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Regional Fractional Optimal Control Problem of a Bilinear Reaction Diffusion Equation Using Distributed Bounded Controls

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Abstract: This paper examines the regional fractional distributed optimal control for a bilinear reaction-diffusion equation that is driven by bounded, distributed controls. We formulate the problem as the minimization of a functional consisting of the difference between the desired and actual fractional values over a time interval, along with an energy term. We prove the existence of an optimal control and characterize it using an optimality system.

Keywords: Infinite dimensional bilinear systems, parabolic systems, distributed controls, bilinear control,fractional optimal control problem, regional optimal control problem, controllability.

1 Introduction, Motivation and Preliminaries

Bilinear control of distributed parameter systems is a crucial aspect of control theory and an area of ongoing research. The significance of this area extends beyond engineering to fields such as biology, chemistry, and socio-economics. A specific type of control system, called the bilinear control system, involves linear systems with control inputs appearing as coefficients. There are two types of controls that can be applied to these systems: locally distributed controls within the system's spatial domain and boundary controls acting on the boundaries.

Fractional calculus has gained recognition as a versatile mathematical tool over the past 60 years, largely due to its successful applications in a variety of fields, such as science and engineering. The theory of fractional differential equations has attracted much interest, as they play a crucial role in modeling processes like diffusion, stochastic processes, economics and hydrology. Additionally, fractional optimal control has been the subject of numerous studies, with researchers like Frederico et al. examining problems in this area using Caputo's sense. In [1], the authors analyzed a fractional optimal control problem with variable constraints. Bahaa in [2] explored fractional optimal control for various systems. When $\alpha = 0$, the problem (1.1) has been widely studied, with researchers like Bradley and Lenhart in [3] demonstrating the existence of an optimal control and providing a characterization through necessary optimality conditions. They also explored an optimal distributed control for a Kirchhoff plate equation in the state position and, in collaboration with Yong [4], for temporal controls on the speed state. For parabolic systems, Lenhart [5] established optimal control of a 1-dimensional fluid flow through a soil-packed tube with a contaminant, using a combined functional criterion of the final contaminant amount and energy. Addou and Benbrik [6] studied a fourth-order parabolic system and found existence and uniqueness of temporal bilinear optimal control. Zerrik and El Kabouss [7] extended this to a more general class of fourth-order parabolic systems with bounded and unbounded controls, and in [8] studied regional optimal control of a bilinear wave equation. They proved existence and characterization of optimal controls. In [9], the authors tackled the optimal control problem for hyperbolic systems with nonlinear boundary control and obtained a characterization through necessary conditions for an optimal control. Additionally, [10] the authors studied the optimal boundary control problem for the Kirchhoff equation, where control acts bilinearly on the boundary and minimizes the deviation from the desired state and energy. They showed the existence and uniqueness of optimal control

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by analyzing the differentiability of the cost function.

The gradient tracking issue of a bilinear reaction-diffusion equation was studied in [11]. The solution is formulated by minimizing a functional that consists of the difference between the desired gradient and the current gradient over a given time interval, as well as an energy term. A generalization of this problem was considered In [12], the authors studied the fractional distributed optimal control of a class of infinite-dimensional parabolic bilinear systems. The existence of an optimal control that was a function of both time and space was established and shown to minimize a quadratic functional. This functional considered the deviation between the desired state and the state on the whole domain.
In this papaer, we focus on the regional optimal control, which in the context of infinite-dimensional systems involves optimizing control inputs within a particular region of the system's state space. This approach offers several benefits over global optimal control. Firstly, it enables a more focused and precise control strategy, leading to improved performance, increased stability, and reduced energy consumption. Secondly, regional optimal control can address specific constraints and requirements that are not considered in global optimal control, resulting in a more tailored and effective solution. This is especially important for infinite-dimensional systems, where the vast and complex state space can make global control strategies difficult to implement. Regional optimal control provides a more manageable approach by focusing on a specific region of the state space.
This paper differs from previous works by examining a regional case fractional optimal control problem for a type of bilinear infinite dimensional systems. Our results demonstrate that for values of α within the range [0,1], the optimal control problem has a solution. Using Frechet differentiability, we provide an expression for the optimal control solution. Furthermore, we consider the case of time-dependent control sets.

