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Inverse Power Modified Chris-Jerry Distribution: Properties, Estimation, Simulation and Medical Application

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Abstract: To provide better fits than the modified Chris-Jerry distribution and some of its known extensions, this study introduces an inverse power modified Chris-Jerry distribution as a generalization of the two parameters of the Chris-Jerry distribution. The essential characteristics of the suggested distribution, including the moments, order statistics, Rényi entropy, stress-strength reliability, moments, and moment-generating function, have been determined. We determine the maximum likelihood estimators of the unknown parameters in the proposed distribution. The simulation study is carried out by producing random samples to estimate distribution parameters. We examine applications to real datasets, where the new distribution is shown to provide a better fit than other distributions.

Keywords: Entropy, Maximum Likelihood estimation, Modified Chris-Jerry, Moments, Order Statistic, Reliability Analyses**,** Stress-Strength Parameter

1 Introduction

Many scholars have worked over the past few decades to propose a distribution that can model real-time data sets with the best fit, and flexibility has been employed. In many fields, including the biological sciences, life testing issues, survey sampling, engineering sciences, etc., inverse distributions have significant applications. Many statistical distributions have lately been presented in research as alternatives to existing distributions. Generating distributions seek to provide the literature with a more flexible distribution than rivals when representing real-world data and statistical features. In this respect, several authors want to propose a novel distribution to add novelty to the literature. This paper introduces the inverse power-modified Chris-Jerry distribution and its distributional features. Some of the papers relating to this research are given below.

The Chris-Jerry distribution is a very simple and flexible model, having features for accommodating various types. Besides providing a suitable model for typical income and wealth data through some more flexible and generalized variants of the classical Chris-Jerry distribution, these are found to be very useful in various problems related to life testing, survival analysis, telecommunication, actuarial science, and economics. The use of the Chris-Jerry distribution as a model for analyzing various sci-economic phenomena is not new in the statistical literature. In fact, the Chris-Jerry distribution and its generalizations give a very flexible family of heavy-tailed distributions that may be used to model income distributions as well as a wide range of other distributions associated with social and economic problems. For more extensive discussion on the use of these models in the context of income distributions, see Villasenor and Arnold [8], Pal et al. [26] and Ali et al. [27]. It was demonstrated that the Chris-Jerry distribution outperformed some well-known Lindley classes of distributions and was more adaptable, see [10]- [14] and [24] for details.

Owing to the difficulty in predicting new parameters that are added by extending current distributions, choosing distributions requires careful consideration of parameter parsimony. Consequently, the need to enhance the one parameter C-JD is what drives the investigation, such that:

• By adding a shape parameter to the C-JD with one parameter, a modified C-J distribution is produced.

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• The parameters of the proposed distribution are tractable using conventional and Bayesian estimates, even with additional parameters.

• Better adaptability and features of the existing distributions.

• The distributions with more favorable fits than the one-parameter C-JD are Weibull, Gamma, Lomax, Burr III, Exponentiated Inverse Exponential, and Generalized Inverse Exponential.

Many writers have devoted their work to inverted distributions and their uses. Numerous continuous distributions, including the Burr [1], Gamma [2], Weibull [3], Exponentiated Inverse Exponential (EIE) [4] , Lomax [5] and Chris-Jerry distribution (C-JD) [7] have been used to model real-time data sets. The modified Chris-Jerry distribution (MCJD) developed by Chinedu et al. [6] is among these two-parameteres Chris-Jerry distribution. For the modified Chris-Jerry distribution distribution with parameters λ and β, the probability density function (pdf) and cumulative distribution function (cdf) of a random variable X are provided by:

$$g(x;\lambda,\beta)=\frac{\beta^2}{\lambda\beta+2}\left(\lambda+\beta\,x^2\right)e^{-\beta x};x>0,\beta>0,\lambda>0, \quad (1)$$

and

$$G(x,\lambda,\beta)=1-\frac{\beta}{\beta\lambda+2}\left(\beta^2x^2+2\beta x+\beta\lambda+2\right)e^{-\beta x};$$

$$x>0,\beta>0,\lambda>0, \quad (2)$$

where λ and β are the shape and scale parameters, respectively.

