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A bivariate compound class of geometric–Poisson and lifetime distributions

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Abstract: Recently, Alkarni and Oraby (2012) obtained general forms for some properties of the compound class of Poisson and lifetime (PL) distributions. In this paper, we obtain some general forms for joint density, cumulative distribution, and survival functions of the bivariate case of PL class. Its conditional distributions are also studied. In addition, the compound class of geometric and lifetime distributions as well as its mixed bivariate case are discussed. For this class some conditional probabilities useful for reliability, biological survey, and engineering are also studied. Our class contains several new mixed bivariate distributions in special cases.

Keywords: Compound distribution, Geometric distribution, Poisson distribution, Mixed bivariate distribution, Random sample size

1 Introduction

Let $X_1, X_2, \ldots, X_N$ be N independent and identically distributed (iid) random variables from an absolutely continuous distribution F, where N has a geometric or Poisson distribution and is independent of X_i's and also let $U = \min\{X_i\}_{i=1}^{N}$. In recent years, several compound lifetime distributions have been introduced by U in the literature. Exponential-geometric (EG) distribution of Adamidis and Loukas (1998), exponentiated exponential-geometric (E2G) distribution of Louzada (2012), Weibull-geometric (WG) distribution of Barreto-Souza et al. (2011), exponential-Poisson (EP) distribution of Kus (2007), and Weibull-Poisson (WP) distribution of Hemmati et al. (2011) are the remarkable distributions in this connection.

Recently, Alkarni and Oraby (2012) obtained some general forms for density, cumulative distribution, survival, and hazard rate functions of U, when N has a Poisson distribution. In this paper, we study its mixed bivariate random variable, i.e., (U,N). In addition, we obtain general forms for density, cumulative distribution, survival, and hazard rate functions of U, when N has a geometric distribution and also study its mixed bivariate case.

2 General expressions for mixed bivariate distributions

2.1 A geometric distribution for N

In this section, we consider a geometric distribution for N with probability mass function

$$P(N=n) = pq^{n-1}, \quad n = 1,2,\ldots, \ (0 < q = 1-p < 1),$$

denoted by $N \sim Ge(p)$, and then we obtain a set of theorems, relations, and general forms for some properties of U and for mixed bivariate random variables (U,N).

Definition 2.1 *Let $X_1, X_2, \ldots, X_N$ be iid random variables from an absolutely continuous distribution with cumulative distribution function (cdf) F, survival function $\overline{F}$, and probability density function (pdf) f, where $N \sim Ge(p)$ is independent of X_i's. Then, we say that $U = \min\{X_i\}_{i=1}^{N}$ has a F-geometric distribution and will be denoted by $U \sim FG(\boldsymbol{\theta}, p)$, where $\boldsymbol{\theta}$ is the parameter vector of F.*

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Theorem 2.1 *Let $U \sim FG(\boldsymbol{\theta}, p)$. Then, the cdf, survival (reliability) function, pdf, and hazard rate function of U, for all $x \in R^+$, are given by*

$$F_U(x) = \frac{F(x)}{1 - q\overline{F}(x)}, \tag{1}$$

$$\overline{F}_U(x) = \frac{p\overline{F}(x)}{1 - q\overline{F}(x)}, \tag{2}$$

$$f_U(x) = \frac{pf(x)}{\{1 - q\overline{F}(x)\}^2} \tag{3}$$

and

$$h_U(x) = \frac{h(x)}{1 - q\overline{F}(x)}, \tag{4}$$

respectively, where $h(x) = \frac{f(x)}{\overline{F}(x)}$ is the hazard rate function of $X \sim F$.

Proof. Using a conditional argument on N, the proof is completed. (See also Marshall and Olkin, 1997).

Remark 2.1 *As we see from 4, $\frac{1}{1-q\overline{F}(x)}$ is a decreasing function for all $x \in \mathscr{R}^+$. Thus, $h_U(x)$ is decreasing if $h(x)$ is decreasing. Indeed, it says a mixture of decreasing hazard rate functions is decreasing. (See Barlow and Proschan, 1975).*

One can show that the properties given in Theorem 2.1 agree with corresponding properties of EG, WG, and E2G distributions as special cases.

Definition 2.2 *Under conditions given in Def. 2.1, a random vector (U,N) with stochastic representation $(U,N) \stackrel{d}{=} (\min\{X_i\}_{i=1}^N, N)$ is said to have a mixed bivariate F-geometric distribution and will be denoted by $(U,N) \sim BFG(\boldsymbol{\theta}, p)$.*

Theorem 2.2 *Let $(U,N) \sim BFG(\boldsymbol{\theta}, p)$. Then, the joint pdf of (U,N) for $x \in \mathscr{R}^+$ and $n = 1,2,...,$ is given by*

$$f_{U,N}(x,n) = nf(x)\overline{F}(x)^{n-1}pq^{n-1}. \tag{5}$$

Proof. We write $f_{U,N}(x,n) = \sum_{n=1}^{\infty} f_{U|N}(x).P(N=n)$ and then the proof is completed.

