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On The Fuzzy Topological Spaces Based On A Fuzzy Space (X,I)

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Abstract: The fuzzy set ($\mathscr{FS}$) X is a class of objects associated by a membership that assigns each element of X a grade value (or values) in the closed interval $I=[0,1]$. Such a set defines a new type of topology called fuzzy topology ($\mathscr{FT}$). There are many definitions for the $\mathscr{FT}$, one of these definitions is Dip's definition that introduced the fuzzy space ($\mathscr{FS}_{\mathscr{P}}$) (X,I) as a set of fuzzy subspaces ($\mathscr{F}_{\mathscr{S}}\mathscr{S}$), and defined $\mathscr{FT}$ on the fuzzy space ($\mathscr{FS}_{\mathscr{P}}$) (X,I) which we study and develop in this paper. Various kinds of fuzzy topological spaces ($\mathscr{FTS}$) on the $\mathscr{FS}_{\mathscr{P}}$ (X,I) are defined and explained in this article, for example, cofinite (and cocountable) $\mathscr{FT}$, left (and right) ray $\mathscr{FT}$, and standard $\mathscr{FT}$. The fuzzy point ($\mathscr{FP}$) is studied and classified. So the exterior, interior, boundary, dense, and isolated $\mathscr{FP}$ are defined, and we apply some theorems on them. Furthermore, fuzzy separation axioms are presented with illustrated examples.

Keywords: finite fuzzy subspace, countable fuzzy subspace, fuzzy point, dense fuzzy subspace, fuzzy separation axioms.

1 Introduction

In 1965, Zadeh defined the $\mathscr{FS}$ X, he presented it as a class of elements with a membership function of each in the interval I, see [11]. This definition of $\mathscr{FS}$ generalizes the definition of classical set, if A is a set of objects in ordinary sense, its membership function can assign one value 0 or 1 according as the object does or does not belong to A. The belonging concept represents a basic job in the ordinary sets, does not work the similar function in the $\mathscr{FS}$. It makes no sense to say that a point x belongs to a $\mathscr{FS}$ A except in the trivial sense of its membership function being positive. One can talk about two levels α and β such that $0<\beta<\alpha<1$, then say that, x belongs to A if its membership function $\mu_A(x)\geq\alpha$, x does not belong to A if $\mu_A(x)\leq\beta$, or x has an indeterminate case relative to A if $\beta\leq\mu_A(x)\leq\alpha$. This takes us to a three-valued logic with three truth values: $T(\mu_A(x)\geq\alpha)$, $F(\mu\leq\beta)$, and $U(\beta\leq\mu_A(x)\leq\alpha)$, see [7].

Chang defined the $\mathscr{FT}$ on a the $\mathscr{FS}$ X, that satisfies the axioms of ordinary topology, see [1]. In [10], Wong suggested a new concepts of clustering and convergence in the $\mathscr{FT}$. Hazra et al. [5] presented a definition of $\mathscr{FT}$ depending on the membership of the ($\mathscr{F}_{\mathscr{S}}\mathscr{S}$) based on the $\mathscr{FS}$ X. In 1999, Dib [2] defined a $\mathscr{FT}$ on a $\mathscr{FS}_{\mathscr{P}}$ (X,I) which is a set of ($\mathscr{F}_{\mathscr{S}}\mathscr{S}$)s that satisfy the axioms of ordinary topology. In [9], Banu and Halis presented the fuzzy soft topological space. Kim et al. [6] defined a bipolar ($\mathscr{FP}$) and introduced a bipolar $\mathscr{FT}$, and explained its properties. Lee and Hur defined a hesitant $\mathscr{FT}$ and base, obtained some of their properties, see [8]. Gholap and Nikumbh [4] studied interrelation between fuzzy graphs and $\mathscr{FTS}$ by adjacency relation.

In this work, we establish and improve a new stage of the Dib's approach that used the $\mathscr{FS}_{\mathscr{P}}$ (X,I) to present the $\mathscr{FT}$. We define the finite and countable ($\mathscr{F}_{\mathscr{S}}\mathscr{S}$). Different kinds of $\mathscr{FTS}$ are defined through examples like , cofinite $\mathscr{FT}$, cocountable $\mathscr{FT}$, right ray $\mathscr{FT}$, left ray $\mathscr{FT}$ and standard (usual) $\mathscr{FT}$ with explanation graphs. Then we define interior, exterior, boundary, dense, and isolated $\mathscr{FP}$. Finally, we present many theorems with properties and fuzzy separation axioms on the $\mathscr{FS}_{\mathscr{P}}$ (X,I).