Let Ω be an open bounded domain of $\mathbb{R}^n$, $n \geq 1$, with regular boundary $\partial\Omega$, and consider a bilinear system described by the equation:

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = Ay(x,t) + u(x,t)By(x,t) & Q = \Omega \times]0,T[, \\ y(x,t) = 0 & \Gamma = \partial\Omega \times]0,T[, \\ y(x,0) = y_0(x) & \Omega, \end{cases} \tag{1.1}$$

Where $A = \Delta$ of domain $\mathscr{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$, B is a bounded operator on $L^2(\Omega)$ and $u \in U = \{u \in L^2(Q) / -m \leq u \leq M\}$ (m and M are constants) is the control where U is the set of admissible controls.
For $y_0 \in H_0^1(\Omega)$ and $u \in U$, system (1.1) has a unique solution $y \in W = \{y \in L^2(0,T;H_0^1(\Omega)) | \dfrac{\partial y}{\partial t} \in L^2(0,T;L^2(\Omega))\}$.
For any non-empty open subregion ω of the domain of evolution Ω and for $y \in L^2(\omega)$, we will set $r_\omega y = y|_\omega$ the restriction operator of y to ω with $y|_\omega$ in $L^2(\omega)$ is defined as $\left(\int_\omega |y(x)|^2 dx\right)^{\frac{1}{2}}$, and $r_\omega^* : L^2(\omega) \longrightarrow L^2(\Omega)$ be the adjoint operator of r_ω as defined by

$$r_\omega^* y(x) = \begin{cases} y(x) & \text{if} \quad x \in \omega \\ 0 & \text{else } x \in \Omega \backslash \omega. \end{cases}$$

We introduce the following cost function to be minimized within U

$$J(u) = \frac{1}{2} \| r_\omega D_x^\alpha y_u(T) - y_d \|_{L^2(\omega)}^2 + \frac{\beta}{2} \|u\|_{L^2(0,T;L^2(\Omega))}^2, \tag{1.2}$$

where D_x^α denotes the fractional spacial derivative of order $\alpha \in [0,1]$, y is a solution of system (1.1), $y_d \in L^2(\omega)$ is a desired derivative and β is a positive constant. This can be represented as the minimization problem as follows

$$j(u^*) = \min_{u \in U} \mathbb{J}(u), \tag{1.3}$$

The rest of the paper is structured as follows: In Section 2, we prove the existence of an optimal control solution for problem (1.3). In Section 3, we provide a characterization of the optimal control by differentiating the cost function with respect to the control, and we also examine the conditions for its uniqueness

2 Existence of an Optimal Control

Here in this section, we are interested in the existence of the optimal control which is a solution to a minimization problem (1.3). First let recall the notion of weak solution of system(1.1).

Definition 1.
Let $T>0$, a continuous function $y\in[0,T]\longrightarrow L^2(\Omega)$ is a weak solution of system (1.1) on $[0,T]$, if it satisfies the following integral equation

$$y_u(t)=S(t)y_0+\int_0^T S(t-s)u(.,s)y(s)ds,\ \text{for all }\ t\in[0,T]$$

where $S(t)$ denotes the C_0 semi-group generated by A in $L^2(\Omega)$.

For fractional Riemann Louiville derivatives we recall the following definition.

Definition 2.
Let $0<\alpha<1$ and $T>0$, the fractional spatial Riemann Liouville derivatives of order α is defined by :

$$D_x^\alpha: H_0^1(\Omega)\longrightarrow L^2(\Omega) \tag{2.1}$$

$$y\longrightarrow D_x^\alpha y=\frac{d}{dx}I_0^{1-\alpha}y, \tag{2.2}$$

where $I_0^{1-\alpha}$ is the Riemann-Liouville integral of $(1-\alpha)$ order defined by :

$$I_0^{1-\alpha}y(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^x(x-\tau)^{-\alpha}y(\tau,t)d\tau$$

with $\Gamma(1-\alpha)=\int_0^{+\infty}\tau^{-\alpha}e^{-\tau}d\tau.$

In the following, we show the existence of an optimal control, solution of problem (1.3).