Theorem 2.3 *Let $(U,N) \sim BFG(\boldsymbol{\theta}, p)$. Then, the conditional pdf of $N|U = x$ for $x \in \mathscr{R}^+$ and $n = 1,2,...,$ is given by*

$$f_{N|U=x}(n) = n[1 - q\overline{F}(x)]^2[q\overline{F}(x)]^{n-1}. \tag{6}$$

Proof. It is clear that $f_{N|U=x}(n) = \frac{f_{U,N}(x,n)}{f_U(x)}$. Replacing $f_{U,N}(x,n)$ and $f_U(x)$ by 5 and 3, respectively, completes the proof.

Corollary 2.1 *$(N|U = x) \stackrel{d}{=} W - 1$, where $W \sim nb(2, 1 - q\overline{F}(x))$. "nb" denotes the negative binomial distribution.*

Corollary 2.2 *The moment generating function (mgf), mean, and variance of $N|U = x$, for $x \in \mathscr{R}^+$, are given by $M_{N|U=x}(t) = e^t\{\frac{1-q\overline{F}(x)}{1-q\overline{F}(x)e^t}\}^2$, $t < -\log q\overline{F}(x)$, $E(N|U = x) = \frac{1+q\overline{F}(x)}{1-q\overline{F}(x)}$, and $Var(N|U = x) = \frac{2q\overline{F}(x)}{\{1-q\overline{F}(x)\}^2}$, respectively.*

Proof. Using Corollary 2.1 completes the proof.

Theorem 2.4 *Let $(U,N) \sim BFG(\boldsymbol{\theta}, p)$. Then, the joint cdf and joint survival function of (U,N), for $x \in \mathscr{R}^+$ and $n = 1,2,...,$ are given by*

$$F_{U,N}(x,n) = F_N(n) - \overline{F}_U(x)F_{N_\circ}(n) \tag{7}$$

and

$$\overline{F}_{U,N}(x,n) = \overline{F}_U(x)\overline{F}_{N_\circ}(n), \tag{8}$$

respectively, where $F_N(n) = 1 - q^n$ is the cdf of $N \sim Ge(p)$, $\overline{F}_U(x)$ is given in (2), and $F_{N_\circ}(n) = 1 - \overline{F}_{N_\circ}(n) = 1 - [q\overline{F}(x)]^n$ is the cdf of $N_\circ \sim Ge(1 - q\overline{F}(x))$.

Proof.

$$F_{U,N}(x,n) = P(U \le x, N \le n) = \sum_{k=1}^{n} \int_0^x f_{(U,N)}(t,k)dt = \sum_{k=1}^{n} \int_0^x kf(t)\overline{F}(t)^{k-1} pq^{k-1} dt$$
$$= \sum_{k=1}^{n} pq^{k-1}\{1-\overline{F}(x)^k\} = F_N(n) - \overline{F}_U(x) F_{N_\circ}(n).$$

A similar proof is applied to $\overline{F}_{U,N}(x,n)$ and then the proof is completed. Here, the identity $\overline{F}_{U,N}(x,n) = 1 - F_U(x) - F_N(n) + F_{U,N}(x,n)$ can also be applied.

Corollary 2.3 *$N_\circ \overset{d}{=} (N = k|U > x)$, for $k = 1,2,...$, where $N_\circ \sim Ge(1-q\overline{F}(x))$.*

Proof. The proof is straightforward and so is omitted.

Some conditional probabilities are useful for reliability, survival analysis, biological survey, and engineering. Here, we present some conditional probabilities below:

Theorem 2.5 *Let $(U,N) \sim BFG(\boldsymbol{\theta}, p)$. Then, for $x \in \mathscr{R}^+$ and n=1,2,...*

$$P(U > x, N > m|N > n) = \begin{cases} q^{m-n}\overline{F}_U(x)\overline{F}(x)^m, & m \ge n > 0 \\ \overline{F}_U(x)\overline{F}(x)^n, & 0 < m \le n. \end{cases}$$

Proof. For $m \ge n > 0$:

$$P(U > x, N > m|N > n) = \frac{P(U > x, N > m, N > n)}{P(N > n)} = \frac{P(U > x, N > m)}{P(N > n)}$$
$$= \frac{\overline{F}_U(x)\overline{F}_{N_\circ}(m)}{q^n} = q^{m-n}\overline{F}_U(x)\overline{F}(x)^m.$$

A similar proof is applied to the case $0 < m \le n$.

