{R}$*;*

(iii) $D^{\alpha}[fg]=fD^{\alpha}[g]+gD^{\alpha}[f]-fgk_1(\alpha,\cdot)$*;*

(iv) $D^{\alpha}\left[\frac{f}{g}\right]=\frac{gD^{\alpha}[f]-fD^{\alpha}[g]}{g^2}+\frac{f}{g}k_1(\alpha,\cdot)$*;*

(v) for $\alpha\in(0,1]$ *and fixed* $s\in\mathbb{R}$*, the exponential function satisfies*

$$D_t^{\alpha}[e_p(t,s)]=p(t)e_p(t,s) \tag{7}$$

for $e_p(t,s)$ *given in (6);*

(vi) for $\alpha\in(0,1]$ *and for the exponential function* e_0 *given in (6), we have*

$$D^{\alpha}\left[\int_a^t\frac{f(s)e_0(t,s)}{k_0(\alpha,s)}ds\right]=f(t). \tag{8}$$

Definition 3. *([1]) (Integrals). Let* $\alpha\in(0,1]$ *and* $t_0\in\mathbb{R}$*. In the light of (6) and Lemma 1 (v) & (vi), define the antiderivative via*

$$\int D^{\alpha}f(t)\,d_{\alpha}t=f(t)+ce_0(t,t_0),\quad c\in\mathbb{R}.$$

Similarly, define the integral of f *over* $[a,b]$ *as*

$$\int_a^t f(s)e_0(t,s)\,d_{\alpha}s:=\int_a^t\frac{f(s)e_0(t,s)}{k_0(t,s)}ds,\quad d_{\alpha}s:=\frac{1}{k_0(\alpha,s)}ds; \tag{9}$$

recall that

$$e_0(t,s):=e^{-\int_s^t\frac{k_1(\alpha,\tau)}{k_0(\alpha,\tau)}d\tau}=e^{-\int_s^t k_1(\alpha,\tau)d_{\alpha}\tau}$$

from (6).

Lemma 2. *([1]) (Basic Integrals). Let the conformable differential operator* D^{α} *be given as in (3), the integral be given as in (9) with* $\alpha\in(0,1]$*. Let the functions* k_0,k_1 *be continuous and satisfy (2), and let* f *and* g *be differentiable as needed. Then*

(i) the derivative of the definite integral of f *is given by*

$$D^{\alpha}\left[\int_a^t f(s)e_0(t,s)\,d_{\alpha}s\right]=f(t); \tag{10}$$

(ii) the definite integral of the derivative of f *is given by*

$$\int_a^t D^{\alpha}[f(s)]e_0(t,s)\,d_{\alpha}s=f(s)e_0(t,s)|_{s=a}^{t}:=f(t)-f(a)e_0(t,a); \tag{11}$$

(iii) an integration by parts formula is given by

$$\int_a^b f(t)\,D^\alpha[g(t)]\,e_0(b,t)\,d_\alpha t = f(t)\,g(t)\,e_0(b,t)\,|_{t=a}^{b} -$$

$$\int_a^b g(t)\,(D^\alpha[f(t)] - k_1(\alpha,t)\,f(t))\,e_0(b,t)\,d_\alpha t; \tag{12}$$

(iv) a version of the Leibniz rule for differentiation of an integral is given by

$$D^\alpha\left[\int_a^t f(t,s)\,e_0(t,s)\,d_\alpha s\right] = \tag{13}$$

$$\int_a^t (D_t^\alpha[f(t,s)] - k_1(\alpha,t)\,f(t,s))\,e_0(t,s)\,d_\alpha s + f(t,t),$$

using (15); or, if e_0 is absent,

$$D^\alpha\left[\int_a^t f(t,s)\,d_\alpha s\right] = f(t,t) + \int_a^t D_t^\alpha[f(t,s)]\,d_\alpha s. \tag{14}$$

Definition 4.*([1]) (Partial Conformable Derivatives). Let $\alpha\in[0,1]$, and let the functions $k_0,k_1:[0,1]\times\mathbb{R}\to[0,\infty)$ be continuous and satisfy (2). Given a function $f:\mathbb{R}^2\to\mathbb{R}$ such that $\frac{\partial}{\partial t}f(t,s)$ exists for each fixed $s\in\mathbb{R}$, define the partial differential operator D_t^α via*

$$D_t^\alpha f(t,s) = k_1(\alpha,t)\,f(t,s) + k_0(\alpha,t)\,\frac{\partial}{\partial t}f(t,s). \tag{15}$$

Next comes the important Taylor's formula.

Theorem 1.*([1]) Let $n\in\mathbb{N}$, $\alpha\in(0,1]$, and suppose f is n times continuously differentiable on $[t_0,\infty)$. Let $t,s\in[t_0,\infty)$, and define the functions h_k by*

$$h_0(t,s)\equiv 1 \quad \text{and} \quad h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\,d_\alpha\tau \ \text{ for } k\in\mathbb{N}_0. \tag{16}$$

Then

$$f(t) = e_0(t,s)\sum_{k=0}^{n-1}(-1)^k h_k(s,t)\,(D^\alpha)^k f(s) +$$

$$(-1)^{n-1}\int_s^t h_{n-1}(\tau,t)\,(D^\alpha)^n f(\tau)\,e_0(t,\tau)\,d_\alpha\tau \tag{17}$$

for $t\in[t_0,\infty)$.