Theorem 1.
There is a optimal control $u^*\in U$ solution to (1.3).

*Proof.*For $u\in U$, the associated solution of system (1.1) is one of the equation

$$y_u(x,t)=S(t)y_0(x)+\int_0^T S(t-s)u(x,s)By(x,s)ds.$$

Using the bound of the semi-group $(S(t))_{t\geq 0}$ over $[0,T]$, we have

$$\|y_u(t)\|_{L^2(\Omega)}\leq C\|y_0\|_{L^2(\Omega)}+C\|B\|_{L^2(\Omega)}\int_0^T\|u(s)y(s)\|_{L^2(\Omega)}ds.$$

It follows

$$\|y_u(t)\|_{L^2(\Omega)}\leq C\|y_0\|_{L^2(\Omega)}+CM\|B\|_{L^2(\Omega)}\int_0^T\|y(s)\|_{L^2(\Omega)}ds.$$

Using the Gronwal inequality, we get

$$\|y_u(t)\|_{L^2(\Omega)}\leq C_1 exp(CM\|B\|_{L^2(\Omega)}T). \tag{2.3}$$

with $C_1=C\|y_0\|_{L^2(\Omega)}$.
The set $\{J(u)/u\in U\}$ is non-empty and is bounded. Therefore, we can choose a minimising sequence $(u_n)_{n\in\mathbb{N}}\in\mathscr{U}$ such that

$$\lim_{n\to\infty}J(u_n)=\inf_{u\in\mathscr{U}}J(u).$$

Then $(J(u_n))_{n\in\mathbb{N}}$ is bounded. Since $\|u_n\|_{L^2(0,T;L^2(\Omega))}\leq\frac{2}{\beta}J(u_n)$ thus $(u_n)_{n\in\mathbb{N}}$ is bounded.
Let us denote the solution corresponding to the system $y_n=y(u_n)$. From a priori estimates, we deduce that

$$\|u_n\|_{L^2([0,T],L^2(\Omega))}\leq M_1,$$
$$\|y_n\|_{L^2([0,T],L^2(\Omega))}\leq M_2,$$
$$\|Ay_n\|_{L^2([0,T],L^2(\Omega))}\leq M_3,$$
$$\|u_nBy_n\|_{L^2([0,T],L^2(\Omega))}\leq M_4,$$
$$\|\frac{\partial y_n}{\partial t}\|_{L^2([0,T],L^2(\Omega))}\leq M_5,$$

where $M_i (i = 1, \dots, 5)$ are positive constants.

We can get a subsequence that satisfies the convergence properties below.

$$\begin{aligned} u_n &\rightharpoonup u^* \quad \text{weakly in } L^2([0,T], L^2(\Omega)), \\ y_n &\rightharpoonup y^* \quad \text{weakly in } L^2([0,T], L^2(\Omega)), \\ Ay_n &\rightharpoonup \varphi \quad \text{weakly in } L^2([0,T], L^2(\Omega)) \\ u_n By_n &\rightharpoonup \psi \quad \text{weakly in } L^2([0,T], L^2(\Omega)) \\ \frac{\partial y_n}{\partial t} &\rightharpoonup \frac{\partial y^*}{\partial t} \quad \text{weakly in } L^2([0,T], L^2(\Omega)), \end{aligned}$$

Using the lower semi-continuity of the norms and the continuity of D_x^α and r_ω, we deduce that

$$J(u^*) \le \frac{1}{2} \lim_{n\to+\infty} \int_\omega (r_\omega D_x^\alpha y_n(x,T) - y_d(x))^2 \, dx + \lim_{n\to\infty} \inf \frac{\beta}{2} \int_0^T \|u_n(.,t)\|^2_{L^2(\Omega)} dt$$

It follows

$$J(u^*) \le \liminf_{n\to+\infty} J(u^n) = \inf_{u\in U} J(u).$$

We therefore find that u^* is the optimum control.