This article is ordered as: Section 2, we go over the preliminaries of $\mathscr{FT}$ that are necessary in this research. In section 3, we give some types of $\mathscr{FTS}$s on the $\mathscr{FS}_{\mathscr{P}}$ (X,I) and its properties of ($\mathscr{FP}$) -($\mathscr{F}_{\mathscr{S}}\mathscr{S}$) of the

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$\mathscr{FS}_{\mathscr{P}}$ (X,I). In section 4, some fuzzy separation axioms of $\mathscr{FTS}$s are explained with examples. Finally, the results are summarized in section 5.

2 Preliminaries

Consider the a closed interval $I=[0,1]$, and the ordinary set X. we present the definitions and theorems that necessary in this research. We start with the $\mathscr{FS}_{\mathscr{P}}$ (X,I) which expresses all fuzzy elements (x,I) such that $x \in X$ and $(x,I)=\{(x,r): r \in I\}$. The subset $W \subset (X,I)$ is $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ if it satisfies the definition 2.

Definition 2.1. [2] The $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ $W \subset (X,I)$ is a set of the elements (x,w_x), such that $x \in W_0 \subset X$, and the membership $w_x \subset I$ has at least one element in addition to $\{0\}$ (I.e. $W=\{(x,w_x): x \in W_0, \{0\} \neq w_x \subset I\}$) where (x,w_x) is the fuzzy element of $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ W. In case w_x is zero only then x is not in W_0 (I.e. $w_x=\{0\} \Leftrightarrow x \notin W_0$). W_0 is the support of W and denoted by $S(W)=W_0$. The support of W is shown mathematically as $S(W)=\{x \in X: w_x=\{0,w_x'\}, w_x' \subset (0,1]\}$. If $w_x=0$ for all $x \in X$ then the support set $S(W)=\emptyset$, and hence W is called empty $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ of the $\mathscr{FS}_{\mathscr{P}}$ (X,I) which is denoted by $\emptyset^F=\{(x,\{0\}):$ for all $x \in X\}$, notice $S(\emptyset^F)=\emptyset$ as its membership is zero for every $x \in X$.

The concept subset between $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s is defined as follows

Definition 2.2.[2] Let $W=\{(x,w_x): x \in W_0\}$ and $V=\{(x,v_x): x \in V_0\}$ be two $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s then $W \subset V$ if $W_0 \subset V_0$ and $w_x \subset v_x$, for all $x \in W_0$.

$\emptyset^F$ is clearly contained in any fuzzy subspace. The point in the $\mathscr{FT}$ is called fuzzy point $(\mathscr{FP})$, which is a $\mathscr{F}_{\mathscr{S}}\mathscr{S}$ of the $\mathscr{FS}_{\mathscr{P}}$ (X,I) with restriction on its membership.

Definition 2.3.[2] Consider the $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ $P=\{(x,p_x): x \in P_0\}$, $\emptyset \neq P_0 \subset X$, if p_x has only one element ρ_x in addition to $\{0\}$, $(p_x=\{0,\rho_x\}: x \in P_0)$ then P is named a fuzzy point $(\mathscr{FP})$ of the $\mathscr{FS}_{\mathscr{P}}$ (X,I) . If $p_x \subset u_x$ for all $x \in P_0$ then $P \in U$.

Intersection $\cap$ and union $\cup$ operations in (X,I) are defined in definition 2.

Definition 2.4.[2] Let $W=\{(x,w_x): x \in W_0\}$ and $V=\{(x,v_x): x \in V_0\}$ be $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s of the $\mathscr{FS}_{\mathscr{P}}$ (X,I). Then

1.$W \cup V=\{(x,w_x \cup v_x): x \in W_0 \cup V_0\}$,
2.$W \cap V=\{(x,w_x \cap v_x): x \in W_0 \cap V_0\}$,
3.$S(W \cup V)=S(W) \cup S(V)=W_0 \cup V_0$,
4.$S(W \cap V) \subset S(W) \cap S(V)=W_0 \cap V_0$.

In the part 4 of the definition 2, the "$\subset$" will be "$=$", if $w_x \cap v_x \neq \{0\}$, for all $x \in W_0 \cap V_0$. The next definition shows that the difference between $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s is a $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$.

Definition 2.5.[2] Consider W and V are two $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s, then $W-V=\{(x,h_x): x \in W_0, h_x=(w_x-v_x) \cup \{0\}\}$

We can note $W_0-V_0 \subset S(W-V)$, but $W_0-V_0=S(W-V)$ if $w_x \subset v_x$, for every $x \in W_0 \cap V_0$.

The definition 2 shows the conditions of the $\mathscr{FTS}$ $\big((X,I),\tau\big)$.

