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The Truncated Unit Chris-Jerry Distribution and Its Applications

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Abstract: This study introduces the truncated Unit Chris-Jerry distribution. It investigates its fundamental characteristics, including moments, moment-generating functions, characteristic functions, incomplete moments, and various statistical measures including order statistics, mean residual lifetimes, mean previous lifetime, and entropy. It exhibits the characteristics of a hazard failure rate function that is on the rise. Various estimation approaches are briefly covered, including maximum likelihood, least squares, weighted least squares, maximum product of spacings method, Cramer-Von-Mises method, Anderson-Darling methods, right-tail Anderson-Darling method, and percentile-based estimations. A simulation study was performed to demonstrate the practical utility of the proposed distribution. Also, the Bayesian procedure to estimate the unknown parameter is applied by using the Markov chain Monte Carlo technique and the distribution was applied to two sets of real data.

Keywords: Truncated distributions; Lifetime distributions; moments; Classical estimation methods; Bayesian procedure. Real data sets.

1 Introduction

Truncated distributions find diverse applications across various scientific domains, including specific communication networks, economics, hydrology, materials science, and physics. A conditional distribution emerges as a truncated distribution when the domain of the parent distribution is restricted to a smaller population region. Truncation implies the exclusion of events beyond a predetermined threshold or outside a specific range, making it impossible to observe or document occurrences in those areas. In the context of reliability, truncated data is acceptable and prevalent, particularly when dealing with small values of the variable of interest. This is notably relevant to the study of failure rates in products. In instances of truncation, information about items beyond the defined constraints is unattainable. For instance, manufacturing truncation occurs when a subset of objects is selected from a larger population for examination, wherein items that did not meet established criteria have been excluded. Numerous truncated distributions have been identified, with one notable example being the generalized exponential distribution introduced by Abid [1]. [2] estimated the mean residual life function using the local linear fitting technique. [3] employed the Alpha power transformation to modify the Kumaraswamy distribution, leading to the introduction of the alpha power Kumaraswamy distribution. Also, Ahmed et al. [4] introduced a truncated variant of the Birnbaum-Saunders (BS) distribution, highlighting its superior performance in modeling financial loss information from a business-oriented bank compared to the conventional BS model. Furthermore, Aldahlan et al. [5] explored the truncated version of the Cauchy power family. Algarni et al. [6] examined the truncated version of the inverse Lomax-G family. Moreover, [7] introduced a generalization of the truncated inverse Weibull-generated family of distributions, incorporating a new shape parameter by applying a power transform. Furthermore, Almarashi et al. [8] are specialists in probability distributions. Almetwally et al. [9] discussed the truncated Cauchy power and the Weibull-G clan, indicating these investigations' potential utility and interest in population research and other fields involving truncated distributions. Conversely, Bantan et al. [11] conducted a study on inverted truncated

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Kumaraswamy-generated distributions, organizing them into families. Moreover, Chesean et al. [16] employed the truncated composite approach on the Burr X distribution, leading to the development of a novel truncated Burr X generated family. Cohen [17] and Patel [29] outlined the characteristics of a truncated Poisson distribution, and they calculated parameter estimators along with their asymptotic deviations. Cohen focused on the case of a single truncation, while Patel investigated Gaussian distributions with two truncations. Genc [18] investigated the truncated inverted generalized exponential distribution in their study. [39] introduced the generalized power Akshaya distribution, and its parameters were estimated using both conventional and Bayesian approaches. [19] introduced a novel truncated distribution, termed the upper-truncated Lomax distribution, which is related to the Lomax distribution. Hassan et al. [20] introduced the truncated Lomax-G family power Lomax distribution, and in a related work, [21] proposed the power truncation Lomax-G family. Additionally, Hassan et al. [22] discussed the truncated Weibull Frechet distribution in their work. Furthermore, Jayakumar and Sankaran [23] deliberated on the distribution of negative binomials. Additionally, Kantar and Usta [24] explored the Weibull distribution. Onyekwere and Obulezi [27] examined the Marshall-Olkin Chris-Jerry distribution including its applications. Additionally, Onyekwere and Obulezi [28] introduced a Chris-Jerry distribution, and they presented the two parameters of Chris-Jerry distribution with Chinedu et al. [43]. Nadarajah [33] discussed various truncated distributions, including the t-distribution and inverted distributions. Moreover, Nadarajah explored the beta distribution and the Levy distribution as two distinct types of distributions. Shapiro [34] studied the sum of independent truncated random variables. Singh et al. [35] introduced a version of Lindley distribution. It discusses its statistical features and demonstrates its superior modeling compared to Weibull, Lindley, and exponential distributions based on actual data. Stevens [36] investigated the truncated normal distribution and [45] presented the truncated Weibull–exponential distribution. [38] presented both classical and Bayesian estimation methods for the Akshaya distribution parameter. ZeinEldin et al. [40] explored the exponentiated truncated inverse Weibull-generated family of distributions. Also, Ramadan et al. [42] introduced the unit half logistic geometric distribution and Gomaa et al. [41] presented the unit alpha-power Kum-modified size-biased Lehmann type II distribution.

The subsequent sections of this paper are organized as follows: Section 2 is dedicated to the derivation of the new distribution. Section 3 focuses on the derivation of various useful characteristics. In Section 4, classical estimation methods, including the classical estimation method, are introduced. Section 5 proposes the application of the Bayesian technique for parameter estimation. Section 6 presents a simulation study to demonstrate the flexibility of the distribution. Section 7 showcases the application of the distribution to two types of real data, illustrating its versatility.

2 The Truncated Unit Chris Jerry (TUCJ) Distribution

The probability density function (pdf) and cumulative distribution function (cdf) of the random variable Y which follows the Chris Jerry (CJ) distribution, as presented by Onyekwere and Obulezi [28], are expressed as follows:

$$f_{CJ}(y;\theta) = \frac{\theta^2}{\theta+2}(1+\theta y^2)\mathrm{e}^{-\theta y}, y>0, \tag{1}$$

and

$$F_{CJ}(y;\theta) = 1-\left[1+\frac{\theta y(\theta y+2)}{\theta+2}\right]\mathrm{e}^{-\theta y}, x>0,\ \theta>0. \tag{2}$$

Many authors used the truncated approach to introduce a new generating family of distributions, as in Refs [6], [7], [48], [49], [11], [46], and [47]. A random variable X is said to follow the right truncated unit Chris-Jerry (TUCJ) distribution if its (pdf) can be given by:

$$f_{TUCJ}(x;\theta) = \frac{f_{CJ}(x)}{\int_0^1 f_{CJ}(x)dx},$$

but

$$\begin{aligned}
\int_0^1 f_{CJ}(x)dx &= \int_0^1 \frac{\theta^2}{\theta+2}(1+\theta x^2)e^{-\theta x} \\
&= \frac{\theta^2}{\theta+2}\left[\int_0^1 e^{-\theta x}dx + \int_0^1 \theta x^2 e^{-\theta x}dx\right] \\
&= \frac{\theta^2}{\theta+2}\left[\frac{-1}{\theta}(e^{-\theta}-1)+\frac{1}{\theta^2}\left(2-e^{-\theta}(2+2\theta+\theta^2)\right)\right] \\
&= e^{-\theta}(e^{\theta}-\theta-1).
\end{aligned}$$

Finally, $f_{TUCJ}(x;\theta)$ can be given as

$$f_{TUCJ}(x;\theta) = \frac{\theta^2(1+\theta x^2)e^{\theta(1-x)}}{(\theta+2)(e^{\theta}-\theta-1)}, \quad \theta>0, 0<x<1. \tag{3}$$

According to Equation (3), the pdf of the TUCJ distribution has two shapes (decreasing and U-shaped, see Figure (1)).