Example 1.([1]) For $\alpha\in(0,1]$, let $\omega_0,\omega_1\in(0,\infty)$, let k_1 satisfy (2), and take

$$k_0(\alpha,t)\equiv\alpha\omega_0^{1-\alpha}. \tag{18}$$

By (9),

$$d_\alpha\tau = \frac{1}{k_0(\alpha,\tau)}d\tau = \frac{1}{\alpha\omega_0^{1-\alpha}}d\tau.$$

Letting $h_0(t,s)\equiv 1$, we calculate h_1 via (16) to get

$$h_1(t,s) = \int_s^t h_0(\tau,s)\,d_\alpha\tau = \frac{1}{\alpha\omega_0^{1-\alpha}}\int_s^t 1\,d\tau = \frac{t-s}{\alpha\omega_0^{1-\alpha}};$$

additionally,

$$h_2(t,s) = \int_s^t h_1(\tau,s)\,d_\alpha\tau = \frac{1}{2!}\left(\frac{t-s}{\alpha\omega_0^{1-\alpha}}\right)^2.$$

In general we have that

$$h_n(t,s) = \frac{1}{n!}\left(\frac{t-s}{\alpha\omega_0^{1-\alpha}}\right)^n. \tag{19}$$

Note that at $\alpha = 1$ we have

$$h_n(t,s) = \frac{1}{n!}(t-s)^n$$

as expected.

Example 2.([1]) For $\alpha \in (0,1]$, let $\omega_0, \omega_1 \in (0,\infty)$, let k_1 satisfy (2), and this time take

$$k_0(\alpha,t) = \alpha(\omega_0 t)^{1-\alpha}, \quad t \in [0,\infty). \tag{20}$$

By (9),

$$d_\alpha \tau = \frac{\tau^{\alpha-1}}{\alpha\omega_0^{1-\alpha}} d\tau.$$

Again starting with $h_0(t,s) \equiv 1$, we see that

$$h_1(t,s) = \int_s^t h_0(\tau,s)\, d_\alpha \tau = \frac{1}{\alpha\omega_0^{1-\alpha}} \int_s^t \tau^{\alpha-1} d\tau = \frac{t^\alpha - s^\alpha}{\alpha^2\omega_0^{1-\alpha}},$$

and

$$h_2(t,s) = \int_s^t h_1(\tau,s)\, d_\alpha \tau = \frac{1}{2!}\left(\frac{t^\alpha - s^\alpha}{\alpha^2\omega_0^{1-\alpha}}\right)^2.$$

Continuing, we find that

$$h_n(t,s) = \frac{1}{n!}\left(\frac{t^\alpha - s^\alpha}{\alpha^2\omega_0^{1-\alpha}}\right)^n, \tag{21}$$

which is just $\frac{1}{n!}(t-s)^n$ at $\alpha = 1$.

2 Main Results

Motivation comes from [3]-[5]. We need

Remark.(to Lemma 2 (iii))
When $f = g$ we get

$$\int_a^b f(t) D^\alpha [f(t)] e_0(b,t)\, d_\alpha t = f^2(b) - f^2(a) e_0(b,a) -$$

$$\int_a^b f(t) D^\alpha [f(t)] e_0(b,t)\, d_\alpha t + \int_a^b k_1(\alpha,t) f^2(t) e_0(b,t)\, d_\alpha t. \tag{22}$$

Therefore it holds

$$\int_a^b f(t) D^\alpha [f(t)] e_0(b,t)\, d_\alpha t = \left(\frac{f^2(b) - f^2(a) e_0(b,a)}{2}\right) + \tag{23}$$

$$\frac{1}{2}\int_a^b f^2(t) k_1(\alpha,t) e_0(b,t)\, d_\alpha t.$$

We present the following Opial's type inequality:

Theorem 2. *All as in Theorem 1. Further assume that $t \geq s$, $(D^{\alpha})^k f(s) = 0$, for $k = 0, 1, ..., n-1$; $\alpha \in (0,1]$, $n \in \mathbb{N}$; and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_s^t |f(w)| \left|(D^{\alpha})^n f(w)\right| e_0(t,w)\, d_{\alpha}w \leq$$

$$\left[\int_s^t \left(\int_s^w |h_{n-1}(\tau,w)|^p e_0(w,\tau)\, d_{\alpha}\tau\right) e_0(t,w)\, d_{\alpha}w\right]^{\frac{1}{p}}$$

$$\left\{ \frac{\left(\int_s^t |(D^{\alpha})^n f(w)|^q e_0(t,w)\, d_{\alpha}w\right)^2}{2} + \right. \tag{24}$$

$$\left. \frac{1}{2}\left[\int_s^t \left(\int_s^w \left|(D^{\alpha})^n f(\tau)\right|^q e_0(w,\tau)\, d_{\alpha}\tau\right)^2 k_1(\alpha,w)\, e_0(t,w)\, d_{\alpha}w\right]\right\}^{\frac{1}{q}}.$$

Proof. Since $(D^{\alpha})^k f(s) = 0$, for $k = 0, 1, ..., n-1$, $t \geq s$, we have

$$f(t) = (-1)^{n-1} \int_s^t h_{n-1}(\tau,t) (D^{\alpha})^n f(\tau)\, e_0(t,\tau)\, d_{\alpha}\tau, \tag{25}$$

and

$$f(w) = (-1)^{n-1} \int_s^w h_{n-1}(\tau,w) (D^{\alpha})^n f(\tau)\, e_0(w,\tau)\, d_{\alpha}\tau, \tag{26}$$

for all $s \leq w \leq t$.