Definition 2.6.[2] The family $\big((X,I),\tau\big)$ of $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s is called $\mathscr{FTS}$ if the following axioms are hold

1.$\emptyset^F,(X,I) \in \tau$,
2.For every $W,V \in \tau$, we have $W \bigcap V \in \tau$,
3.$\bigcup_{W \in \tau_1} W \in \tau$ for every $\tau_1 \subset \tau$.

The elements of τ are called open $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s.

The trivial $\mathscr{FT}$ has only the elements $\emptyset^F$ and (X,I). Another example, the discrete $\mathscr{FT}$ contains all the $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s of (X,I). The following definition explains the interior $\mathscr{FP}$ in the $\mathscr{FTS}$. We will use the abbreviation nbhd for neighborhood in this paper.

Definition 2.7.[2] The nbhd of the $\mathscr{FP}$ P in the $\mathscr{FT}$ τ is a $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ U, that has an element of τ, containing P. If U is a nbhd of the $\mathscr{FP}$ P, then P is called an interior point of U.

$U^0=Int(U)$ is the set of all its interior points. In the following theorem, some properties of the interior of subspaces are mentioned.

Theorem 2.8.[2] Let $\big((X,I),\tau\big)$ be a $\mathscr{FTS}$:

1.The $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ W is open iff it is a nbhd of all its $\mathscr{FP}$s.
2.W^0 is the largest open $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$, which is contained in W.
3.If $W \subset V$ then $W^0 \subset V^0$, and $(W \cap V)^0=W^0 \cap V^0$ for any $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s W and V.

The $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ W is a closed if $((X,I),\tau)$ if $W^c=(X,I)-W$ is open. The collection of closed $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s is closed under finite unions and arbitrary intersections. Moreover, $\emptyset^F$ and (X,I) are clopen $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s. The closure and limit point in the $\mathscr{FTS}$ $\big((X,I),\tau\big)$ are defined in the following definitions.

Definition 2.9.[2] The closure $\overline{W}$ of the $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ W is the intersection of all closed $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s containing W in the $\mathscr{FTS}$ $\big((X,I),\tau\big)$.

Definition 2.10.[2] The $\mathscr{FP}$ P is named a limit point of a $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ W, if every nbhd of P contains $\mathscr{FP}$s of W other than P.

The triangular fuzzy number is used in this research which is defined as follows

Definition 2.11. [3] A fuzzy number $(\mathscr{FN})$ $\bar{E}=(a,b,c)$ is named a triangular $\mathscr{FN}$ if its membership:

$$\mu_{\bar{E}}(x)=\begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a<x<b \\ \frac{c-x}{c-b}, & b<x<c \\ 0, & x \geq a \end{cases}$$

3 Fuzzy Topological Spaces

Many definitions, examples, and theorems are defined and presented depending on the $\mathscr{FS}_{\mathscr{P}}$ (X,I). In order to define the cofinite and cocountable topologies, we need to define the finite and the countable $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s of the $\mathscr{F}\mathscr{S}_{\mathscr{P}}$ (X,I).

Definition 3.1. The $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ W of the $\mathscr{F}\mathscr{S}_{\mathscr{P}}$ (X,I) is finite (countable) $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ if w_x is finite (countable) for all $x \in W_0$.

Notice that, the finite (countable) in the definition 3 is different from the case in ordinary set, we use it to introduce the cofinite (cocountable) $\mathscr{F}\mathscr{T}$.

Example 3.2. (The cofinite (cocountable) $\mathscr{F}\mathscr{T}$ on (X,I)). Let (X,I) be a $\mathscr{F}\mathscr{S}_{\mathscr{P}}$ and $\tau = \{\emptyset^F, (X,I), U \subset (X,I) :$ where U^c is finite (countable) $(\mathscr{F}_{\mathscr{S}}\mathscr{S})\}$, then τ is called cofinite (cocountable) $\mathscr{F}\mathscr{T}$ on (X,I). We show only that if $U,V \in \tau$ then $U \cap V \in \tau$. According to the definition 2, if $U \in \tau$ then $U = \emptyset^F$, $U = (X,I)$ or $(X,I) - U$ is a finite (countable) $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$. The same holds true for $V \in \tau$. Thus, if either U or V is empty, $U \cap V = \emptyset^F \in \tau$. If $U = (X,I)$, then $U \cap V = V \in \tau$, and if $V = (X,I)$ then $U \cap V = U \in \tau$. Finally, consider the remaining case where non of U or V is empty $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ or (X,I). Then $(X,I) - (U \cap V)$ is finite (countable) because its membership
$h_x = (I - (u_x \cap v_x)) \cup \{0\} = (I - u_x) \cup (I - v_x) \cup \{0\}$ finite (countable) for each $x \in X$. Hence, $(U \cap V) \in \tau$. In the same way, we can show the the union condition. It is clear that every cofinite $\mathscr{F}\mathscr{T}$ is cocountable $\mathscr{F}\mathscr{T}$.