Remark.
If we consider the system (1.1) with a source term $f \in L^2\left(0,T; L^2(\Omega)\right)$

$$\frac{\partial y}{\partial t} = Ay + u(t)By + f \text{ on } Q$$

the same well posedness and regularity results as hold, but the constant C_1 in Equation (2.3) takes the form as follows:

$$C_1 = C\left(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2\left(0,T;L^2(\Omega)\right)}\right).$$

3 Characterization of an optimal control

This section discusses the characteristics of an optimum control solution of problem (1.3). We describe the optimal control to a given desired state only on a ω subregion for the reaction-diffusion equation using a bilinear control. First we demonstrate the next lemma.

Lemma 1.
The solution map $u \in U \longrightarrow y = y(u) \in W$ is differentiable in the sense:

$$\frac{y_{u+\varepsilon h} - y_u}{\varepsilon} \rightharpoonup z \quad \textit{weakly in } W,$$

as $\varepsilon \to 0$, where u, $u + \varepsilon h \in U$. Furthermore, z is the weak solution of this system:

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = Az(x,t) + u(x,t)Bz(x,t) + h(x,t)By_u(x,t) & Q, \\ z(x,t) = 0 & \Gamma, \\ z(x,0) = 0 & \Omega, \end{cases} \tag{3.1}$$

Proof
We let $y_{u+\varepsilon h}$ be the solution of the system (1.1) with control $u + \varepsilon h$, so $\varphi = \dfrac{y_{u+\varepsilon h} - y_u}{\varepsilon}$ is the weak solution of the system below

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(x,t) = A\varphi(x,t) + u(x,t)B\varphi(x,t) + h(x,t)By_u(x,t) & Q, \\ \varphi(x,t) = 0 & \Gamma, \\ \varphi(x,0) = 0 & \Omega, \end{cases} \tag{3.2}$$

Applying Remark 2 to System (3.2), we get

$$\|\frac{y_{u+\varepsilon h}-y_u}{\varepsilon}\|_W \leq C\|hBy_u\|_{L^2(0,T;L^2(\Omega))} \leq C_3\|y_u\|_{L^2(0,T;L^2(\Omega))} \tag{3.3}$$

with C_1 is independent of ε. Hence, on a subsequence as $\varepsilon \longrightarrow 0$, we have:

$$\frac{y_{u+\varepsilon h}-y_u}{\varepsilon} \rightharpoonup z \quad \text{weakly in } W.$$

In a similar way to the proof of the theorem (1), we can see that z is the weak solution of (3.2).

□

Proposition 1.
Let consider the adjoint system given by:

$$\begin{cases} \frac{\partial p}{\partial t}(x,t) = -A^* p(x,t) + B^*(up)(x,t) & Q \\ p(x,t) = 0 & \Gamma, \\ p(x,T) = (rD_x^\alpha)^* rD_x^\alpha y^*(T) - (rD_x^\alpha)^* y_d & \Omega. \end{cases} \tag{3.4}$$

Then the Frechet derivative of J at $u \in U$ is given by:

$$J^{'}(u)(t) = p(t)By_u(t) + \varepsilon u(t). \tag{3.5}$$

Proof.
Let $u \in U$ represent an optimal control and $y = y(u)$ represent the associated optimal solution. For $\varepsilon > 0$, let $h + \varepsilon u \in U$. Then the derivative of $\mathbb{J}(u)$ in the direction from h is satisfied