By Hölder's inequality we obtain

$$|f(w)| \leq \int_s^w |h_{n-1}(\tau,w)| \left|(D^{\alpha})^n f(\tau)\right| e_0(w,\tau)\, d_{\alpha}\tau =$$

$$\int_s^w |h_{n-1}(\tau,w)| (e_0(w,\tau))^{\frac{1}{p}} \left|(D^{\alpha})^n f(\tau)\right| (e_0(w,\tau))^{\frac{1}{q}}\, d_{\alpha}\tau \leq$$

$$\left(\int_s^w |h_{n-1}(\tau,w)|^p (e_0(w,\tau))\, d_{\alpha}\tau\right)^{\frac{1}{p}} \left(\int_s^w \left|(D^{\alpha})^n f(\tau)\right|^q (e_0(w,\tau))\, d_{\alpha}\tau\right)^{\frac{1}{q}}. \tag{27}$$

Call

$$z(w) := \int_s^w \left|(D^{\alpha})^n f(\tau)\right|^q (e_0(w,\tau))\, d_{\alpha}\tau, \quad z(s) = 0, \tag{28}$$

$s \leq w \leq t$.

Then (by (10))

$$D^{\alpha}z(w) = \left|(D^{\alpha})^n f(w)\right|^q, \tag{29}$$

and

$$\left|(D^{\alpha})^n f(w)\right| = (D^{\alpha}z(w))^{\frac{1}{q}}, \text{ all } s \leq w \leq t. \tag{30}$$

Therefore we have (all $s \leq w \leq t$)

$$|f(w)| \left|(D^{\alpha})^n f(w)\right| e_0(t,w) \leq$$

$$\left[\left(\int_s^w |h_{n-1}(\tau,w)|^p e_0(w,\tau)\, d_{\alpha}\tau\right) e_0(t,w)\right]^{\frac{1}{p}} \left[z(w)\, D^{\alpha}z(w)\, e_0(t,w)\right]^{\frac{1}{q}}. \tag{31}$$

Next, we apply again Hölder's inequality and finally we use (23) to get that

$$\int_s^t |f(w)| \left|(D^{\alpha})^n f(w)\right| e_0(t,w)\, d_{\alpha}w \leq$$

$$\int_s^t \left\{ \left[\left(\int_s^w |h_{n-1}(\tau,w)|^p e_0(w,\tau)\, d_{\alpha}\tau\right) e_0(t,w)\right]^{\frac{1}{p}} \left[z(w)\, D^{\alpha}z(w)\, e_0(t,w)\right]^{\frac{1}{q}} \right\} d_{\alpha}w$$

$$\leq\left(\int_s^t\left(\int_s^w\left|h_{n-1}(\tau,w)\right|^p e_0(w,\tau)\,d_\alpha\tau\right)e_0(t,w)\,d_\alpha w\right)^{\frac{1}{p}} \tag{32}$$
$$\left(\int_s^t z(w)\,D^\alpha z(w)\,e_0(t,w)\,d_\alpha w\right)^{\frac{1}{q}}=:(\xi).$$

By (23), we derive that

$$\int_s^t z(w)\,D^\alpha z(w)\,e_0(t,w)\,d_\alpha w=$$
$$\frac{z^2(t)}{2}+\frac{1}{2}\int_s^t z^2(w)\,k_1(\alpha,w)\,e_0(t,w)\,d_\alpha w=$$
$$\frac{\left(\int_s^t\left|(D^\alpha)^n f(w)\right|^q e_0(t,w)\,d_\alpha w\right)^2}{2}+ \tag{33}$$
$$\frac{1}{2}\left[\int_s^t\left(\int_s^w\left|(D^\alpha)^n f(\tau)\right|^q e_0(w,\tau)\,d_\alpha\tau\right)^2 k_1(\alpha,w)\,e_0(t,w)\,d_\alpha w\right].$$

Consequently, we get that

$$(\xi)=\left[\int_s^t\left(\int_s^w\left|h_{n-1}(\tau,w)\right|^p e_0(w,\tau)\,d_\alpha\tau\right)e_0(t,w)\,d_\alpha w\right]^{\frac{1}{p}}$$
$$\left\{\frac{\left(\int_s^t\left|(D^\alpha)^n f(w)\right|^q e_0(t,w)\,d_\alpha w\right)^2}{2}+\right. \tag{34}$$
$$\left.\frac{1}{2}\left[\int_s^t\left(\int_s^w\left|(D^\alpha)^n f(\tau)\right|^q e_0(w,\tau)\,d_\alpha\tau\right)^2 k_1(\alpha,w)\,e_0(t,w)\,d_\alpha w\right]\right\}^{\frac{1}{q}}.$$

The claim is proved.