In the next examples we define the right ray $\mathscr{F}\mathscr{T}$ and the left ray $\mathscr{F}\mathscr{T}$.

Example 3.3. (The right ray $\mathscr{F}\mathscr{T}$ on $(\mathbb{R},I)$). Let $(\mathbb{R},I)$ be a $\mathscr{F}\mathscr{S}_{\mathscr{P}}$, $\tau_r = \{\emptyset^F, (\mathbb{R},I), U^r = ((r,\infty), u_x) : u_x \subset I, x \in (r,\infty), r \in \mathbb{R}\}$, then τ is named the right ray $\mathscr{F}\mathscr{T}$ on $(\mathbb{R},I)$. The set $\tau \subset \mathscr{P}(\mathbb{R})$ forms a topology for $\mathbb{R}$. To verify this, first note that the condition (1) of the definition 2 is satisfied. Next let U^a and V^b be any two of the open right rays in τ_r. Then $U^a \cap V^b = \{(x, u_x \cap v_x) : x \in U_0^a \cap V_0^b\} = \{((max\{a,b\},\infty), w_x) : w_x = u_x \cap v_x \subset I, x \in (max\{a,b\},\infty)\}$ is also an open right ray fyzzy subspace with membership w_x. Furthermore, $U^a \cap (\mathbb{R},I) = U^a$, $U^a \cap \emptyset^F = \emptyset^F$, and $(\mathbb{R},I) \cap \emptyset^F = \emptyset^F$. Finally, consider the collection $\{U^{r_i} : i \in \Delta\}$ of nonempty open right rays $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s. Then the union is either all of $(\mathbb{R},I)$ which is in τ_r or the union is not all of $(\mathbb{R},I)$. In the latter case, the left end points of the support sets of open right rays $\{U_0^{r_i} : i \in \Delta\}$ form a set which has lower bound, and hence, a greatest lower bound which we call r_l. So $\bigcup_{i\in\Delta} U^{r_i} = U^{r_l}$ is an open right ray $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ with supported set $U_0^{r_l}$ and membership $\bigcup_{i\in\Delta} u_{x_i}$. So that $\bigcup_{i\in\Delta} U^{r_i} \in \tau_r$.

In the following example, we explain a special case of open right rays $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s U^a, V^b and W^c with their memberships in $xy-$plane where $U^a \subset V^b \subset W^c$.

Example 3.4. Consider the elements U^a, V^b and W^c in a right ray $\mathscr{F}\mathscr{T}$ on $(\mathbb{R},I)$ such that $S(U^a) = (a,\infty)$, $S(V^b) = (b,\infty)$, $S(W^c) = (c,\infty)$, where $a < b < c$ with fuzzy membership values are given in the figure 1. It is clear that $U^a \cap V^b \cap W^c = W^c$, and $U^a \cup V^b \cup W^c = U^a$

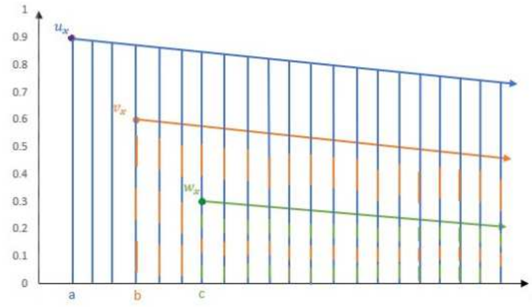

Fig. 1: Fuzzy membership values of U^a, V^b and $W^c \subset (\mathbb{R},I)$

Similarly, we can define the left ray $\mathscr{F}\mathscr{T}$.

Example 3.5. (The left ray $\mathscr{F}\mathscr{T}$ on $(\mathbb{R},I)$). Let $(\mathbb{R},I)$ be a $\mathscr{F}\mathscr{S}_{\mathscr{P}}$, $\tau_l = \{\emptyset^F, (\mathbb{R},I), U^l = ((-\infty,l), u_x) : u_x \subset I, x \in (-\infty,l), l \in \mathbb{R}\}$, then τ is called the left ray $\mathscr{F}\mathscr{T}$ on $(\mathbb{R},I)$. We can verify that τ_l satisfies the definition 2 using similar steps in the example 3.

In the same way, we can show a simple sample of open left rays $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s U^a, V^b and W^c with their memberships in $xy-$plane where $W^a \subset V^b \subset U^a$. as in the following example.

Example 3.6. Consider U^a, V^b and W^c are in a left ray $\mathscr{F}\mathscr{T}$ on (R,I) such that $S(U^a) = (-\infty,a), S(V^b) = (-\infty,b), S(W^c) = (-\infty,c)$, where $a > b > c$ with fuzzy membership values are given in figure 2. This example explains that $W^c \subset V^b \subset U^a$.