The support of $y=x^{-\frac{1}{\alpha}}$as $(0,\infty)$ is then obtained by solving for y at $x=0$ and $x=\infty$, respectively. Assuming that $y=x^{-\alpha}$ represents the observed value of Y, then $x=y^{-\frac{1}{\alpha}}$ and $(\frac{dx}{dy}=-\frac{1}{\alpha}y^{-\frac{1}{\alpha}-1}$).

The inverse scheme's goal is to evaluate data with a potentially non-monotone HRF, i.e., which could be present in unimodal forms, bathtubs, or upside-down bathtubs (UBT). This is due to the fact that a monotone hazard rate may not be present in many applications, such as cancer and mortality studies. The identification of the heavy-tail qualities is also added by this kind of change. As a result, long-tailed and non-monotone hazard rate models must be created. Numerous issues pertaining to econometrics, biological sciences, survey sampling, engineering sciences, medical research, and life testing can benefit from the use of inverted distributions.

This paper is arranged to investigate the proposed inverse power modified Chris-Jerry (IPMC-J) distribution. The reliability, hazard rate, cumulative hazard, and reversed hazard of the proposed distribution are obtained in Section 2. Properties of the IPMC-J distribution, such as moments, moment-generating functions, distributions of order statistics, and R´enyi entropy, are obtained in Section 3. In Section 4, the maximum likelihood estimates of the distribution parameters are introduced, and the approximate confidence interval estimators of $\left(\hat{\alpha},\hat{\beta}\ and\ \hat{\lambda}\right)$ and their asymptotic distribution for the IPMC-J distribution are presented. The simulation research produces random samples that follow the IPMC-J distribution in Section 5. In Section 6, we present a real-world example of infant mortality rate data to demonstrate our model's value. Finally, the conclusions are given in Section 7.

2 Inverse Power Modified Chris-Jerry Distribution

The random variable X of the modified C-J distribution defined in Eq. (1) supposes that another random Y is related to X by the inverse power function $Y=g(x)=x^{\frac{-1}{\alpha}}$. The derivation of the pdf of the inverse modified C-J distribution finds the distribution of the random variable Y. Mathematically, the distribution is determined the pdf by solving Eq. (1), we obtain

$$f(y;\alpha,\beta,\lambda)=\frac{\alpha\beta^2}{\beta\lambda+2}y^{-\alpha-1}\left(\lambda+\beta y^{-2\alpha}\right)e^{-\beta y^{-\alpha}}. \quad (3)$$

Also, the cdf of the IPMC-J distribution can be given as

$$F(y;\alpha,\beta,\lambda)=\frac{\beta^2y^{-2\alpha}+2\beta y^{-\alpha}+\beta\lambda+2}{\beta\lambda+2}e^{-\beta y^{-\alpha}}. \quad (4)$$

It's interesting to note that the IPMC-J distribution, which can be used to depict failure rates for bathtub and upside-down bathtub forms, as well as datasets with heavy-tailed data, gives greater flexibility than the modified C-J distribution. Noteworthy is the fact that the IPMC-J distribution presented in this study is known as a heavy-tailed distribution since it has polynomial tails for every value of α,β and λ also known as the heavy-tailed parameter in this distribution, which regulates how quickly the upper tail decays. As the value of α increases, the upper tail's decay rate drops. As α falls in value, the tail gets heavier.

Figures 1 and 2 show the pdf and cdf functions of the IPMC-J distribution for different values of α,β and λ.

3 Reliability Analyses

This section displays the survival, hazard rate, reversed hazard, and cumulative hazard functions of the IPMC-J distribution with three parameters. These functions are useful for analyzing reliability.

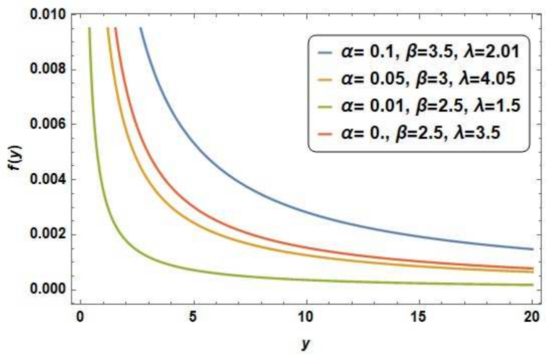

Fig. 1: Plot of the pdf of IPMC-J distribution for parameters α, β and λ.