Theorem 2.6 *Let $(U,N) \sim BFG(\boldsymbol{\theta}, p)$. Then, for $x \in \mathscr{R}^+$ and n=1,2,...*

$$P(U > x, N > n|U > u) = \begin{cases} \frac{\overline{F}_U(x)\overline{F}_{N_\circ}(n)}{\overline{F}_U(u)}, & 0 < u \le x \\ [q\overline{F}(u)]^n = P(U > u|N > n), & 0 < x \le u. \end{cases}$$

Proof. The proof is similar to the proof of Theorem 2.5 and so is omitted.

2.1.1 A special case

Let $X_1, X_2, ..., X_N$ be N iid random variables from an exponential distribution with the cdf $F(x) = 1 - e^{-\beta x}$, $x > 0$, $\beta > 0$, where $N \sim Ge(p)$ is independent of X_i's. According to given relations in the previous section, we have the following results.

1. According to Theorem 2.1, the cdf, survival function, and pdf of $U = \min\{X_i\}_{i=1}^N$ are given by $F_U(x) = \frac{1-e^{-\beta x}}{1-qe^{-\beta x}}$, $x > 0$, $\overline{F}_U(x) = \frac{pe^{-\beta x}}{1-qe^{-\beta x}}$, and $f_U(x) = \frac{p\beta e^{-\beta x}}{\{1-qe^{-\beta x}\}^2}$, respectively. Further, Eq. (4) implies that the hazard rate function of U is decreasing, since $h(x)$ of exponential distribution is constant and $\frac{1}{1-q\overline{F}(x)}$ is always decreasing.

Remark 2.2 *As we see, the above results agree with the EG distribution of Adamidis and Loukas (1998).*

In the sequel, we consider the mixed bivariate exponential-geometric distribution and denote it by $(U,N) \sim BEG(\beta, p)$. It is clear that the EG distribution of Adamidis and Loukas (1998) is the marginal distribution of $BEG(\beta, p)$ distribution.

2. According to Theorem 2.2, the joint pdf of $BEG(\beta, p)$ distribution is given by

$$f_{(U,N)}(x,n) = \frac{n\beta}{pq}(qe^{-\beta x})^n, \quad x > 0,\ n = 1,2,\dots.$$

3. According to Corollary 2.1, the conditional pdf of $N|U=x$ is given by

$$f_{U|N}(x)=n(1-qe^{-\beta x})^2(qe^{-\beta x})^{n-1}, \quad x>0,\ n=1,2,\dots .$$

Indeed $(N|U=x)\stackrel{d}{=}W-1$, where $W\sim nb(2,1-qe^{-\beta x})$.

4. According to Theorem 2.4, the joint cdf and survival function of (U,N) are given by

$$F_{U,N}(x,n)=\frac{(1-q^n)(1-qe^{-\beta x})-pe^{-\beta x}(1-(qe^{-\beta x})^n)}{1-qe^{-\beta x}},\ x>0,\ n=1,2,\dots$$

and

$$\overline{F}_{U,N}(x,n)=\frac{pq^n e^{-(n+1)\beta x}}{1-qe^{-\beta x}},\ x>0,\ n=1,2,\dots,$$

respectively.

In addition, we can obtain the following conditional probabilities having applications in reliability and survival analysis.

5.

$$P(U>x,N>m|N>n)=\begin{cases}\frac{pq^{m-n}e^{-(m+1)\beta x}}{1-qe^{-\beta x}}, & m\geq n>0\\ \frac{pe^{-(n+1)\beta x}}{1-qe^{-\beta x}}, & 0<m\leq n.\end{cases}$$

6.

$$P(U>x,N>n|U>u)=\begin{cases}\frac{e^{-(\beta-u)x}(1-qe^{-\beta u})q^n e^{-n\beta x}}{1-qe^{-\beta x}}, & 0<u\leq x\\ (qe^{-\beta u})^n=P(U>u|N>n), & 0<x\leq u.\end{cases}$$

Remark 2.3 *The above properties can be easily applied to $(U,N)\sim BWG(\alpha,\beta,p)$, where BWG denotes the mixed bivariate Weibull-geometric distribution. It is enough that $e^{-\beta x}$ is replaced by $e^{(-\beta x)^\alpha}$, which is the cdf of a Weibull distribution. In addition, if $F(x)=(1-e^{-\beta x})^\alpha$, i.e., the cdf of exponentiated exponential distribution, then we have a mixed bivariate exponentiated exponential-geometric distribution, which one can check all above mentioned properties for this new distributions.*

2.2 A Poisson distribution for N

Here, we consider a zero truncated Poisson distribution with rate λ for N with pmf

$$\pi_N(n)=\frac{e^{-\lambda}\lambda^n}{n!(1-e^{-\lambda})}, \quad n=1,2,\dots . \tag{9}$$

We denote it by $N\sim P_o(\lambda)$. The properties of $U=\min\{X_i\}_{i=1}^N$ has already been studied by Alkarni and Oraby (2012). In this section, we study the properties of the mixed bivariate random variable (U,N).