Corollary 1._(to Theorem 2, for $n=1$) Here f is continuously differentiable on $[t_0,\infty)$, $t,s\in[t_0,\infty)$, $t\geq s$, $f(s)=0$, $\alpha\in(0,1]$; and $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then_

$$\int_s^t|f(w)|\,|D^\alpha f(w)|\,e_0(t,w)\,d_\alpha w\leq$$
$$\left[\int_s^t\left(\int_s^w e_0(w,\tau)\,d_\alpha\tau\right)e_0(t,w)\,d_\alpha w\right]^{\frac{1}{p}}$$
$$\left\{\frac{\left(\int_s^t|D^\alpha f(w)|^q e_0(t,w)\,d_\alpha w\right)^2}{2}+\right. \tag{35}$$
$$\left.\frac{1}{2}\left[\int_s^t\left(\int_s^w|D^\alpha f(\tau)|^q e_0(w,\tau)\,d_\alpha\tau\right)^2 k_1(\alpha,w)\,e_0(t,w)\,d_\alpha w\right]\right\}^{\frac{1}{q}}.$$

Corollary 2._(to Theorem 2, for $n=2$) Let f be twice continuously differentiable on $[t_0,\infty)$, $t,s\in[t_0,\infty)$, $t\geq s$, $f(s)=D^\alpha f(s)=0$, $\alpha\in(0,1]$; and $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Here it is $h_1(t,s)=\int_s^t\frac{d\tau}{k_0(\alpha,\tau)}$. Then_

$$\int_s^t|f(w)|\left|(D^\alpha)^2 f(w)\right|e_0(t,w)\,d_\alpha w\leq$$
$$\left[\int_s^t\left(\int_s^w|h_1(\tau,w)|^p e_0(w,\tau)\,d_\alpha\tau\right)e_0(t,w)\,d_\alpha w\right]^{\frac{1}{p}}$$

$$\left\{ \frac{\left(\int_s^t \left| (D^\alpha)^2 f(w) \right|^q e_0(t,w)\, d_\alpha w \right)^2}{2} + \right. \tag{36}$$

$$\left. \frac{1}{2} \left[\int_s^t \left(\int_s^w \left| (D^\alpha)^2 f(\tau) \right|^q e_0(w,\tau)\, d_\alpha \tau \right)^2 k_1(\alpha,w)\, e_0(t,w)\, d_\alpha w \right] \right\}^{\frac{1}{q}}.$$

It follows an Ostrowski like inequality.

Theorem 3. *Let $n \in \mathbb{N}$, and f is n times continuously differentiable over $[a.b] \subset \mathbb{R}$; h_k as in (16). For fixed $s \in [a,b]$, assume that $(D^\alpha)^k f(s) = 0$, $k = 1, \ldots, n-1$, $\alpha \in (0,1]$, and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \int_a^b (f(t) - e_0(t,s) f(s))\, dt \right| \le \int_a^b |f(t) - e_0(t,s) f(s)|\, dt \le$$

$$\left[\left\| (D^\alpha)^n f \right\|_{L_q([a,s], e_0(\alpha,\cdot) d_\alpha \tau)} \left(\int_a^s \left(\int_t^s (h_{n-1}(\tau,t))^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{p}} dt \right) + \right. \tag{37}$$

$$\left. \left\| (D^\alpha)^n f \right\|_{L_q([s,b], d_\alpha \tau)} \left(\int_s^b \left(\int_s^t (|h_{n-1}(\tau,t)|)^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{p}} dt \right) \right].$$

Proof. Here $(D^\alpha)^k f(s) = 0$, $k = 1, \ldots, n-1$; $t, s \in [a,b]$, s is fixed.
We have by (17) that

$$f(t) - e_0(t,s) f(s) = (-1)^{n-1} \int_s^t h_{n-1}(\tau,t) (D^\alpha)^n f(\tau)\, e_0(t,\tau)\, d_\alpha \tau.$$

Let $t \ge s$, then by Hölder's inequality, we get

$$|f(t) - e_0(t,s) f(s)| \le \int_s^t |h_{n-1}(\tau,t)| \left| (D^\alpha)^n f(\tau) \right| e_0(t,\tau)\, d_\alpha \tau =$$

$$\int_s^t \left(|h_{n-1}(\tau,t)| (e_0(t,\tau))^{\frac{1}{p}} \right) \left(\left| (D^\alpha)^n f(\tau) \right| e_0(t,\tau)^{\frac{1}{q}} \right) d_\alpha \tau \le$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{p}} \left(\int_s^t \left| (D^\alpha)^n f(\tau) \right|^q e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{q}} \le \tag{38}$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{p}} \left(\int_s^t \left| (D^\alpha)^n f(\tau) \right|^q d_\alpha \tau \right)^{\frac{1}{q}} =$$

$$\left\| (D^\alpha)^n f \right\|_{L_q([s,b], d_\alpha \tau)} \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{p}}.$$