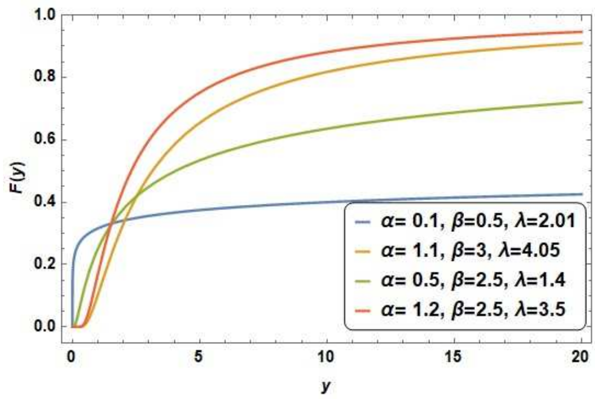

Fig. 2: Plot of the cdf of IPMC-J distribution for parameters α, β and λ.

3.1 Survival function

Let Y be a continuous random variable. Assuming that the IPMC-J distribution of parameters α, β and λ allows us to define the survival function of Y as follows:

$$S(y;\alpha,\beta,\lambda) = 1 - F(y;\alpha,\beta,\lambda) = 1 - \frac{\beta^2 y^{-2\alpha} + 2\beta y^{-\alpha} + \beta\lambda + 2}{\beta\lambda + 2}. \tag{5}$$

The survival function, sometimes referred to as the reliability function, shows the likelihood of living to be y years old or older. Reliability theory and survival analysis are important in the investigation of $S(y)$. It is crucial for characterizing component systems or for figuring out how reliable a system.

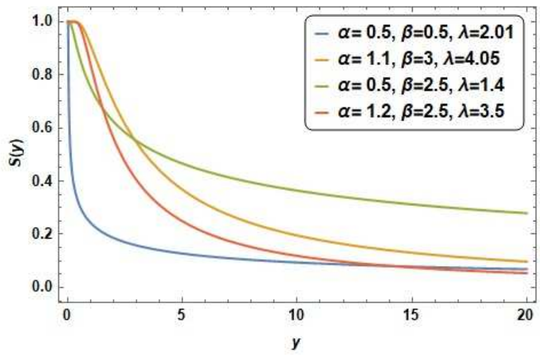

Fig. 3: Survival function of the of IPMC-J distribution for parameters α, β and λ.

3.2 Hazard rate function

The hazard rate function of a statistical distribution can be found mathematically by dividing the survival function $S(y)$ by the probability density function $f(y)$. Thus, the hazard rate function of the distribution is defined as follows:

$$h(y;\alpha,\beta,\lambda) = \frac{f(y;\alpha,\beta,\lambda)}{S(y;\alpha,\beta,\lambda)}$$

$$= \frac{\alpha\beta^2 y^{-\alpha-1}\left(\lambda + \beta y^{-2\alpha}\right)}{\beta\lambda + 2\left[\beta^2 y^{-2\alpha} + 2\beta y^{-\alpha} + \beta\lambda + 2\right]}. \tag{6}$$

Figures (3) and (4), respectively, depict the behavior of $S(y)$ and $h(y)$ for various values of α, β and λ for the IPMC-J distribution. It appears that the hazard rate function is always decreasing. As α increases, its shape gets less peaked, and as β and λ increase, it becomes more peaked.

3.3 Reversed hazared rate function

Reliability analysis and maintenance management both benefit from the use of the reversed hazard rate (RHR) function, see [29]-[33]. It is especially helpful for evaluating left-censored lifespan statistics, hidden failures, and waiting times. For significant statistical distributions, the RHR function is demonstrated to be a declining function, which qualifies it for use in maintenance engineering applications. The reversed hazard rate, which is the ratio of the probability density function (pdf) to the cumulative distribution function (cdf), is the mathematical representation of the hazard rate function of a statistical distribution. It is provided by