The cdf related to Equation (3) is given by:

$$F_{TUCJ}(x;\theta) = \frac{F_{CJ}(x;\theta)-F_{CJ}(0;\theta)}{F_{CJ}(1,\theta)-F_{CJ}(0,\theta)}.$$

$$F_{TUCJ}(x;\theta) = \frac{\theta x(\theta x+2)e^{\theta}}{(\theta+2)\left((\theta+1)e^{\theta x}-e^{\theta}\right)}, \theta>0, x\in(0,1). \tag{4}$$

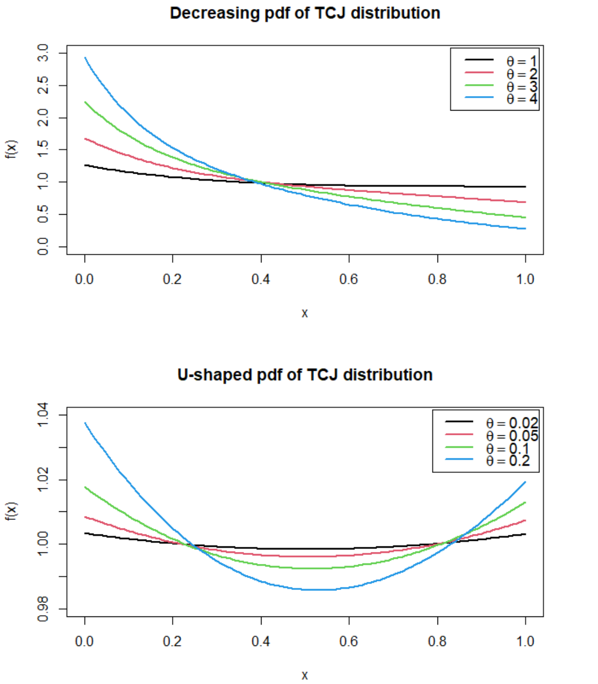

Fig. 1: The pdf plots of TUCJ distribution at different values of θ.

Depending on pdf (3) and cdf (4), the survival and hazard rate functions are given respectively by:

$$\bar{F}_{TUCJ}(x;\theta) = 1 - F_{TUCJ}(x;\theta) = 1 - \frac{\theta x(\theta x+2)e^{\theta}}{(\theta+2)\left((\theta+1)e^{\theta x} - e^{\theta}\right)},$$

and

$$h_{\ TUCJ}(x;\theta) = \frac{f_{TUCJ}(x;\theta)}{\bar{F}_{TUCJ}(x;\theta)} = \frac{e^{\theta}(-1+e^{\theta}-\theta)\theta^2(1+\theta x^2)}{(-2+e^{\theta}-\theta)(-e^{\theta x}(\theta+1)(\theta+2)+e^{\theta}(2+\theta+\theta x(2+\theta x)))}.$$

Figure (2) shows the TUCJ distribution's cdf and the survival function respectively at various θ values.

Additionally, the reversed and cumulative reversed hazard rate functions are obtained respectively as follows:

$$\tau_{\ TUCJ}(x;\theta) = \frac{\frac{\theta^2(1+\theta x^2)e^{\theta(1-x)}}{(\theta+2)(e^{\theta}-\theta-1)}}{\frac{\theta x(\theta x+2)e^{\theta}}{(\theta+2)\left((\theta+1)e^{\theta x}-e^{\theta}\right)}} = \frac{e^{-x\theta}\theta(-e^{\theta}+e^{x\theta}(1+\theta)(1+\theta x^2)}{(-1+e^{\theta}-\theta)x(2+\theta x))}, \quad (5)$$

and

$$H_{\ TUCJ}(x;\theta) = -\ln \bar{F}_{TUCJ}(x;\theta) = \ln\left(\frac{(\theta+2)\left((\theta+1)e^{\theta x}-e^{\theta}\right)}{\theta x(\theta x+2)e^{\theta}+(\theta+2)\left((\theta+1)e^{\theta x}-e^{\theta}\right)}\right).$$

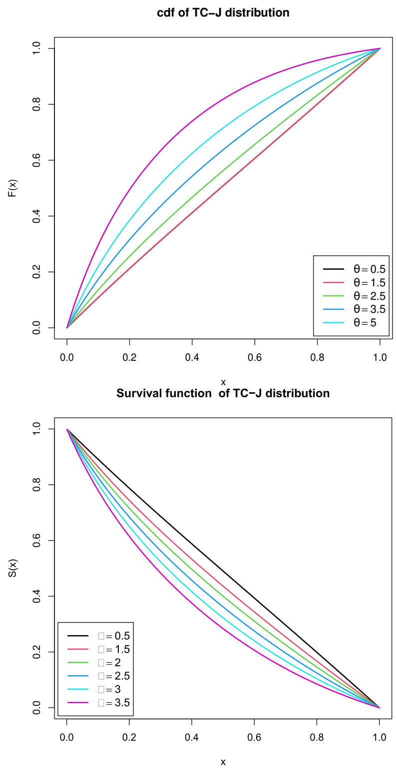

Fig. 2: The cdf and survival function plots of the TUCJ distribution at various values of θ.

Figure (3) illustrates the hazard rate and reversed hazard functions of the TUCJ distribution at various values of θ. It's clear that the hazard rate function is increasing function at different values of θ and the reversed hazard rate function is decreasing.

3 Fundamental Characteristics

This section explores various statistical features of the TUCJ distribution. These characteristics include quantiles, moments, quantile-producing process, and incomplete moments function, which are considered sequentially.

3.1 Moments and related metrics

As moments play a crucial role in statistical analysis, their calculation is essential. The moments of the TUCJ distribution are then obtained as follows:

$$\mu'_r = E(x^r) = \int_0^1 x^r \frac{\theta^2(1+\theta x^2)e^{\theta(1-x)}}{(\theta+2)(e^{\theta}-\theta-1)}dx = \frac{e^{\theta}\theta^{-r}\left(\theta\Gamma[1+r]+\Gamma[3+r]-\theta\Gamma[1+r,\theta]-\Gamma[3+r,\theta]\right)}{\left(-1+e^{\theta}-\theta\right)(2+\theta)}. \quad (6)$$

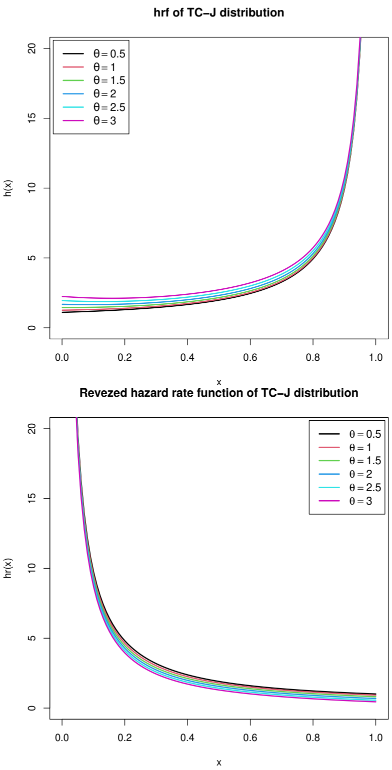

Fig. 3: The hazard and reversed hazard functions plot of the TUCJ distribution at various values of θ.

The first four moments and the variance of the TUCJ distribution can be written as:

$$\mu_1' = \frac{e^{\theta}(-6-\theta+\theta\Gamma[2,\theta]+\Gamma[4,\theta])}{\theta(-1+e^{\theta}+\theta)(2+\theta)}, \tag{7}$$

$$\mu_2' = \frac{e^{\theta}(-2(12+\theta)+\theta\Gamma[3,\theta]+\Gamma[5,\theta])}{(1-e^{\theta}+\theta)(2+\theta)\theta^2}, \tag{8}$$

$$\mu_3' = \frac{e^{\theta}(-6(2\theta+\theta)+\theta\Gamma[4,\theta]+\Gamma[6,\theta])}{(1-e^{\theta}+\theta)(2+\theta)\theta^3}, \tag{9}$$

$$\mu_4' = \frac{e^{\theta}(-24(3\theta+\theta)+\theta\Gamma[5,\theta]+\Gamma[7,\theta]}{(1-e^{\theta}+\theta)(2+\theta)\theta^4}, \tag{10}$$

and

$$\begin{aligned} var &= \mu_2' - (\mu_1')^2 \\ &= \frac{e^{\theta}(-e^{\theta}(-6-\theta+\theta\Gamma[2,\theta]+\Gamma[4,\theta])^2)}{(1-e^{\theta}+\theta)^2(2+\theta)^2\theta^2} + \\ &\frac{e^{\theta}((2+\theta)(1-e^{\theta}+\theta)(-2(12+\theta)+\theta\Gamma[3,\theta]+\Gamma[5,\theta]))}{(1-e^{\theta}+\theta)^2(2+\theta)^2\theta^2}. \end{aligned} \tag{11}$$