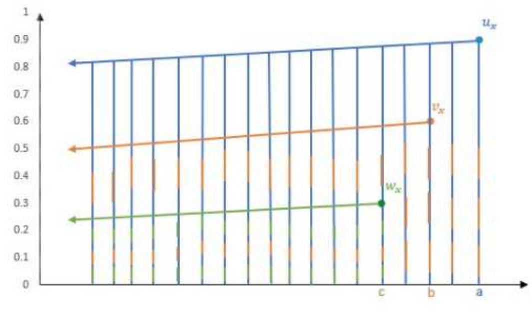

Fig. 2: Fuzzy membership values of U^a, V^b and $W^c \subset (\mathbb{R},I)$

Example 3.7. (The standard $\mathscr{F}\mathscr{T}$ on $(\mathbb{R},I)$). Let $(\mathbb{R},I)$ be a $\mathscr{F}\mathscr{S}_{\mathscr{P}}$, $\tau_s = \{\emptyset^F, (\mathbb{R},I), U \subset (\mathbb{R},I)\}$: for each $\mathscr{F}\mathscr{P}$ $P \in U$, there exists $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ $((a,b), o_x)$ containing P such that $P \in ((a,b), o_x) \subset U$ then τ is called the standard (or usual) $\mathscr{F}\mathscr{T}$ on $(\mathbb{R},I)$. To see that τ_s is standard $\mathscr{F}\mathscr{T}$ on $(\mathbb{R},I)$, we first note that $\emptyset^F$ and $(\mathbb{R},I) \in \tau_s$. Next, consider

U and V any two elements in τ_s. If either of these elements is $\emptyset^F$ or $(\mathbb{R},I)$ or it happens that $U\cap V=\emptyset^F$, the resulting intersection belongs to τ_s. Otherwise, let $P\in U\cap V=\{(x,u_x\cap v_x): x\in U_0\cap V_0\}$. Since $P\in U$, there exists $(\mathscr{F}_{\mathscr{G}}\mathscr{S})$ $\left((a,b),o_x^{(a,b)}\right)$ where $P\in\left((a,b),o_x^{(a,b)}\right)\subset U$ which means

$$P_x\subset o_x^{(a,b)}\subset u_x \quad \text{for all} \quad x\in P_0 \quad \text{and} \quad (a,b)\subset U_0 \quad (1)$$

Similarly, as $P\in V$, there exists $(\mathscr{F}_{\mathscr{G}}\mathscr{S})$ $\left((c,d),o_x^{(c,d)}\right)$ such that $P\in\left((c,d),o_x^{(c,d)}\right)\subset V$ which means

$$P_x\subset o_x^{(c,d)}\subset v_x \quad \text{for all} \quad x\in P_0 \quad \text{and} \quad (c,d)\subset V_0 \quad (2)$$

From the equations 1 and 2, we get that

$$P_x\subset o_x^{(a,b)}\cap o_x^{(c,d)}\subset u_x\cap v_x \quad (3)$$

and

$$\big(max\{a,c\},min\{b,d\}\big)=(a,b)\cap(c,d)\subset U_0\cap V_0 \quad (4)$$

Then

$$\begin{aligned} P &\in \left((a,b),o_x^{(a,b)}\right)\cap\left((c,d),o_x^{(c,d)}\right)\\ &=\left((max\{a,c\},min\{b,d\}),o_x^{(a,b)}\cap o_x^{(c,d)}\right) \quad (5)\\ &\subset\left(U_0\cap V_0,u_x\cap v_x\right)=U\cap V. \end{aligned}$$

Therefore, there exists $(\mathscr{F}_{\mathscr{G}}\mathscr{S})$ $\left((max\{a,c\},min\{b,d\}),o_x^{(a,b)}\cap o_x^{(c,d)}\right)$ contains P and is a subset of $U\cap V$, hence $U\cap V\in\tau_s$. Finally, let $\{U_\alpha\in\tau_s:\alpha\in\Delta\}$ be a family of nonempty elements of τ_s. If $U_\alpha=(\mathbb{R},I)$ for some $\alpha\in\Delta$, or the union is empty, then $\cup_{\alpha\in\Delta}U_\alpha\in\tau_s$. Otherwise, let $P\in\cup_{\alpha\in\Delta}U_\alpha$. Then $P\in U_\alpha$ for some $\alpha\in\Delta$, and there exists $(\mathscr{F}_{\mathscr{G}}\mathscr{S})$ $\big((a,b),o_x\big)$ containing P such that $P\in\big((a,b),o_x\big)\subset U_\alpha\subset\cup_{\alpha\in\Delta}U_\alpha$. Thus the union belongs to τ_s.