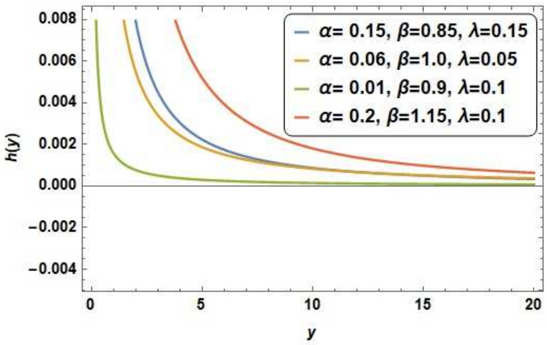

Fig. 4: Hazard rate function of the of IPMC-J distribution for parameters α, β and λ.

and expands the idea of hazard rate to a reverse time direction.

$$h_r(y) = \frac{f(y)}{F(y)} = \frac{\left(\alpha\beta^2\right) y^{-\alpha-1}\left(\lambda + \beta y^{-2\alpha}\right)}{\beta^2 y^{-2\alpha} + 2\beta y^{-\alpha} + \beta\lambda + 2}. \quad (7)$$

The probability of an immediate past failure, provided that the unit has already failed at time y, defined by $h(y)$, is described by the reversed hazard $h_r(y)$.

3.4 Cumulative hazared rate function

In lifetime data analysis, the cumulative hazard rate function is an essential concept, [34]. It offers a means of calculating the total risk of an event happening up until a specific moment in time. Additionally, the inverted hazard rate function is discussed, which is especially helpful for left-censored observations and lives with reversed time scales [35]. The cumulative hazard rate (chr) is defined as follows:

$$\begin{aligned} H(y) &= -\ln\left[1 - F(y)\right] \\ &= -\ln\left[1 - \frac{\beta^2 y^{-2\alpha} + 2\beta y^{-\alpha} + \beta\lambda + 2}{\beta\lambda + 2}\right]. \end{aligned} \quad (8)$$

4 Properties of the IPMC-J Distribution

In this section, the IPMC-J distribution statistical and mathematical characteristics are explained. These characteristics are crucial, particularly when using the distribution to analyze data from real-world sources.

4.1 Moments

Some of a model's most crucial characteristics like kurtosis, skewness, and dispersion are described by the moments of distributions. As a result, the IPMC-J distribution distribution is the k^{th} moment provided by

$$\mu_k' = E\left(y^k\right) = \int_0^\infty y^k f(y)\, dy,$$

then

$$\begin{aligned} \mu_k' &= \frac{\alpha\beta^2}{\beta\lambda+2}\int_0^\infty y^{k-\alpha-1}\left(\lambda + \beta y^{-2\alpha}\right) e^{-\beta y^{-\alpha}} dy \\ &= \frac{\alpha\beta^2}{\beta\lambda+2}\int_0^\infty \left[\lambda y^{k-\alpha-1} e^{-\beta y^{-\alpha}} + \beta y^{k-3\alpha-1} e^{-\beta y^{-\alpha}}\right] dy. \end{aligned} \quad (9)$$

Putting $\beta y^{-\alpha} = z$, $y^{-\alpha} = \frac{Z}{\beta}$ then $y = z^{-\frac{1}{\alpha}}\beta^{\frac{1}{\alpha}}$ and $\frac{dz}{dy} = -\frac{1}{\alpha}\beta^{\frac{1}{\alpha}} y^{-\frac{1}{\alpha}-1}$into Eq. (9).

By using the Gamma function, we obtain as

$$\mu_k' = \frac{\beta^2}{\beta\lambda+2}$$

$$\left[-\lambda\beta^{k-\alpha}\Gamma\left(\frac{-k+\alpha}{\alpha}\right) + \beta^{k-2\alpha}\Gamma\left(\frac{-k+2\alpha}{\alpha}\right)\right]. \quad (10)$$