So when, $s \le t \le b$, we got that

$$|f(t) - e_0(t,s) f(s)| \le \tag{39}$$

$$\left\| (D^\alpha)^n f \right\|_{L_q([s,b], d_\alpha \tau)} \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{p}}.$$

Let now $t \le s$, then

$$-(f(t) - e_0(t,s) f(s)) = (-1)^{n-1} \int_t^s h_{n-1}(\tau,t) (D^\alpha)^n f(\tau)\, e_0(t,\tau)\, d_\alpha \tau. \tag{40}$$

Hence, it holds (again by Hölder's inequality)

$$|f(t)-e_0(t,s)f(s)| \leq \int_t^s h_{n-1}(\tau,t)\left|(D^{\alpha})^n f(\tau)\right| e_0(t,\tau)\,d_{\alpha}\tau = \tag{41}$$

$$\int_t^s h_{n-1}(\tau,t)(e_0(t,\tau))^{\frac{1}{p}}\left|(D^{\alpha})^n f(\tau)\right| e_0(t,\tau)^{\frac{1}{q}}\,d_{\alpha}\tau \leq$$

$$\left(\int_t^s (h_{n-1}(\tau,t))^p e_0(t,\tau)\,d_{\alpha}\tau\right)^{\frac{1}{p}}\left(\int_t^s \left|(D^{\alpha})^n f(\tau)\right|^q e_0(t,\tau)\,d_{\alpha}\tau\right)^{\frac{1}{q}} \leq$$

$$\left(\int_t^s (h_{n-1}(\tau,t))^p e_0(t,\tau)\,d_{\alpha}\tau\right)^{\frac{1}{p}}\left(\int_a^s \left|(D^{\alpha})^n f(\tau)\right|^q e_0(\alpha,\tau)\,d_{\alpha}\tau\right)^{\frac{1}{q}} =$$

$$\left(\int_t^s (h_{n-1}(\tau,t))^p e_0(t,\tau)\,d_{\alpha}\tau\right)^{\frac{1}{p}}\left\|(D^{\alpha})^n f\right\|_{L_q([a,s],e_0(\alpha,\cdot)d_{\alpha}\tau)}. \tag{42}$$

Thus, when, $a \leq t \leq s$, we found that

$$|f(t)-e_0(t,s)f(s)| \leq$$

$$\left\|(D^{\alpha})^n f\right\|_{L_q([a,s],e_0(\alpha,\cdot)d_{\alpha}\tau)}\left(\int_t^s (h_{n-1}(\tau,t))^p e_0(t,\tau)\,d_{\alpha}\tau\right)^{\frac{1}{p}}. \tag{43}$$

Consequently, it holds

$$\left|\int_a^b (f(t)-e_0(t,s)f(s))\,dt\right| \leq \int_a^b |f(t)-e_0(t,s)f(s)|\,dt =$$

$$\int_a^s |f(t)-e_0(t,s)f(s)|\,dt + \int_s^b |f(t)-e_0(t,s)f(s)|\,dt \leq \tag{44}$$

$$\left[\left(\int_a^s \left(\int_t^s (h_{n-1}(\tau,t))^p e_0(t,\tau)\,d_{\alpha}\tau\right)^{\frac{1}{p}} dt\right)\left\|(D^{\alpha})^n f\right\|_{L_q([a,s],e_0(\alpha,\cdot)d_{\alpha}\tau)} +\right.$$

$$\left.\left(\int_s^b \left(\int_s^t (|h_{n-1}(\tau,t)|)^p e_0(t,\tau)\,d_{\alpha}\tau\right)^{\frac{1}{p}} dt\right)\left\|(D^{\alpha})^n f\right\|_{L_q([s,b],d_{\alpha}\tau)}\right].$$

The claim is proved.

Next comes a Poincaré type inequality.

Theorem 4.*Let all as in Theorem 3, including $f(s)=0$, and $a \leq s \leq t \leq b$. Then*

$$\|f\|_{L_q([s,b])} \leq \left(\int_s^b \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\,d_{\alpha}\tau\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}\left\|(D^{\alpha})^n f\right\|_{L_q([s,b],d_{\alpha}\tau)}. \tag{45}$$

*Proof.*Since $(D^{\alpha})^k f(s)=0$, for $k=0,1,...,n-1$; $t \geq s$, we have

$$f(t) = (-1)^{n-1}\int_s^t h_{n-1}(\tau,t)(D^{\alpha})^n f(\tau)\,e_0(t,\tau)\,d_{\alpha}\tau. \tag{46}$$

By Hölder's inequality we obtain

$$|f(t)| \leq \int_s^t |h_{n-1}(\tau,t)|\left|(D^{\alpha})^n f(\tau)\right| e_0(t,\tau)\,d_{\alpha}\tau =$$

$$\int_s^t |h_{n-1}(\tau,t)|(e_0(t,\tau))^{\frac{1}{p}}\left|(D^{\alpha})^n f(\tau)\right| e_0(t,\tau)^{\frac{1}{q}}\,d_{\alpha}\tau \leq$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} \left(\int_s^t \left|(D^\alpha)^n f(\tau)\right|^q e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{q}} \leq \tag{47}$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} \left(\int_s^b \left|(D^\alpha)^n f(\tau)\right|^q e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{q}},$$

$\forall\, t \in [s,b]$.