$$\begin{aligned} dJ(u^*).h &= \lim_{\varepsilon \to 0^+} \frac{J(u^*+\varepsilon h) - J(u^*)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} \int_\omega \frac{1}{\varepsilon}\Big((r_\omega D_x^\alpha y_\varepsilon(T) - y_d)^2 - (r_\omega D_x^\alpha y_{u^*}(T) - y_d)^2 \Big) dx + \frac{\beta}{2}\int_Q (2hu^* + \varepsilon h^2)\, dQ \\ &= \lim_{\varepsilon \to 0^+} \int_\omega \left(\frac{y_\varepsilon - y_{u^*}}{\varepsilon}\right)\left(\frac{(r_\omega D_x^\alpha)^*(r_\omega D_x^\alpha)y_\varepsilon + (r_\omega D_x^\alpha)^*(r_\omega D_x^\alpha)y_{u^*} - 2(rD_x^\alpha)^* y_d}{2}\right) dx + \frac{\beta}{2}\int_Q (2hu^* + \varepsilon h^2)\, dQ \\ &= \int_\omega z(x,T)((r_\omega D_x^\alpha)^*(r_\omega D_x^\alpha)y_{u^*} - (rD_x^\alpha)^* y_d) dx + \beta \int_Q u^* h\, d\Gamma. \end{aligned}$$

We substitute the term $((r_\omega D_x^\alpha)^*(r_\omega D_x^\alpha)y_{u^*} - (rD_x^\alpha)^* y_d)$ from (3.4), we obtain the following equality

$$\begin{aligned} dJ(u^*).h &= \int_\omega z(x,T)p(x,T)dx + \beta\int_Q u^* hdQ \\ &= \int_\omega \int_0^T \left(\langle \frac{\partial p}{\partial t}, z\rangle + \langle \frac{\partial z}{\partial t}, p\rangle \right) dtdx + \beta \int_Q u^* hdQ. \\ &= \int_\omega \int_0^T \langle \frac{\partial p}{\partial t}, z\rangle + \int_\omega \int_0^T \Big[\langle Az + u^* Bz + hBy^*, p\rangle \Big] dtdx + \beta\int_Q u^* hdQ. \\ &= \int_\omega \int_0^T \langle \frac{\partial p}{\partial t} + A^* p + u^* B^* p, z\rangle + \int_\omega \int_0^T \langle hBy^*, p\rangle dtdx + \beta \int_Q u^* hdQ. \end{aligned}$$

Therefore, we can deduce that

$$dJ(u^*).h = \int_\omega \int_0^T \langle hBy^*, p\rangle dtdx + \beta \int_Q u^* hdQ.$$

This completes the proof.

□

The following results characterize and gives an expression of an optimal control solution of problem (1.3) in several cases of admissible controls sets.

Proposition 2.
An optimal control solution of problem (1.3) is given by

$$u^* = \max\Big(-M, \min\Big(\frac{1}{\beta}\langle By^*, p\rangle_{L^2(\omega)}, M\Big)\Big). \tag{3.6}$$

Proof.
The Frechet differential of J is given by

$$J'(u).h = \int_\omega \int_0^T \langle hBy^*, p\rangle dtdx + \beta \int_Q u^* hdQ.$$

Since J achieves its minimum at u^*, we have

$$0 \leq \int_\omega \int_0^T \langle hBy^*, p\rangle dtdx + \beta \int_Q u^* hdQ.$$

Taking $h = \max\Big(m, \min\Big(-\frac{1}{\beta}Bz(x,t)p(x,t), M\Big)\Big) - u^*$, we show that $h(u^* + \frac{1}{\beta}Bzp)$ is negative and then

$$(\max\Big(m, \min\Big(-\frac{1}{\beta}Bz(x,t)p(x,t), M\Big)\Big) - u^*)(u^* + \frac{1}{\beta}Bzp) = 0.$$

If $M \leq -\frac{1}{\beta}Bzp$ we have $(M - u^*)(u^* + \frac{1}{\beta}Bzp) = 0$,thus $u^* = M$.
If $m \leq -\frac{1}{\beta}Bzp \leq M$ we have $(-\frac{1}{\beta}Bzp - u^*)(u^* + \frac{1}{\beta}Bzp) = 0$.
Therefore $u^* = -\frac{1}{\beta}Bzp$.
Now, if $m \geq -\frac{1}{\beta}Bzp$, we have $(m - u^*)(u^* + \frac{1}{\beta}Bzp) = 0$ and then $u^* = m$.
We conclude that

$$u^*(x,t) = \max\Big(m, \min\Big(-\frac{1}{\beta}Bz(x,t)p(x,t), M\Big)\Big).$$

The next proposition shows a necessary optimality condition.