3.2 Moment generating function

Consider X following the truncated unit Chris-Jerry distribution. The moment-generating function of X is:

$$\begin{aligned} \mu_x(t) &= E(e^{tx}) = \int_0^1 e^{tx} f_{TUCJ}(x;\theta)dx \\ &= \int_0^1 e^{tx} \frac{\theta^2(1+\theta x^2)e^{\theta(1-x)}}{(\theta+2)(e^{\theta}-\theta-1)}dx \\ &= \frac{\theta^2(e^{\theta}((t-\theta)^2+2\theta))}{(-1+e^{\theta}-\theta)(2+\theta)(-t+\theta)^3} \\ &- \frac{\theta^2(e^{\theta}(e^{t}(t^2(1+\theta)-2t\theta(2+\theta)+\theta(1+\theta)(2+\theta)))}{(-1+e^{\theta}-\theta)(2+\theta)(-t+\theta)^3}. \end{aligned} \tag{12}$$

3.3 Measures of incomplete moments and inequality

In many cases, partial intervals are utilized to evaluate statistical domains, particularly in the measurement of income inequality through various distributions like the Pietra and Lorenz curves, income quintiles, and Gini coefficients. An incomplete Chris-Jerry truncated moment is employed to derive the resulting distribution:

$$\begin{aligned} \phi_s(t) &= \int_0^t x^s \frac{\theta^2(1+\theta x^2)e^{\theta(1-x)}}{(\theta+2)(e^{\theta}-\theta-1)}dx = e^{\theta}t^r(t\theta)^{-r}\times \\ &\frac{(\theta\Gamma[1+r]+\Gamma[3+r]-\theta\Gamma[1+r,\theta r]-\Gamma[(3+r),t\theta])}{(-1+e^{\theta}-\theta)(2+\theta)}. \end{aligned}$$

3.4 Entropy

Entropy serves as a measure of the uncertainty linked to the distribution of a random variable X. The entropy of a random variable X is defined as:

$$I_{\delta}(x)=\frac{1}{1-\delta}\log[\int_{0}^{1}f(x)^{\delta}\,dx]=\frac{1}{1-\delta}\log\int_{0}^{1}\left(\frac{\theta^{2}(1+\theta x^{2})e^{\theta(1-x)}}{(\theta+2)(e^{\theta}-\theta-1)}\right)^{\delta}dx$$

$$=\frac{1}{1-\delta}\log A^{\delta}\left[\int_{0}^{\infty}(1+\theta x^{2})^{\delta}e^{-\delta\theta x}dx-\int_{1}^{\infty}(1+\theta x^{2})^{\delta}e^{-\delta\theta x}dx\right]$$

$$=\frac{1}{1-\delta}\log A^{\delta}\left(\frac{2}{\delta}\int_{0}^{\infty}x\, e^{-\theta x\delta}dx-\left[(1+\theta x^{2})\frac{e^{-\theta\delta x}}{-\theta\delta}\right]_{1}^{\infty}-\frac{2}{\delta}\int_{1}^{\infty}2x^{-\theta x\delta}\,dx\right)$$

$$=\frac{1}{1-\delta}\log A^{\delta}\left(\frac{2}{\delta^{2}\theta^{2}}-\left[(1+\theta x^{2})\frac{e^{-\theta x\delta}}{-\theta\,\delta}\right]_{1}^{\infty}-\frac{2}{\delta}e^{-\theta\delta}\frac{(\theta\,\delta+1)}{\theta\delta^{2}}\right)$$

$$=\frac{1}{1-\delta}\log A^{\delta}\left[\frac{2}{\theta^{2}\delta^{3}}-\left(\frac{(1+\theta)e^{-\theta\delta}}{\theta\,\delta}-\frac{2}{\delta}e^{-\theta\delta}\frac{(\theta\delta+1)}{\theta\delta^{2}}\right)\right].$$

3.5 Mean Residual Life Function (MRL)

The mean residual life function for the TUCJ distribution is defined as:

$$m(x)=E(x-X/X\leq x)=\frac{1}{1-F(x)}\int_{x}^{1}(1-F(t))dt$$

$$=\frac{1}{1-\frac{\theta x(\theta x+2)e^{\theta}}{(\theta+2)((\theta+1)e^{\theta x}-e^{\theta})}}\int_{0}^{1}\left(1-\frac{\theta t(\theta t+2)e^{\theta}}{(\theta+2)\left((\theta+1)e^{\theta t}-e^{\theta}\right)}\right)dt \tag{13}$$

$$=\frac{1}{\theta(2+\theta)\left(1-\frac{\theta x(\theta x+2)e^{\theta}}{(\theta+2)((\theta+1)e^{\theta x}-e^{\theta})}\right)}\times\left(-\theta(2+\theta)(-1+x+log\theta-log(1+\theta)+x\theta(2+x\theta)log(1-\frac{e^{(\theta-x\theta)}}{1+\theta})\right) \tag{14}$$

$$+\frac{1}{\theta(2+\theta)\left(1-\frac{\theta x(\theta x+2)e^{\theta}}{(\theta+2)((\theta+1)e^{\theta x}-e^{\theta})}\right)}\cdot\left(2(1+\theta)Polylog[2,\frac{1}{1+\theta}]-2(1+x\theta)Polylog[2,\frac{e^{(\theta-x\theta)}}{1+\theta}]\right)$$

$$+\frac{1}{\theta(2+\theta)\left(1-\frac{\theta x(\theta x+2)e^{\theta}}{(\theta+2)((\theta+1)e^{\theta x}-e^{\theta})}\right)}\cdot\left(+2Polylog[3,\frac{1}{1+\theta}]-2Polylog[3,\frac{e^{(\theta-x\theta)}}{1+\theta}]\right).$$

3.6 Mean past lifetime function

In practical scenarios where systems are not continuously monitored, there is a need to understand the history of these systems, particularly when various components have experienced failures. Assume one element with a lifespan denoted as x has failed at or before time x, where $x\geq 0$. Consider the conditional random variable $(x-X\mid X\leq x)$. This conditional random variable represents the time elapsed since the component failed, given that its lifespan is less than or equal to x. Consequently, the mean previous lifetime (MPL) of the component can be denoted as:

$$K(x)=E(x-X/X\leq x)=\frac{1}{F(x)}\int_{0}^{x}F(t)dt$$

$$=\frac{1}{\frac{\theta x(\theta x+2)e^{\theta}}{(\theta+2)((\theta+1)e^{\theta x}-e^{\theta})}}\int_{0}^{x}\frac{\theta t(\theta t+2)e^{\theta}}{(\theta+2)\left((\theta+1)e^{\theta t}-e^{\theta}\right)}dt$$

$$=\frac{e^{-\theta}\left(-e^{-\theta}+e^{x\theta}(1+\theta)\right)\left(x\theta(2+x\theta)log(1-\frac{e^{(\theta-x\theta)}}{1+\theta})\right)}{x\theta^{2}(2+x\theta)}$$

$$-\frac{2(1+x\theta)Polylog[2,\frac{e^{(\theta-x\theta)}}{1+\theta}]+2\left(Polylog[2,\frac{e^{\theta}}{1+\theta}]\right)}{x\theta^{2}(2+x\theta)}$$

$$+\frac{2\left(Polylog[3,\frac{e^{\theta}}{1+\theta}]\right)-2Polylog[3,\frac{e^{(\theta-x\theta)}}{1+\theta}]}{x\theta^{2}(2+x\theta)},$$

where $Polylog[a,b]=\sum_{k=1}^{\infty}\frac{b^{k}}{k^{a}}$.