Hence it holds

$$|f(t)|^q \leq$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{q}{p}} \left(\int_s^b \left|(D^\alpha)^n f(\tau)\right|^q e_0(t,\tau)\, d_\alpha\tau\right) \leq$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{q}{p}} \left(\int_s^b \left|(D^\alpha)^n f(\tau)\right|^q d_\alpha\tau\right) = \tag{48}$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{q}{p}} \left\|(D^\alpha)^n f\right\|^q_{L_q([s,b],d_\alpha\tau)}.$$

Consequently, we get that

$$\int_s^b |f(t)|^q\, dt \leq$$

$$\left(\int_s^b \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{q}{p}} dt\right) \left\|(D^\alpha)^n f\right\|^q_{L_q([s,b],d_\alpha\tau)}. \tag{49}$$

Thus, we derive

$$\left(\int_s^b |f(t)|^q\, dt\right)^{\frac{1}{q}} \leq$$

$$\left(\int_s^b \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} \left\|(D^\alpha)^n f\right\|_{L_q([s,b],d_\alpha\tau)}. \tag{50}$$

The claim is proved.

It follows a Sobolev type inequality.

Theorem 5.*All as in Theorem 4, $r \geq 1$. Then*

$$\|f\|_{L_r([s,b])} \leq \left(\int_s^b \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{r}{p}} dt\right)^{\frac{1}{r}} \left\|(D^\alpha)^n f\right\|_{L_q([s,b],d_\alpha\tau)}. \tag{51}$$

*Proof.*As in (47) we obtain

$$|f(t)| \leq \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} \left(\int_s^b \left|(D^\alpha)^n f(\tau)\right|^q e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{q}} \tag{52}$$

$$\leq \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} \left\|(D^\alpha)^n f\right\|_{L_q([s,b],d_\alpha\tau)}, \tag{53}$$

$\forall\, t \in [s,b]$.

Hence, by $r \geq 1$, we obtain

$$|f(t)|^r \leq \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{r}{p}} \left\|(D^\alpha)^n f\right\|^r_{L_q([s,b],d_\alpha\tau)}, \tag{54}$$

$\forall\, t \in [s,b]$.

Consequently, it holds

$$\int_s^b |f(t)|^r dt \leq$$

$$\left(\int_s^b \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{r}{p}} dt \right) \left\| (D^\alpha)^n f \right\|^r_{L_q([s,b],d_\alpha \tau)}. \quad (55)$$

Finally we get

$$\left(\int_s^b |f(t)|^r dt \right)^{\frac{1}{r}} \leq$$

$$\left(\int_s^b \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{r}{q}} dt \right)^{\frac{1}{r}} \left\| (D^\alpha)^n f \right\|_{L_q([s,b],d_\alpha \tau)}. \quad (56)$$

The claim is proved.

We continue with Polya type inequalities.

Corollary 3._(to Theorem 3) Let_ $n \in \mathbb{N}$, _and_ $f \in C^n([a,b])$; h_k _as in (16). For fixed_ $s \in [a,b]$, _assume that_ $(D^\alpha)^k f(s) = 0$, $k = 0,1,...,n-1$, $\alpha \in (0,1]$, _and let_ $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. _Then_

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt \leq$$

$$\left[\left\| (D^\alpha)^n f \right\|_{L_q([a,s],e_0(\alpha,\cdot) d_\alpha \tau)} \left(\int_a^s \left(\int_t^s (h_{n-1}(\tau,t))^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{p}} dt \right) + \right.$$

$$\left. \left\| (D^\alpha)^n f \right\|_{L_q([s,b],d_\alpha \tau)} \left(\int_s^b \left(\int_s^t (|h_{n-1}(\tau,t)|)^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{p}} dt \right) \right]. \quad (57)$$

_Proof._By (37), just set $f(s) = 0$.

We give the following result.

Theorem 6._Let_ $n \in \mathbb{N}$, $f \in C^n([a,b])$; h_k _as in (16). Assume that_ $(D^\alpha)^k f(a) = (D^\alpha)^k f(b) = 0$, $k = 0,1,...,n-1$, $\alpha \in (0,1]$, _and let_ $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. _Then_

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt \leq$$

$$\left\| (D^\alpha)^n f \right\|_{L_q\left(\left[a,\frac{a+b}{2}\right],d_\alpha \tau\right)} \left(\int_a^{\frac{a+b}{2}} \left(\int_a^t (|h_{n-1}(\tau,t)|)^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{p}} dt \right) +$$

$$\left\| (D^\alpha)^n f \right\|_{L_q\left(\left[\frac{a+b}{2},b\right],e_0(\alpha,\cdot) d_\alpha \tau\right)} \left(\int_{\frac{a+b}{2}}^b \left(\int_t^b (h_{n-1}(\tau,t))^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{p}} dt \right).$$

_Proof._Let $s = a$ and instead of b we choose $\frac{a+b}{2}$. Then, by (57), we derive

$$\left| \int_a^{\frac{a+b}{2}} f(t)\, dt \right| \leq \int_a^{\frac{a+b}{2}} |f(t)|\, dt \leq$$

$$\left\| (D^\alpha)^n f \right\|_{L_q\left(\left[a,\frac{a+b}{2}\right],d_\alpha \tau\right)} \left(\int_a^{\frac{a+b}{2}} \left(\int_a^t (|h_{n-1}(\tau,t)|)^p e_0(t,\tau)\, d_\alpha \tau \right)^{\frac{1}{p}} dt \right). \quad (59)$$

Next, assume that $s = b$ and instead of a we choose $\frac{a+b}{2}$.