Proposition 3.
Let $u^* \in U$ be an optimal control, then:

$$\forall v \in U, \quad < J'(u), u^* - v >_{L^2(0,T;L^2(\Omega))} \geq 0.$$

Proof.
If $v = u$ we get the condition.
If v is different that u, and since U is convex we have

$$u^* + \lambda(v - u^*) \in U, \quad \text{for any} \lambda \in]0,1[$$

It follows

$$J(u^*) \leq J(u^*) + \lambda(v - u^*)$$

which gives

$$J(u^*) \leq J(u^*) + \lambda < J'(u^*), v - u^* >_{L^2(O,T;L^2(\Omega))} + o(\lambda(v - u^*))$$

Then,

$$< J'(u^*), v - u^* >_{L^2(O,T;L^2(\Omega))} \geq \frac{1}{\lambda}(\lambda(v - u^*)).$$

Since $o(\lambda(v-u^*)) = \|\lambda(v-u^*)\|\varphi(\lambda(v-u^*))$, with $\lim\limits_{\|z\|\to 0}\varphi(z) = 0$. Then

$$\lim_{\lambda\to 0}\frac{1}{\lambda}o(\lambda(v-u^*)) = \lim_{\|z\|\to 0}\|\lambda(v-u^*)\|\varphi(\lambda(v-u^*)) = \|\, v-u^*\|\lim_{\lambda\to 0}\varphi(\lambda(v-u^*)) = 0.$$

we conclude that

$$< J'(u), u^*-v >_{L^2(O,T;L^2(\Omega))} \geq \lim_{\lambda\to 0}\frac{1}{\lambda}o(\lambda(v-u^*)) = 0.$$

Corollary 1.
Let $g\in L^2(\Omega)$ such that $|g|\neq 0$ and assuming that $U = L^2(0,T)$.
Then an optimal control is given by

$$u^*(x,t) = v^*(t)g(x) \tag{3.7}$$

with $v^*(t) = -\frac{1}{\beta\|g\|_{L^2(\Omega)}}\int_\Omega By(x,t)p(x,t)$
Particularly, if $g(x) = \mathbb{1}_D(x)$, with $D\subset\Omega$ is the actuator location and $\mathbb{1}_D$ is the characteristic function such that its measure $\mu(D)$ is non-zero, then an optimal control $v^*(t)$ is given by

$$v^*(t) = \max\left(m, \min\left(-\frac{1}{\beta\mu(D)}\int_\Omega By(x,t)\,p(x,t)dx, M\right)\right). \tag{3.8}$$

Proof.
Let $v\in L^2(0,T)$, such that $w(x,t) = v(t)g(x)$ it follows from (3) that $\langle J'(u^*), w\rangle_{L^2(0,T;L^2(\Omega)))} = 0$ which gives

$$\int_0^T v(t)\int_\Omega g(x)J'(u^*)(x,t)dxdt = 0 \quad \forall v\in L^2(0,T)$$

Hence

$$\int_\Omega g(x)J'(u^*)(x,t)dx = 0 \quad \forall t\in]0,T[$$

Then $\langle J'(u^*)(t), g\rangle_{L^2(\Omega)} = 0$, It means

$$\langle Bz(t)p(t), g\rangle_{L^2(\Omega)} + \beta v^*(t)\langle g,g\rangle_{L^2(\Omega)}, \quad \forall t\in]0,T[$$

which leads to formula (3.7).

4 Conclusion

This study focuses on the issue of regional fractional optimal control for parabolic bilinear systems with bounded controls. We derive a distributed control solution that minimizes a quadratic functional. This work opens up avenues for further exploration, such as the regional fractional optimal control of hyperbolic systems, which will be the subject of a future research paper.

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