The following example shows a sample of elements of the standard $\mathscr{F}\mathscr{T}$ on $(\mathbb{R},I)$ with their memberships in $xy-$plane.

Example 3.8. Let $U_1=\left((a,b),o_x^{(a,b)}\right)$, $U_2=\left((c,d),o_x^{(c,d)}\right)$ $U_3=\left((e,f),o_x^{(e,f)}\right)$ be $(\mathscr{F}_{\mathscr{G}}\mathscr{S})$ of $(\mathbb{R},I)$, where

$$\mu_{U_I}=\begin{cases}u_x, & x\in U_i,\ i=1,2,3.\\ 0, & o.w\end{cases}$$

Using triangular $\mathscr{F}\mathscr{N}$, we can find the value of $\mu(x)$. It is clear that $U_3\subset U_2\subset U_1$, See figure 3.

In the following definitions, we add more properties that enrich the $\mathscr{F}\mathscr{T}\mathscr{S}$ $((X,I),\tau)$ such as exterior, boundary, dense, and isolated point. The interior $\mathscr{F}\mathscr{P}$ was explained in the definition 2, in the following definitions we define the exterior and boundary $\mathscr{F}\mathscr{P}$ of a $(\mathscr{F}_{\mathscr{G}}\mathscr{S})$, and to achieve this we explain the complement of a $(\mathscr{F}_{\mathscr{G}}\mathscr{S})$ U in figure 4.

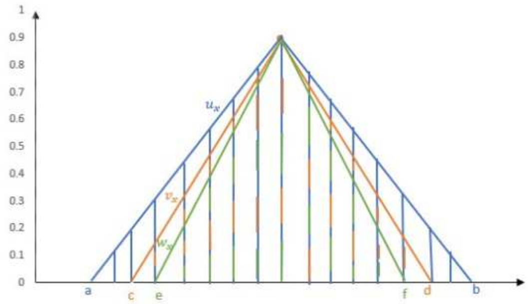

Fig. 3: Fuzzy membership values of U_1, U_2, $U_3\subset(\mathbb{R},I)$

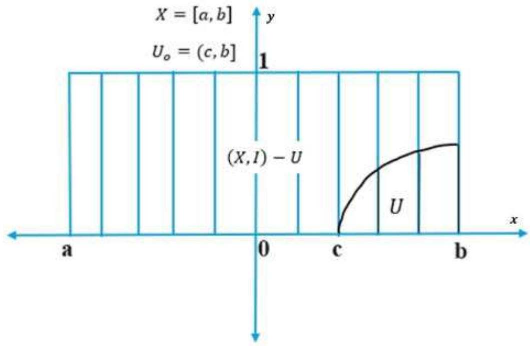

Fig. 4: Complement of a $\mathscr{F}\mathscr{S}_{\mathscr{P}}$ U

Definition 3.9. A $\mathscr{F}\mathscr{P}$ P of the $\mathscr{F}\mathscr{S}_{\mathscr{P}}$ (X,I) is an exterior $\mathscr{F}\mathscr{P}$ of a subspace B, if there exists a nbhd G of P such that $G\cap B=\emptyset^F$. The union of all exterior $\mathscr{F}\mathscr{P}$s of B denoted by $Ext(B)$.

Definition 3.10. A $\mathscr{F}\mathscr{P}$ P of the $\mathscr{F}\mathscr{S}_{\mathscr{P}}$ (X,I) is a boundary $\mathscr{F}\mathscr{P}$ of a subspace B, if every nbhd G of P has at least one $\mathscr{F}\mathscr{P}$ of B and at least one $\mathscr{F}\mathscr{P}$ of $(X,I)-B$. The union of all boundary $\mathscr{F}\mathscr{P}$s of B denoted by $Bd(B)$.

The definitions 2, 3, and 3 tell us directly that $Int(B)\subset B$, and $Ext(B)\subset(X,I)-B$ while boundary $\mathscr{F}\mathscr{P}$s of B may locate in either B or $((X,I)-B)$, It also tell us that these three $(\mathscr{F}_{\mathscr{G}}\mathscr{S})$s are pairwise disjoint. In the following theorem, we can see the relation between $Bd(B)$ and B when B is open or closed.

Theorem 3.11. Let B be a $(\mathscr{F}_{\mathscr{G}}\mathscr{S})$ of the $\mathscr{F}\mathscr{S}_{\mathscr{P}}$ (X,I). Then

1. B is open iff B contains non of all boundary $\mathscr{F}\mathscr{P}$s.
2. B is closed iff B contains all boundary $\mathscr{F}\mathscr{P}$s.