Then, again by (57), we obtain

$$\left|\int_{\frac{a+b}{2}}^{b} f(t)\,dt\right| \leq \int_{\frac{a+b}{2}}^{b} |f(t)|\,dt \leq$$

$$\left\|(D^{\alpha})^{n} f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right],e_0(\alpha,\cdot)d_\alpha\tau\right)} \left(\int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}(h_{n-1}(\tau,t))^{p} e_0(t,\tau)\,d_\alpha\tau\right)^{\frac{1}{p}} dt\right). \tag{60}$$

Finally, we have that (by (59), (60))

$$\left|\int_{a}^{b} f(t)\,dt\right| \leq \int_{a}^{b} |f(t)|\,dt = \int_{a}^{\frac{a+b}{2}} |f(t)|\,dt + \int_{\frac{a+b}{2}}^{b} |f(t)|\,dt \leq$$

$$\left\|(D^{\alpha})^{n} f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right],d_\alpha\tau\right)} \left(\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}(|h_{n-1}(\tau,t)|)^{p} e_0(t,\tau)\,d_\alpha\tau\right)^{\frac{1}{p}} dt\right)+$$

$$\left\|(D^{\alpha})^{n} f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right],e_0(\alpha,\cdot)d_\alpha\tau\right)} \left(\int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}(h_{n-1}(\tau,t))^{p} e_0(t,\tau)\,d_\alpha\tau\right)^{\frac{1}{p}} dt\right). \tag{61}$$

The claim is proved.

We finish with a Hilbert-Pachpatte type inequality.

Theorem 7. *Here $j = 1,2$. Let $n_j \in \mathbb{N}$, and $f_j \in C^{n_j}([a_j,b_j])$; h_{k_j} as in (16). For fixed $s_j \in [a_j,b_j]$, assume that $(D^{\alpha})^{k_j} f_j(s_j) = 0$, $k_j = 0,1,...,n_j-1$, $\alpha \in (0,1]$, and let $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$. Then*

$$\int_{s_1}^{b_1}\int_{s_2}^{b_2} \frac{|f_1(t_1)|\,|f_2(t_2)|\,dt_1dt_2}{\left[\frac{\left(\int_{s_1}^{t_1}(h_{n_1-1}(\tau_1,t_1))^{p} e_0(t_1,\tau_1)d_\alpha\tau_1\right)}{p}+\frac{\left(\int_{s_2}^{t_2}(h_{n_2-1}(\tau_2,t_2))^{q} e_0(t_2,\tau_2)d_\alpha\tau_2\right)}{q}\right]} \leq$$

$$(b_1-s_1)(b_2-s_2)\left(\int_{s_1}^{b_1}\left|(D^{\alpha})^{n_1} f_1(\tau_1)\right|^{q} d_\alpha\tau_1\right)^{\frac{1}{q}}\left(\int_{s_2}^{b_2}\left|(D^{\alpha})^{n_2} f_2(\tau_2)\right|^{p} d_\alpha\tau_2\right)^{\frac{1}{p}}. \tag{62}$$

Proof. Here $j = 1,2$. Since $(D^{\alpha})^{k_j} f_j(s_j) = 0$, $k_j = 0,1,...,n_j-1$; $t_j \geq s_j$, we have

$$f_j(t_j) = (-1)^{n_j-1}\int_{s_j}^{t_j} h_{n_j-1}(\tau_j,t_j)(D^{\alpha})^{n_j} f_j(\tau_j)\,e_0(t_j,\tau_j)\,d_\alpha\tau_j. \tag{63}$$

As in (27) we get

$$|f_1(t_1)| \leq \left(\int_{s_1}^{t_1}|h_{n_1-1}(\tau_1,t_1)|^{p} e_0(t_1,\tau_1)\,d_\alpha\tau_1\right)^{\frac{1}{p}}$$

$$\left(\int_{s_1}^{t_1}\left|(D^{\alpha})^{n_1} f_1(\tau_1)\right|^{q} e_0(t_1,\tau_1)\,d_\alpha\tau_1\right)^{\frac{1}{q}}, \tag{64}$$

and

$$|f_2(t_2)| \leq \left(\int_{s_2}^{t_2}|h_{n_2-1}(\tau_2,t_2)|^{q} e_0(t_2,\tau_2)\,d_\alpha\tau_2\right)^{\frac{1}{q}}$$

$$\left(\int_{s_2}^{t_2}\left|(D^{\alpha})^{n_2} f_2(\tau_2)\right|^{p} e_0(t_2,\tau_2)\,d_\alpha\tau_2\right)^{\frac{1}{p}}. \tag{65}$$