4 Methods of Estimation

In this section, we will discuss the maximum likelihood estimation (MLE) procedure, least squares estimation (LSE), weighted least square method, maximum product of spacing estimation (MPS), Cramer-Von-Mises estimation (CVME), Anderson-Darling estimation (ADE), Right tail Anderson-Darling estimation (RTADE) and Percentile estimation (PE) estimate the unknown parameter.

4.1 Maximum likelihood estimation

To estimate its parameters, we consider using the distribution. Let $x_1,x_2,\ldots,x_n$ represent a sample of n values selected from the random variables $X_1,X_2,\ldots,X_n$, which denote the lifespan of the data. Consequently, the likelihood function L is formulated as:

$$L=\prod_{i=1}^{n}f(x_i;\theta)=\prod_{i=1}^{n}\frac{\theta^{2}(1+\theta x_i^{2})e^{\theta(1-x_i)}}{(\theta+2)(e^{\theta}-\theta-1)}. \tag{15}$$

In this case, the log-likelihood function is expressed as:

$$\ln(L) = 2n\ln\theta + \sum_{i=1}^{n}\ln(1+x_i^2\theta) + \sum_{i=1}^{n}\theta(1-x_i) - n\left[\ln(\theta+2) + \ln(e^{\theta}-\theta-1)\right].$$

As a result, the first and second Log likelihood function derivatives concerning θ are as follows:

$$\frac{\partial \ln L}{\partial \theta} = \frac{2n}{\theta} + \sum_{i=1}^{n}\frac{x_i^2}{1+\theta x_i^2} + \sum_{i=1}^{n}(1-x_i) - n\left(\frac{1}{\theta+2} + \frac{e^{\theta}-1}{e^{\theta}-\theta-1}\right), \quad (16)$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-2n}{\theta^2} + \sum_{i=1}^{n}\frac{x_i^4}{(1+\theta x_i^2)^2} - n\left(\frac{-1}{(\theta+2)^2} + \frac{e^{\theta}(e^{\theta}-\theta-1)-(e^{\theta}-1)^2}{(e^{\theta}-\theta-1)^2}\right). \quad (17)$$

4.2 Estimation using least squares and the weighted least squares

Swain et al. [37] introduced least squares and weighted least squares estimators for calculating the parameters of distributions. In this study, a similar method is employed for the TUCJ distribution technique. The least squares TUCJ estimator for the parameter θ is obtained by minimizing the distribution.

$$\sum_{j=1}^{n}\left[F(x_j) - \frac{j}{n+1}\right]^2.$$

About the unknown parameter θ, assume $F(X_j)$ is the distribution function of the ordered random variables, where $X_1, X_2, \ldots X_n$ is an n-th random sample from $F(\cdot)$. Consequently, in this example, the least square estimator of θ, denoted as $\hat{\theta}_{LSE}$, can be obtained by minimizing:

$$\sum_{j=1}^{n}\left[\frac{\theta x(\theta x+2)e^{\theta}}{(\theta+2)\left((\theta+1)e^{\theta x}-e^{\theta}\right)} - \frac{j}{n+1}\right]^2.$$

4.3 Cramer-Von-Mises estimation (CVME)

The CVME method is derived by minimizing the discrepancy between the cumulative and empirical distribution functions, as summarized below:

$$CMV(\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_{(i)};\theta) - \frac{2i-1}{2n}\right)^2, \quad (18)$$

The first derivative with respect to θ is expressed as:

$$\frac{\partial CMV(\theta)}{\partial \theta} = 2\sum_{i=1}^{n}\left(F(x_{(i)};\theta) - \frac{2i-1}{2n}\right)F'_{\theta}(x_{(i)};\theta) = 0$$
$$= 2\sum_{i=1}^{n}\left(\frac{\theta x(\theta x+2)e^{\theta}}{(\theta+2)\left((\theta+1)e^{\theta x}-e^{\theta}\right)} - \frac{2i-1}{2n}\right) \times \frac{\theta^2(1+\theta x^2)e^{\theta(1-x)}}{(\theta+2)(e^{\theta}-\theta-1)} = 0. \quad (19)$$

The value $\hat{\theta}$ of θ minimizes Equation (19), and this can be analytically solved using various mathematical programs.

4.4 Maximum product of spacings

The Maximum Product of Spacings (MPS) estimation method, pioneered by Cheng and Amin [14] and separately by Ranneby [32], provides an alternative approach to Maximum Likelihood (ML) estimation. They explored various properties of MPS estimators, demonstrating that MPS offers consistent and asymptotically efficient estimates in scenarios where ML estimators may face challenges. This is particularly evident when the likelihood function lacks an upper bound, in situations involving heavy-tailed distributions with unspecified scale and location parameters, as discussed by [30], and in the context of mixture distributions. Consequently, the MPS method overcomes some drawbacks associated with the ML method while retaining nearly all its properties in large samples [15]. Consider an ordered sample $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ of size n from the TUCJ distribution. The uniform spacings for this ordered sample are given by:

$$D_i(\theta) = F(x_{(i)}|\theta) - F(x_{(i-1)}|\theta); i = 1, \ldots, n+1. \quad (20)$$

where $F(x_{(0)}|\theta) = 0, F(x_{(n+1)}|\theta) = 1$ and $\sum_{i=1}^{n+1} D_i(\theta) = 1$. Then, the product spacings function can be expressed as:

$$G(\theta) = \left(\prod_{i=1}^{n+1} D_i(\theta)\right)^{1/n+1}. \quad (21)$$

The logarithm of the product spacing is

$$\log G = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[F(x_{(i)}|\theta) - F(x_{(i-1)}|\theta)\right]. \quad (22)$$

The MPS estimator, θ_{MPS}, of θ is obtained by solving the non-linear equation $\partial \log G/\partial\theta = 0$ numerically as it has no analytical solution. The residual methods such as Anderson-Darling estimation (ADE), Right tail Anderson-Darling estimation (RTADE) and Percentile estimation (PE) are discussed by [38], [39] and [43].

5 Bayesian estimation method

In this section, Bayesian estimation (BE) is used to estimate the parameter θ, assumed to be independent and following a gamma prior distribution with parameters a and b as follows:

$$g(u;a,b) = \frac{b^a}{\Gamma(a)} u^{a-1} e^{-ub}, \quad u,a,b > 0. \tag{23}$$

Subsequently, the joint prior density function of θ is expressed as:

$$g(\theta,\alpha) = \prod_{i=1}^{n} g(\theta) \propto (\theta)^{a-1} e^{-\theta b}. \tag{24}$$

The joint posterior distribution function, as per the Bayesian procedure, is provided by:

$$g(\theta|\underline{x}) = \frac{g(\theta)L(\underline{x})}{\int g(\theta)L(\underline{x})} \propto g(\theta)L(\underline{x}). \tag{25}$$

Substituting from Equations (15) and (24) into Equation (25), we get

$$g(\theta|\underline{x}) \propto (\theta)^{a-1}. \tag{26}$$

The Markov Chain Monte Carlo (MCMC) method [12] is employed to numerically summarize the posterior distribution without the need to calculate the normalized constant.

6 Simulation Study

The performance evaluation of estimators considers varying sample sizes n. A numerical analysis assesses the performance of estimates for the TUCJ model, focusing on biases and mean square errors (MSE). The simulation utilizes R and Open Bugs programs. The algorithm for generating random samples from the TUCJ distribution using the inversion method is outlined as follows:

Random samples $(X_1, X_2, \ldots, X_n)$ of sizes $n = 25, 50, 75, 100, 200$, and 300 are generated from the TUCJ distribution. Parameter values $\theta = 2, 1.5, 0.5$, and 0.25 are considered. Estimates of the TUCJ model are evaluated based on these parameter values and sample sizes. Biases and mean squared errors (MSEs) of the estimates are computed across different parameter values. Empirical results are presented in Tables 1 to 4.

The tables from the simulation study (Tables 1 to 4) enable the following conclusions to be drawn.