Proof. For first part.
$(\Rightarrow)$ Let B be open. Then $Int(B)=B$ by theorem 2, so $Bd(B)\cap B=\emptyset^F$.
$(\Leftarrow)$ Let $x\in B$ and $x\notin Bd(B)$. Since $x\notin Ext(B)$, it follows

that $x \in Int(B)$, and hence $B \subset Int(B)$. As $Int(B) \subset B$, we get $B = Int(B)$ open.
For second part.
$(\Rightarrow)$ Let B be closed, then $(X,I) - B$ is open. By using first part, we have $\Big((X,I) - B\Big) \cap Bd\Big((X,I) - B\Big) = \emptyset^F$. But $Bd\Big((X,I) - B\Big) = Bd(B)$, so $Bd(B) \subset B$.
$(\Leftarrow)$ Let $Bd(B) \subset B$. Then $Bd(B) \not\subset \Big((X,I) - B\Big)$, leads to $\Big((X,I) - B\Big)$ is open, and hence B is closed.

In the next theorem, we express the exterior of a subspace in terms of the interior of a subspace.

Theorem 3.12. If B is $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ of the $\mathscr{F}\mathscr{T}$ $((X,I),\tau)$, then

1. $Ext(B) = Int((X,I) - B)$.
2. $Ext(\emptyset^F) = Int\Big((X,I)\Big)$ and $Int(\emptyset^F) = Ext\Big((X,I)\Big)$.

Proof. For part 1, because $B \subset \overline{B} \xrightarrow{complement} \Big((X,I) - \overline{B}\Big) \subset \Big((X,I) - B\Big)$. Notice that, the $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ $\Big((X,I) - \overline{B}\Big)$ is open, and by taking $Int()$ for both sides, we get $Ext(B) = \Big((X,I) - \overline{B}\Big) \subset Int\Big((X,I) - B\Big) \subset \Big((X,I) - B\Big)$. On the other hand, take the complement of the part $Int\Big((X,I) - B\Big) \subset \Big((X,I) - B\Big)$, we get $B \subset (X,I) - Int\Big((X,I) - B\Big) \xrightarrow{closure} \overline{B} \subset (X,I) - Int\Big((X,I) - B\Big)$
$\xrightarrow{complement} Int\Big((X,I) - B\Big) \subset (X,I) - \overline{B} = Ext(B)$.
Part 2 is direct from part 1.

Using definition 2 of closure, we can define the dense $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ as following.

Definition 3.13. Let $((X,I),\tau)$ be a $\mathscr{F}\mathscr{T}\mathscr{S}$. A $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ $D \subseteq (X,I)$ is said to be dense if $\overline{D} = (X,I)$.

Example 3.14. Let $(\mathbb{R},I)$ be a $\mathscr{F}\mathscr{S}_{\mathscr{P}}$, τ_s be the standard $\mathscr{F}\mathscr{T}$ on $(\mathbb{R},I)$, and $U = \{(\mathbb{Q},Q_x) : Q_x is$ all rational numbers in $I\}$ be a $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$. Then U is dense in τ_s.

The limit $\mathscr{F}\mathscr{P}$ was covered in definition 2, but if the $\mathscr{F}\mathscr{P}$ of a $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ U is not limit $\mathscr{F}\mathscr{P}$, then it is called an isolated $\mathscr{F}\mathscr{P}$ according to the next definition.

Definition 3.15. Let U be a $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$ of (X,I), then a $\mathscr{F}\mathscr{P}$ $P \in U$ is an isolated of U if there exists nbhd G containing P such that $G \cap U = P$.

Definition 3.16. The two $(\mathscr{F}_{\mathscr{S}}\mathscr{S})$s U and V are disjoint if $U \cap V = \emptyset^F$ $(U_0 \cap V_0 = \emptyset)$.

4 Fuzzy Separation Axioms

Fuzzy separation axioms on $\mathscr{F}\mathscr{T}$ $((X,I),\tau)$ are presented and studied with examples.

Definition 4.1. Let τ be a $\mathscr{F}\mathscr{T}$ on the $\mathscr{F}\mathscr{S}_{\mathscr{P}}$ (X,I). Then