Definition 2.3 *Under same conditions given in Def. 2.1, let $N\sim P_o(\lambda)$ is independent of X_i's. Then, the random vector (U,N) is said to have a mixed bivariate F-Poisson distribution and will be denoted by $(U,N)\sim BFP(\boldsymbol{\theta},\lambda)$, where $\boldsymbol{\theta}$ is the parameter vector of F.*

Theorem 2.7 *Let $(U,N)\sim BFP(\boldsymbol{\theta},\lambda)$. Then, the joint pdf of (U,N) for $x\in\mathscr{R}^+$ and $n=1,2,\dots$, is given by*

$$f_{U,N}(x,n)=nf(x)[\overline{F}(x)]^{n-1}\frac{e^{-\lambda}\lambda^n}{n!(1-e^{-\lambda})}. \tag{10}$$

Proof. Writing $f_{U,N}(x,n)=\sum_{n=1}^{\infty}f_{U|N}(x)P(N=n)$ completes the proof.

Theorem 2.8 *Let $(U,N)\sim BFP(\boldsymbol{\theta},\lambda)$. Then, the conditional pdf of $N|U=x$ for $x\in\mathscr{R}^+$ and $n=1,2,\dots$, is given by*

$$f_{N|U=x}(n)=\frac{[\lambda\overline{F}(x)]^{n-1}e^{-\lambda\overline{F}(x)}}{(n-1)!}. \tag{11}$$

Corollary 2.4 *$(N|U=x)\stackrel{d}{=}W_1+1$, where $W_1\sim P(\lambda\overline{F}(x))$.*

Corollary 2.5 *The mgf, mean, and variance of $N|U=x$, for $x\in\mathscr{R}^+$, are given by $M_{N|U=x}(t)=e^{t+\lambda\overline{F}(x)(e^t-1)}$, $E(N|U=x)=1+\lambda\overline{F}(x)$, and $Var(N|U=x)=\lambda\overline{F}(x)$, respectively.*

2.2.1 Two special cases

Let $X_1, X_2, ..., X_N$ be N iid exponential random variables, where $N \sim P_o(\lambda)$. According to mentioned properties in previous section, we can have a new mixed bivariate distribution, called *mixed bivariate exponential-Poisson* ($BEP(\beta, \lambda)$) distribution. Further, by replacing a Weibull distribution with parameters α and β instead of exponential distribution, we have another new mixed bivariate distribution, called *mixed bivariate Weibull-Poisson* ($BWP(\alpha, \beta, \lambda)$) distribution. It is clear that the marginal continuous distributions of $BEP(\beta, \lambda)$ and $BWP(\alpha, \beta, \lambda)$ are, respectively, the EP distribution of Kus (2007) and WP distribution of Hemmati et al. (2011).

Remark 2.4 *All properties given in previous sections can be extended to the mixed bivariate random variable* (V, N)*, where* $V = \max\{X_i\}_{i=1}^N$.

3 Concluding remarks

In this paper, we studied the mixed bivariate random variable (U, N). Many applications can be considered when the random sample size is itself a random variable. For example, suppose that N is the number attacks of a forward player during a soccer which has a discrete distribution. Let X_i's be the used times for every attack having an absolutely continuous distribution F. If forward player attacks m times in a soccer and $U = \min\{X_i\}_{i=1}^N$, then $(u_1, n_1), (u_2, n_2), ..., (u_m, n_m)$ are m joint observations of this example. As an alternative example, suppose that the advertising number of a big trading company in media has a discrete distribution and the amount of earned gain arising the advertising has an absolutely continuous distribution F. It is here interesting to consider $V = \max\{X_i\}_{i=1}^N$, then for m times attempt we have $(v_1, n_1), (v_2, n_2), ..., (v_m, n_m)$ observations. As other examples, we can see such applications in water resources and climate studies focus on the understanding and prediction of properties of hydroclimatic episodes. For more real examples and also for some applications of BFG models, see Kozubowski and Panorska (2008).

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