- The findings from Tables (1-4) demonstrate the stability of the TUCJ distribution, as indicated by the small bias and root mean square error (RMSE) observed for its parameters.
- With increasing sample size, there is occasionally a reduction in bias and RMSE across all estimations.
- This suggests that different estimation techniques produce reliable bias and RMSE results for large sample sizes.
- The MPSE estimation method provides superior metrics compared to the LSE, WLSE, CVME, ADE, and RTADE approaches.
- As the sample size increases, the bias and RMSE values of all estimators decrease, indicating enhanced accuracy in estimating model parameters.
- LSE, WLSE, CVME, ADE, and RTADE exhibit consistently lower bias than other parameters across various sample sizes.
- All sample sizes exhibit a positive bias in the estimators.
- From Tables (1-4), it is observed that the WLSE, LSE, CVME, ADE, RTADE, MLE, and Bayesian methods consistently yield smaller values, indicating accurate bias and RMSE results for large sample sizes.

Figures (4), (5), (6), and (7) show the trace and the posterior density plots for various values of θ and various sample sizes when the number of iterations is 20000. These figures are generated using the OpenBUGS software program.

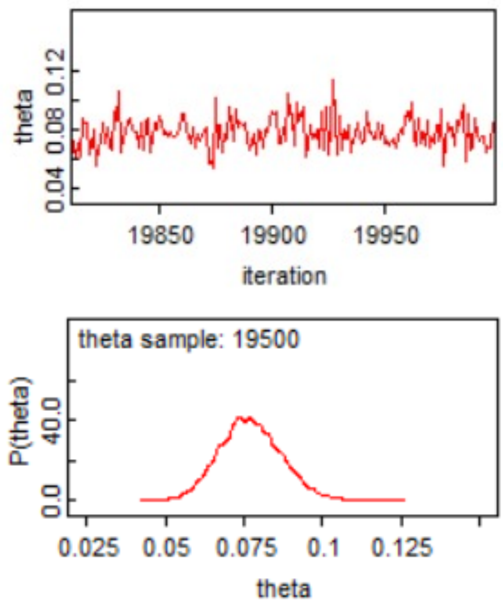

Fig. 4: The trace and the density plots of $\theta = 2$ when the sample size = 200.

Table 1: Results of simulation study for $\theta = 2$

		$\theta = 2$								
N	Name	MLE	MPS	LSE	WLSE	CVME	ADE	RADE	PE	BE
25	Mean	1.9158	2.1614	2.1215	2.1098	2.1242	2.1237	2.0582	2.4975	1.9768
	RBias	0.0421	0.0807	0.0608	0.0549	0.0621	0.0619	0.0291	0.2488	0.0116
	MSE	1.6387	1.6054	1.4252	1.4038	1.4598	1.3789	1.4346	0.9311	1.1586
	RMSE	1.2801	1.2671	1.1938	1.1848	1.2082	1.1743	1.1977	0.9650	1.0764
50	Mean	1.9496	1.9975	1.9978	1.9900	1.9988	1.9973	1.9970	2.2724	1.9005
	RBias	0.0252	0.0012	0.0011	0.0050	0.0006	0.0013	0.0015	0.1362	0.0497
	MSE	0.8952	1.0138	0.8718	0.8598	0.8844	0.8591	0.8568	0.5363	0.7045
	RMSE	0.9461	1.0069	0.9337	0.9273	0.9404	0.9269	0.9256	0.7323	0.8394
75	Mean	1.9556	1.9954	1.9719	1.9671	1.9692	1.9553	1.9421	2.1647	1.8100
	RBias	0.0222	0.0023	0.0141	0.0165	0.0154	0.0224	0.0290	0.0824	0.0950
	MSE	0.5754	0.6371	0.6193	0.6023	0.6388	0.6196	0.6601	0.3593	0.5260
	RMSE	0.7586	0.7982	0.7870	0.7761	0.7992	0.7871	0.8125	0.5994	0.7253
100	Mean	1.9639	1.9932	1.9847	1.9797	1.9852	1.9745	1.9549	2.1201	1.8030
	RBias	0.0181	0.0034	0.0077	0.0101	0.0074	0.0127	0.0226	0.0600	0.0985
	MSE	0.4363	0.4797	0.4441	0.4325	0.4489	0.4339	0.4941	0.2913	0.4282
	RMSE	0.6605	0.6926	0.6664	0.6577	0.6700	0.6587	0.7029	0.5397	0.6544
200	Mean	1.9860	1.9621	1.9668	1.9646	1.9676	1.9603	1.9437	1.9711	1.8478
	RBias	0.0070	0.0189	0.0166	0.0177	0.0162	0.0199	0.0281	0.0145	0.0761
	MSE	0.1884	0.1901	0.1723	0.1646	0.1726	0.1675	0.1724	0.1487	0.2407
	RMSE	0.4341	0.4360	0.4150	0.4057	0.4155	0.4093	0.4152	0.3856	0.4906
300	Mean	1.9896	1.9697	1.9645	1.9653	1.9651	1.9629	1.9521	1.9509	1.8587
	RBias	0.0052	0.0152	0.0177	0.0173	0.0174	0.0185	0.0239	0.0246	0.0706
	MSE	0.1122	0.1181	0.1254	0.1178	0.1255	0.1197	0.1231	0.1256	0.1746
	RMSE	0.3350	0.3436	0.3541	0.3432	0.3542	0.3460	0.3508	0.3544	0.4179

Table 2: Results of simulation study for $\theta = 1.5$

		$\theta = 1.5$								
N	Name	MLE	MPS	LSE	WLSE	CVME	ADE	RADE	PE	BE
25	Mean	1.4460	1.7069	1.6848	1.6733	1.6815	1.6777	1.6454	2.1516	1.6236
	RBias	0.0360	0.1379	0.1232	0.1155	0.1210	0.1184	0.0969	0.4344	0.0824
	MSE	1.5281	1.4533	1.3083	1.2927	1.3432	1.2893	1.2668	1.0533	1.0530
	RMSE	1.2362	1.2055	1.1438	1.1370	1.1590	1.1355	1.1255	1.0263	1.0261
50	Mean	1.4232	1.5092	1.5172	1.5087	1.5155	1.5121	1.5192	1.9343	1.5184
	RBias	0.0512	0.0061	0.0115	0.0058	0.0104	0.0081	0.0128	0.2895	0.0123
	MSE	0.9753	0.9559	0.8502	0.8432	0.8616	0.8503	0.8503	0.6064	0.6579
	RMSE	0.9876	0.9777	0.9220	0.9183	0.9282	0.9221	0.9221	0.7787	0.8111
75	Mean	1.4210	1.4738	1.4850	1.4766	1.4827	1.4685	1.4600	1.7759	1.3970
	RBias	0.0526	0.0175	0.0100	0.0156	0.0116	0.0210	0.0267	0.1839	0.0686
	MSE	0.6722	0.6796	0.6107	0.6017	0.6204	0.6032	0.6492	0.3822	0.4675
	RMSE	0.8199	0.8244	0.7815	0.7757	0.7876	0.7767	0.8057	0.6183	0.6837
100	Mean	1.4353	1.4675	1.4773	1.4691	1.4751	1.4618	1.4436	1.6950	1.3675
	RBias	0.0432	0.0217	0.0152	0.0206	0.0166	0.0255	0.0376	0.1300	0.0883
	MSE	0.5169	0.5395	0.4871	0.4827	0.4950	0.4894	0.5415	0.3069	0.3869
	RMSE	0.7190	0.7345	0.6979	0.6948	0.7035	0.6995	0.7359	0.5540	0.6220
200	Mean	1.4601	1.4199	1.4485	1.4450	1.4481	1.4408	1.4147	1.5230	1.3650
	RBias	0.0266	0.0534	0.0343	0.0367	0.0346	0.0395	0.0569	0.0153	0.0900
	MSE	0.2580	0.2798	0.2211	0.2159	0.2224	0.2181	0.2440	0.1488	0.2437
	RMSE	0.5080	0.5290	0.4703	0.4646	0.4716	0.4670	0.4940	0.3857	0.4937
300	Mean	1.4667	1.4375	1.4447	1.4369	1.4444	1.4342	1.4285	1.5189	1.3388
	RBias	0.0222	0.0417	0.0369	0.0421	0.0371	0.0439	0.0477	0.0126	0.1074
	MSE	0.1721	0.1942	0.1711	0.1854	0.1721	0.1885	0.1744	0.1059	0.1969
	RMSE	0.4148	0.4407	0.4137	0.4305	0.4149	0.4342	0.4177	0.3255	0.4437