(a) $\Big((X,I),\tau\Big)$ is a $T_0 - \mathscr{F}\mathscr{T}\mathscr{S}$ if for any distinct $\mathscr{F}\mathscr{P}$s $p,q \in (X,I)$, there is a nbhd $H \subset (X,I)$ such that H contains one of p or q but not the other.
(b) $\Big((X,I),\tau\Big)$ is a $T_1 - \mathscr{F}\mathscr{T}\mathscr{S}$ if for any distinct $\mathscr{F}\mathscr{P}$s $p,q \in (X,I)$, there are two nbhds $G,H \subset (X,I)$ such that $p \in G$ but $q \notin G$ or $q \in H$ but $p \notin H$.
(c) $\Big((X,I),\tau\Big)$ is a $T_2 - \mathscr{F}\mathscr{T}\mathscr{S}$ if for any distinct $\mathscr{F}\mathscr{P}$s $p,q \in X$, there are two disjoint nbhds $G,H \subset X$ such that $p \in G, q \in H$.
(d) $\Big((X,I),\tau\Big)$ is a fuzzy regular space ($\mathscr{F}\mathscr{R}\mathscr{S}$) if for any $\mathscr{F}\mathscr{P}$ $p \in (X,I)$ and any closed fuzzy subspace $F \subset (X,I)$ such that $p \notin F$, there are two disjoint nbhds $G,H \subset (X,I)$ such that $p \in G, F \subset H$. A $\mathscr{F}\mathscr{R}\mathscr{S}$ and $T_1 - \mathscr{F}\mathscr{T}\mathscr{S}$ is called a $T_3 - \mathscr{F}\mathscr{T}\mathscr{S}$.
(e) $\Big((X,I),\tau\Big)$ is a fuzzy normal space ($\mathscr{F}\mathscr{N}\mathscr{S}$) if for any pair F_1,F_2 of disjoint closed fuzzy subspaces of (X,I), there are two disjoint nbhds G,H, so that $F_1 \subset G, F_2 \subset H$. A $\mathscr{F}\mathscr{N}\mathscr{S}$ and $T_1 - \mathscr{F}\mathscr{T}\mathscr{S}$ is called a $T_4 - \mathscr{F}\mathscr{T}\mathscr{S}$.

Example 4.2. Let (X,I) be a $\mathscr{F}\mathscr{S}_{\mathscr{P}}$ with $X = \{a\}$, and let $\tau = \{\emptyset^F, (X,I)$,
$U = (a,u_y) : u_y = [0,y)$ for some $y \in [0,1]\}$. It is easy to check that τ is $\mathscr{F}\mathscr{T}$ which is explained graphically in figure 5. For any two $\mathscr{F}\mathscr{P}$s $p = (a,\{0,c\})$ and $q = (a,\{0,d\})$ we can find a nbhd $U = (a,u_{y_1})$ such that $p \in (a,u_{y_1})$ and $q \notin (a,u_{y_1})$ where $c < y_1 < d$. But for any nbhd $V = (a,u_{y_2})$ such that $q \in V$, we have $p \in V$ as $c < d < y_2$.

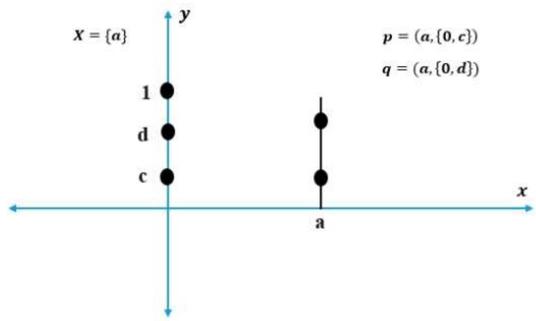

Fig. 5: Fussy T_0 space but not T_1

Example 4.3. The cofinite $\mathscr{F}\mathscr{T}$ on (X,I) is T_1 but not T_2. For any two $\mathscr{F}\mathscr{P}$s $\{(x,p_x) : x \in p_0\} = p \neq q = \{(x,q_x) : x \in q_0\}$, where $p_x = \{0,p'_x : x \in p_0\}$ and $q_x = \{0,q'_x : x \in q_0\}$. Take the nbhd $U = \{(x,u_x) : x \in U_0\}$ of p such that $u_x = I - q'_x$ and the nbhd $V = \{(x,v_x) : x \in V_0\}$ of q where $v_x = I - p'_x$. Then $p \in U$ but $q \notin U$ or $q \in V$ but $p \notin V$. This $\mathscr{F}\mathscr{S}_{\mathscr{P}}$ is not T_2 because U and V are not necessary disjoint.

5 Conclusion

The fuzzy space $(\mathscr{F}\mathscr{S}_{\mathscr{P}})$ (X,I) has significant potential for the fuzzy topology $(\mathscr{F}\mathscr{T})$. In this paper, many concepts, definitions, and theorems are established toward the fuzzy topological space $(\mathscr{F}\mathscr{T}\mathscr{S})$ $\big((X,I),\ \tau\big)$ like; exterior, interior, boundary, isolated fuzzy points, and dense subspace. In addition to some types of fuzzy topology $(\mathscr{F}\mathscr{T})$ and fuzzy separation axioms are illustrated with examples. For future work, more concepts, definitions, and theorems on the bases and subbases on the $\mathscr{F}\mathscr{T}\mathscr{S}$ $\big((X,I),\ \tau\big)$ are expected to introduce and study.

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