Hence

$$|f_1(t_1)|\,|f_2(t_2)| \leq$$

$$\left(\int_{s_1}^{t_1}|h_{n_1-1}(\tau_1,t_1)|^p e_0(t_1,\tau_1)\,d_\alpha\tau_1\right)^{\frac{1}{p}}\left(\int_{s_2}^{t_2}|h_{n_2-1}(\tau_2,t_2)|^q e_0(t_2,\tau_2)\,d_\alpha\tau_2\right)^{\frac{1}{q}}$$

$$\left(\int_{s_1}^{t_1}\left|(D^\alpha)^{n_1}f_1(\tau_1)\right|^q e_0(t_1,\tau_1)\,d_\alpha\tau_1\right)^{\frac{1}{q}}\left(\int_{s_2}^{t_2}\left|(D^\alpha)^{n_2}f_2(\tau_2)\right|^p e_0(t_2,\tau_2)\,d_\alpha\tau_2\right)^{\frac{1}{p}} \tag{66}$$

(using Young's inequality for $a^*,b^*\geq 0$, $a^{*\frac{1}{p}}b^{*\frac{1}{q}}\leq\frac{a^*}{p}+\frac{b^*}{q}$)

$$\leq\left[\frac{\left(\int_{s_1}^{t_1}|h_{n_1-1}(\tau_1,t_1)|^p e_0(t_1,\tau_1)\,d_\alpha\tau_1\right)}{p}+\frac{\left(\int_{s_2}^{t_2}|h_{n_2-1}(\tau_2,t_2)|^q e_0(t_2,\tau_2)\,d_\alpha\tau_2\right)}{q}\right]$$

$$\left(\int_{s_1}^{t_1}\left|(D^\alpha)^{n_1}f_1(\tau_1)\right|^q e_0(t_1,\tau_1)\,d_\alpha\tau_1\right)^{\frac{1}{q}}\left(\int_{s_2}^{t_2}\left|(D^\alpha)^{n_2}f_2(\tau_2)\right|^p e_0(t_2,\tau_2)\,d_\alpha\tau_2\right)^{\frac{1}{p}}. \tag{67}$$

Thus, it holds

$$\int_{s_1}^{b_1}\int_{s_2}^{b_2}\frac{|f_1(t_1)|\,|f_2(t_2)|\,dt_1dt_2}{\left[\frac{\left(\int_{s_1}^{t_1}(|h_{n_1-1}(\tau_1,t_1)|)^p e_0(t_1,\tau_1)d_\alpha\tau_1\right)}{p}+\frac{\left(\int_{s_2}^{t_2}(|h_{n_2-1}(\tau_2,t_2)|)^q e_0(t_2,\tau_2)d_\alpha\tau_2\right)}{q}\right]}$$

(denominator can be zero only when both $t_1=s_1$ and $t_2=s_2$)

$$\leq\left(\int_{s_1}^{b_1}\left(\int_{s_1}^{t_1}\left|(D^\alpha)^{n_1}f_1(\tau_1)\right|^q e_0(t_1,\tau_1)\,d_\alpha\tau_1\right)^{\frac{1}{q}}dt_1\right)$$

$$\left(\int_{s_2}^{b_2}\left(\int_{s_2}^{t_2}\left|(D^\alpha)^{n_2}f_2(\tau_2)\right|^p e_0(t_2,\tau_2)\,d_\alpha\tau_2\right)^{\frac{1}{p}}dt_2\right)\leq$$

$$\left(\int_{s_1}^{b_1}\left(\int_{s_1}^{t_1}\left|(D^\alpha)^{n_1}f_1(\tau_1)\right|^q d_\alpha\tau_1\right)^{\frac{1}{q}}dt_1\right) \tag{68}$$

$$\left(\int_{s_2}^{b_2}\left(\int_{s_2}^{t_2}\left|(D^\alpha)^{n_2}f_2(\tau_2)\right|^p d_\alpha\tau_2\right)^{\frac{1}{p}}dt_2\right)\leq$$

$$\left(\int_{s_1}^{b_1}\left(\int_{s_1}^{b_1}\left|(D^\alpha)^{n_1}f_1(\tau_1)\right|^q d_\alpha\tau_1\right)^{\frac{1}{q}}dt_1\right)$$

$$\left(\int_{s_2}^{b_2}\left(\int_{s_2}^{b_2}\left|(D^\alpha)^{n_2}f_2(\tau_2)\right|^p d_\alpha\tau_2\right)^{\frac{1}{p}}dt_2\right)=$$

$$(b_1-s_1)(b_2-s_2)\left(\int_{s_1}^{b_1}\left|(D^\alpha)^{n_1}f_1(\tau_1)\right|^q d_\alpha\tau_1\right)^{\frac{1}{q}}\left(\int_{s_2}^{b_2}\left|(D^\alpha)^{n_2}f_2(\tau_2)\right|^p d_\alpha\tau_2\right)^{\frac{1}{p}},$$

proving the claim.

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