Table 3: Results of simulation study for $\theta = 0.5$

		$\theta = 0.5$								
N	Name	MLE	MPS	LSE	WLSE	CVME	ADE	RADE	PE	BE
25	Mean	0.7280	0.9802	1.0710	1.0362	1.0462	1.0330	1.0577	1.5578	1.1030
	RBias	0.4559	0.9603	1.1421	1.0725	1.0924	1.0661	1.1153	2.1155	1.2060
	MSE	1.3115	1.3400	1.2295	1.2372	1.2488	1.2227	1.2274	1.7488	1.2117
	RMSE	1.1452	1.1576	1.1088	1.1123	1.1175	1.1057	1.1079	1.3224	1.1008
50	Mean	0.6384	0.8139	0.8363	0.8295	0.8283	0.8338	0.8638	1.3491	0.9739
	RBias	0.2769	0.6277	0.6726	0.6590	0.6567	0.6675	0.7276	1.6983	0.9479
	MSE	0.8313	0.7503	0.7246	0.7023	0.7285	0.7116	0.7777	1.1252	0.7696
	RMSE	0.9118	0.8662	0.8512	0.8380	0.8535	0.8436	0.8819	1.0608	0.8773
75	Mean	0.5660	0.7293	0.7651	0.7485	0.7585	0.7469	0.7684	1.2299	0.8030
	RBias	0.1320	0.4587	0.5303	0.4970	0.5169	0.4938	0.5368	1.4599	0.6059
	MSE	0.6122	0.4986	0.5078	0.4899	0.5107	0.4820	0.5355	0.7427	0.4605
	RMSE	0.7824	0.7061	0.7126	0.6999	0.7146	0.6942	0.7318	0.8618	0.6786
100	Mean	0.5279	0.6725	0.6866	0.6730	0.6747	0.6687	0.6931	1.1191	0.7487
	RBias	0.0559	0.3450	0.3732	0.3461	0.3494	0.3374	0.3861	1.2382	0.4974
	MSE	0.5224	0.4314	0.4287	0.4201	0.4352	0.4170	0.4508	0.6300	0.3563
	RMSE	0.7228	0.6568	0.6547	0.6482	0.6597	0.6457	0.6714	0.7938	0.5969
200	Mean	0.4694	0.5384	0.5348	0.5319	0.5259	0.5295	0.5452	0.8833	0.6782
	RBias	0.0613	0.0769	0.0695	0.0638	0.0518	0.0590	0.0905	0.7666	0.3564
	MSE	0.3488	0.2386	0.2476	0.2374	0.2525	0.2402	0.2343	0.2989	0.2315
	RMSE	0.5906	0.4885	0.4976	0.4872	0.5025	0.4901	0.4841	0.5467	0.4812
300	Mean	0.4488	0.4750	0.5254	0.5061	0.5211	0.5175	0.5275	0.7724	0.5991
	RBias	0.1024	0.0500	0.0507	0.0122	0.0422	0.0350	0.0550	0.5448	0.1982
	MSE	0.2789	0.1945	0.1818	0.1804	0.1832	0.1760	0.1816	0.1832	0.1482
	RMSE	0.5281	0.4410	0.4263	0.4248	0.4280	0.4195	0.4261	0.4280	0.3850

7 Applications and Goodness of Fit

In this section, we propose evaluating the goodness of fit of the TUCJ distribution to real lifetime data and comparing it with several one- and two-parameter distributions.

7.1 Data set I

In this section, we show real data analysis to demonstrate the utility of our process. The data set is used by Kumari et al. (2019) [26]. The information shows how many hours, on average, a fleet of thirteen Boeing 720 jet planes' air conditioning systems fail. According to Canavos and Tsokos[13], exponential distributions provide a reasonably accurate representation of the failure time distribution for the air cooling systems on each plane. The planes "7913" and "7914" have been considered for our illustrative needs. The information is displayed as follows: $0.0046, 0.0184, 0.0507, 0.0737, 0.0829, 0.1106, 0.1429,$

Table 4: Results of simulation study for $\theta = 0.25$

					$\theta = 0.25$					
N	Name	MLE	MPS	LSE	WLSE	CVME	ADE	RADE	PE	BE
25	Mean	0.5313	0.7762	0.9791	0.9528	0.9655	0.9467	0.9734	1.4797	1.0367
	RBias	1.1250	2.1049	2.9163	2.8112	2.8619	2.7869	2.8934	4.9188	3.1467
	MSE	1.3543	1.3211	1.3952	1.3913	1.4006	1.3741	1.4086	2.1533	1.4488
	RMSE	1.1637	1.1494	1.1812	1.1795	1.1835	1.1722	1.1869	1.4674	1.2037
50	Mean	0.4505	0.6075	0.7566	0.7510	0.7471	0.7521	0.7745	1.1977	0.8986
	RBias	0.8019	1.4302	2.0264	2.0038	1.9883	2.0082	2.0981	3.7909	2.5945
	MSE	0.8548	0.7072	0.8150	0.7882	0.8153	0.8005	0.8901	1.3452	0.9594
	RMSE	0.9246	0.8410	0.9028	0.8878	0.9030	0.8947	0.9435	1.1598	0.9795
75	Mean	0.3702	0.5311	0.6794	0.6587	0.6722	0.6508	0.6996	1.0857	0.7182
	RBias	0.4809	1.1242	1.7175	1.6346	1.6889	1.6031	1.7985	3.3430	1.8727
	MSE	0.6216	0.4794	0.5733	0.5475	0.5719	0.5395	0.6220	0.9421	0.5786
	RMSE	0.7884	0.6924	0.7572	0.7399	0.7562	0.7345	0.7887	0.9706	0.7606
100	Mean	0.3572	0.5140	0.5968	0.5783	0.5898	0.5815	0.6054	0.9733	0.6628
	RBias	0.4287	1.0562	1.3874	1.3132	1.3594	1.3258	1.4215	2.8933	1.6512
	MSE	0.5200	0.4144	0.4730	0.4572	0.4727	0.4538	0.5125	0.7985	0.4467
	RMSE	0.7211	0.6437	0.6878	0.6762	0.6875	0.6736	0.7159	0.8936	0.6683
200	Mean	0.2886	0.3960	0.4344	0.4294	0.4293	0.4313	0.4388	0.7313	0.5915
	RBias	0.1546	0.5838	0.7375	0.7176	0.7173	0.7253	0.7552	1.9253	1.3660
	MSE	0.3377	0.2218	0.2468	0.2372	0.2464	0.2375	0.2476	0.4012	0.3047
	RMSE	0.5811	0.4709	0.4968	0.4870	0.4964	0.4873	0.4976	0.6334	0.5520
300	Mean	0.2704	0.3408	0.3883	0.3706	0.3834	0.3715	0.4079	0.6418	0.5031
	RBias	0.0814	0.3631	0.5531	0.4822	0.5337	0.4859	0.6318	1.5672	1.0125
	MSE	0.2614	0.1621	0.1803	0.1728	0.1801	0.1741	0.1883	0.2781	0.1901
	RMSE	0.5113	0.4027	0.4246	0.4157	0.4244	0.4172	0.4340	0.5274	0.4360

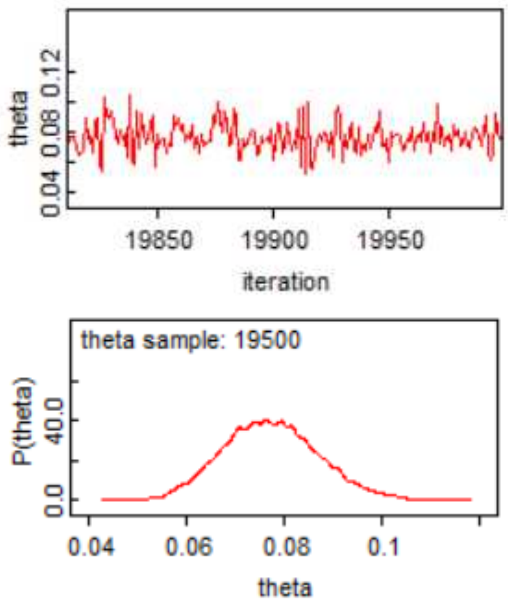

Fig. 5: The trace and the density plots of $\theta = 1.5$ when the sample size = 100.

0.1797, 0.2120, 0.2350, 0.2488, 0.2903, 0.3134, 0.3548, 0.3687, 0.3779, 0.4470, 0.4885, 0.5115, 0.6498, 0.6544, 0.7512, 0.8802, 0.9493, 0.9954. Let (n) be the sample size, and (k) represent the number of parameters. Some statistics like, $-2\ln L$, $AIC = -2\ln L + 2k + \frac{2k(k+1)}{n-k-1}$, $CAIC = -2\ln L + k\left(1+\frac{\ln n}{n}\right)$, $BIC = -2\ln L + K\ln L$, $HQIC = -2\ln L + 2K\ln(\ln n)$, K-S, and P-value for this data are calculated to assess differences among various distributions (including TUCJ, Chris-Jerry (CJ), truncated moment exponential (TME), Beta, Exponential(exp), Bur XII, and gamma distributions), and the corresponding statistics are presented in Table (5).

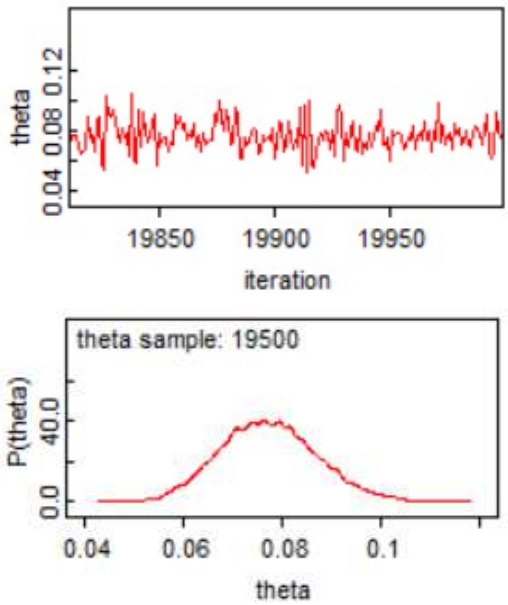

Fig. 6: The trace and the density plots of $\theta = 0.5$ when the sample size = 50.

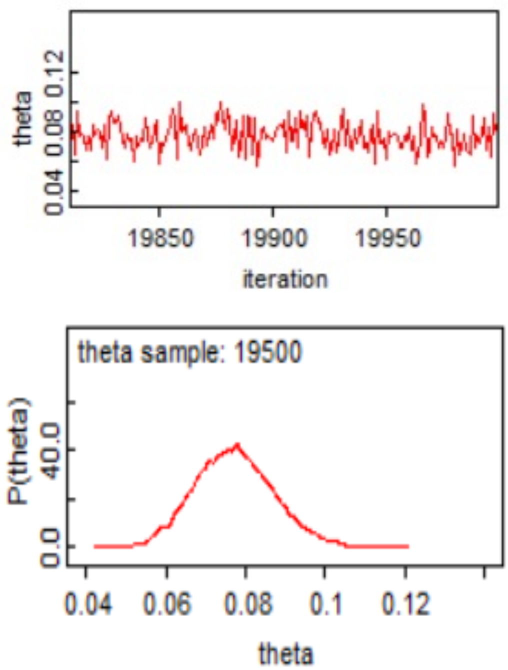

Fig. 7: The trace and the density plots of $\theta = 0.25$ when the sample size = 25.

Table 5: Some statistics for data set I for various distributions.

Model	$\hat{\theta}$	$\hat{\alpha}$	$-2\ln L$	AIC	CAIC	BIC	HQIC	K-S	P-value
TUCJ	2.98	-	4.74	2.74	2.57	1.52	2.41	0.0806	0.9924
CJ	4.40	-	0.73	2.73	2.91	3.95	3.07	0.1183	0.8355
TME	0.21	-	4.12	6.12	6.29	7.34	6.47	0.1361	0.6931
exp	2.66	-	1.05	3.05	3.22	4.27	3.39	0.1113	0.8829
beta	0.68	0.98	4.17	0.17	0.38	2.27	0.51	0.1304	0.7408
Bur XII	1.30	4.00	1.87	5.87	6.42	8.31	6.55	0.0850	0.9866
gamma	1.17	0.32	0.69	4.69	5.24	7.13	5.37	0.0849	0.9867

In Figure (8), the empirical cdf, pdf, and the (P-P) plots for data set I are illustrated across different distributions. The TUCJ distribution is identified as the most suitable fit for data set I, as indicated by the information presented in Table (5) and Figure (8).

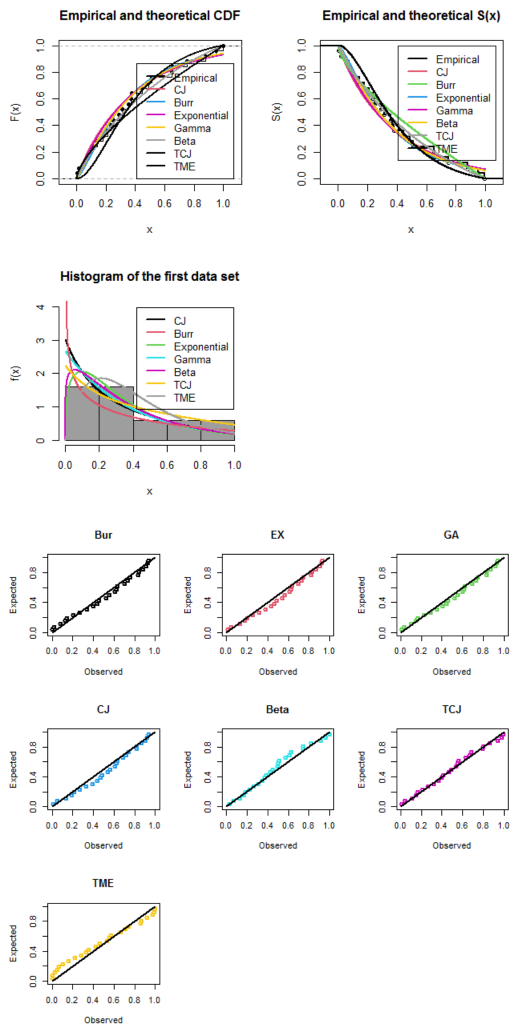

Fig. 8: The empirical cdf, pdf, and (P-P) plots for data set I for various distributions.

Table 6: Bayesian and non-Bayesian estimates of the TUCJD's parameters for the first real data set.

Method	Parameter	
	Estimated value	Standard Error
MLE	2.9769	1.0176
MPS	2.8916	0.9994
LSE	3.1146	2.3131
WLSE	3.0926	0.2048
CVM	3.1229	2.3004
ADE	3.0392	1.0380
RTADE	2.9823	1.4375
BE	2.7183	0.7222

The Bayes Estimates (BE) and Weighted Least Squares Estimates (WLSE) performed better than the others, as evidenced in Table 6. This is attributed to these estimates having the lowest standard errors. Table 6 presents the standard errors (Std. Error) for classical and Bayesian estimates of the parameters for TUCJD.

7.2 Data set II

The data set below includes 30 measurements of the tensile strength of polyester fibers; see Mazuchli et al. [44].
$0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926.$

This data is used before again by [10]. Table (7) summarizes some statistics and compares some distributions. Figure (9) illustrates the empirical cdf, pdf, and the (P-P) plots for data set II across different distributions.

Table 7: Some statistics for data set II for various distributions.

Model	$\hat{\theta}$	$\hat{\alpha}$	$-2\ln L$	AIC	CAIC	BIC	HQIC	K-S	P-value
TUCJ dist.	3.19	-	6.45	4.45	4.31	3.05	4.00	0.0596	0.9997
CJ dist.	4.50	-	0.60	1.40	1.54	2.80	1.84	0.1250	0.6902
TME dist.	0.20	-	3.14	1.14	1.00	0.26	0.69	0.1319	0.6261
exp	2.73	-	0.33	1.67	1.81	3.07	2.12	0.1272	0.6701
beta	0.97	1.62	6.61	2.61	2.17	0.19	1.71	0.0669	0.9979
Bur XII	1.45	4.51	2.05	1.95	2.40	4.76	2.85	0.1037	0.8710
gamma	1.49	0.25	2.88	1.12	1.56	3.92	2.02	0.1028	0.8776

The TUCJ distribution emerges as the most appropriate fit for dataset II, as evident from the details presented in Table (7) and Figure (9).

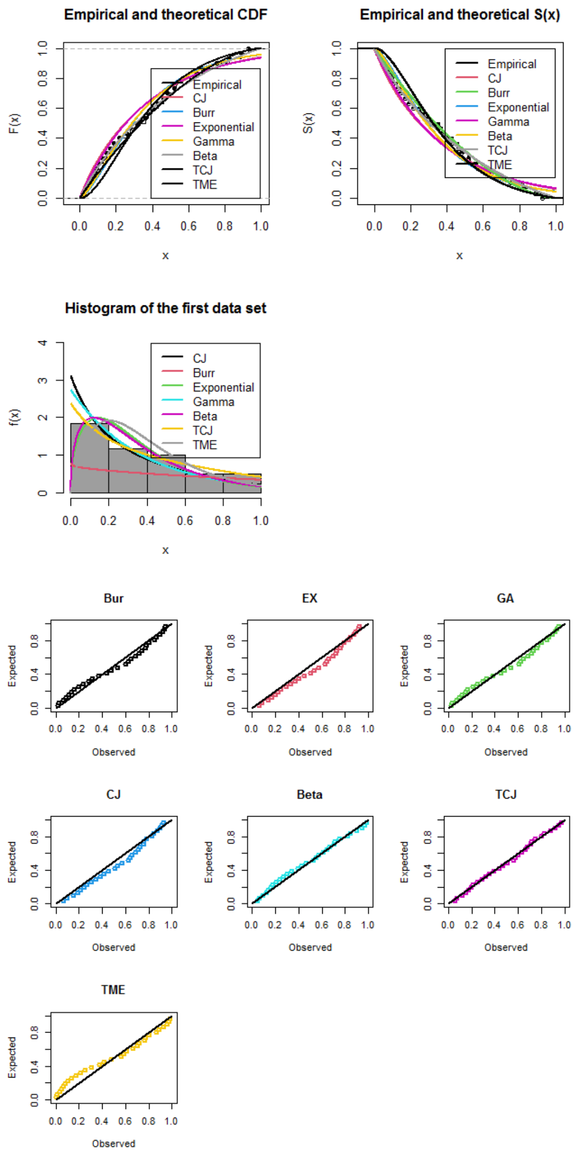

Fig. 9: The empirical cdf, pdf, and (P-P)-plots for data set II for various distributions.

Table 8: Bayesian and non-Bayesian estimates of the TUCJD's parameters for the second real data set.

Method	Parameter	
	Estimated value	Standard Error
MLE	3.1943	0.9225
MPS	3.1123	0.9081
LSE	3.1117	2.0668
WLSE	3.1005	0.1610
CVM	3.1229	2.0556
ADE	3.1251	0.9254
RTADE	3.2266	1.2281
BE	3.6565	0.033

The Bayes Estimates (BE) and Weighted Least Squares Estimates (WLSE) performed better than the others, as evidenced in Table 8. This is attributed to these estimates having the lowest standard errors. Table 8 presents the standard errors (Std. Error) for classical and Bayesian estimates of the parameters for the TUCJD.

The TTT plots for the generated random samples and the real data sets:

Using the R program, we generate three random samples from the quantile function of the TUCJD with sample size $n = 30$ at $\theta = 0.5, 1, 1.5$ respectively. The generated data sets are given as the following:
The first data set at $\theta = 0.5$ is

$x_1 = 0.89, 0.22, 0.17, 0.76, 0.90, 0.69, 0.43, 0.24, 0.53, 0.03, 0.62, 0.02, 0.72, 0.73, 0.29, 0.85, 0.48, 0.62, 0.98, 0.10, 0.57, 0.21, 0.66, 0.35, 0.85, 0.32, 0.65, 0.44, 0.28, 0.26,$

the second data set at $\theta = 1$ is
$x_2 = 0.76, 0.49, 0.72, 0.25, 0.50, 0.33, 0.83, 0.56, 0.72, 0.48, 0.54, 0.35, 0.56, 0.35, 0.04, 0.41, 0.59, 0.60, 0.46, 0.24, 0.02, 0.52, 0.32, 0.73, 0.25, 0.42, 0.13, 0.10, 0.20, 0.13,$

and the third data set at $\theta = 0.5$ is
$x_3 = 0.66, 0.30, 0.31, 0.10, 0.26, 0.82, 0.32, 0.09, 0.05, 0.23, 0.60, 0.37, 0.63, 0.07, 0.45, 0.33, 0.72, 0.27, 1.00, 0.59, 0.25, 0.18, 0.25, 0.59, 0.53, 0.36, 0.98, 0.50, 0.04, 0.94.$

Figure (10) shows an increasing hazard rate function to the truncated unit Chris-Jerry distribution for the three generating random samples and the real data sets.

8 Conclusion

A novel life distribution termed the truncated unit Chris-Jerry (TUCJ) distribution, is introduced and thoroughly examined in this study. The distribution's characteristics are derived, including moments, moment-generating function, and order statistics. The TUCJ distribution exhibits an increasing hazard rate function. Various estimation techniques such as Maximum Likelihood Estimators (MLEs), Least Squares Estimators (LSEs), Weighted Least Squares Estimators (WLSEs), Conditional Variance Moment Estimators (CVMEs), Approximate Distribution Estimators (ADEs), and Robust Truncated Approximate Distribution Estimators (RTADEs) are derived and applied. Bayesian inference is explored using squared error loss function, with parameter simulations conducted using classical and Bayesian methods. The proposed TUCJ distribution demonstrates superior performance compared to other fitted distributions namely CJ, TME, exponential, beta, Burr XII, and gamma distributions based on fitness

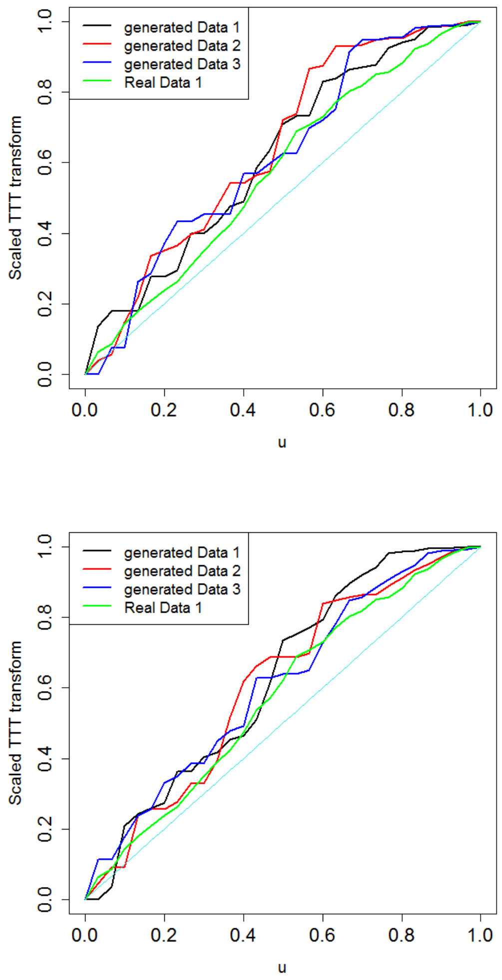

Fig. 10: The TTT plots for the three generated and the two real data sets.

metrics (K-S and P-value) and performance indices (-2ln L, AIC, CAIC, BIC, and HQIC).

Data Availability:

All data generated or analyzed during this study are included in this article.

Conflicts of Interest:

The authors declare no conflict of interest.

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