When $k = 1,2,3$ and 4 are substituted into Eq. (10), the first four moments of the $IPMC-J$ distribution are obtained as

$$\mu_1' = \tfrac{\beta^2}{\beta\lambda+2}\left[-\lambda\beta^{-\beta-1}\Gamma\left(\tfrac{\alpha+1}{\alpha}\right) - \beta^{-2\alpha-1}\Gamma\left(\tfrac{2\alpha-1}{\alpha}\right)\right], \quad \alpha > 1, \quad (11)$$

$$\mu_2' = \tfrac{\beta^2}{\beta\lambda+2}\left[-\lambda\beta^{-\beta-2}\Gamma\left(\tfrac{\alpha+2}{\alpha}\right) - \beta^{-2\alpha-2}\Gamma\left(\tfrac{2\alpha-2}{\alpha}\right)\right], \quad \alpha > 2, \quad (12)$$

$$\mu_3' = \tfrac{\beta^2}{\beta\lambda+2}\left[-\lambda\beta^{-\beta-3}\Gamma\left(\tfrac{\alpha+3}{\alpha}\right) - \beta^{-2\alpha-3}\Gamma\left(\tfrac{2\alpha-3}{\alpha}\right)\right], \quad \alpha > 3, \quad (13)$$

and

$$\mu_4' = \tfrac{\beta^2}{\beta\lambda+2}\left[-\lambda\beta^{-\beta-4}\Gamma\left(\tfrac{\alpha+4}{\alpha}\right) - \beta^{-2\alpha-4}\Gamma\left(\tfrac{2\alpha-4}{\alpha}\right)\right], \quad \alpha > 4. \quad (14)$$

In partical, the first moment μ_1' is the mean (μ), while the variance can be written as $\sigma^2 = \mu_2' - (\mu_1')^2$.

4.2 Moments generating function

The moment-generating function (mgf) of a statistical distribution can also be used to determine many of its traits and characteristics, in addition to moments. Let Y represent a random variable with three parameters. With parameters α, β and λ, the $IPMC-J$ distribution has the following moment-generating function (mgf) :

$$M_Y(t) = E(e^{ty}) = \int_0^\infty e^{ty} f(y) dy. \quad (15)$$

Using the series expansion $e^{ty} = \sum_0^\infty \left(\frac{ty^k}{k!}\right)$, then Eq. (15) becomes

$$M_Y(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E\left(y^k\right). \quad (16)$$

By using the result of the Moment in the Eq. (10), then

$$M_Y(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\beta^2}{\beta\lambda+2}[-\lambda\beta^{k-\alpha}\Gamma\left(\frac{-k+\alpha}{\alpha}\right) + \beta^{k-2\alpha}\Gamma\left(\frac{-k+2\alpha}{\alpha}\right)]. \quad (17)$$

4.3 Order statistics

Assume that $Y_1, Y_2, Y_3 ......Y_n$ is a random sample of size n drawn from the $IPMC-J$ distribution with cdf $F(y)$ and pdf $f(y)$. Next, order statistics are indicated by $Y_{(1)}, Y_{(2)}, Y_{(3)} ....., Y_{(n)}$,

where $Y_{(n)} = \max\left(Y_{(1)}, Y_{(2)}, Y_{(3)} ....., Y_{(n)}\right)$ and $Y_{(1)} = \min\left(Y_{(1)}, Y_{(2)}, Y_{(3)} ....., Y_{(n)}\right)$.

The k^{th} order statistics of pdf is obtained as

$$\begin{aligned} f_{y_{(k)}}(y;\alpha,\beta,\lambda) &= \frac{n!}{(k-1)!(n-k)!} \times f_{IPMC-J}(y;\alpha,\beta,\lambda) \\ &\times [F_{IPMC-J}(y;\alpha,\beta,\lambda)]^{k-1} \\ &[1 - F_{IPMC-J}(y:\alpha,\beta,\lambda)]^{n-k} \\ &= \frac{n!}{(k-1)!(n-k)!} \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i \\ &[F(y)]^{k+i-1} f(y). \quad (18) \end{aligned}$$

where $f_{IPTPC-J}(y;\alpha,\beta,\lambda)$ and $F_{IPTPC-J}(y;\alpha,\beta,\lambda)$ are defined in Eq. (3) and Eq. (4), respectively. If $k = n$ in Eq. (18), the pdf of the nth order statistic $Y_{(n)}$ IPMC-J distribution is given as

$$f_{k=n}(y;\alpha,\beta,\lambda) = \frac{n!}{(k-1)!(n-k)!} \frac{n\alpha\beta^2}{\beta\lambda+2} \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i \mathbf{y}^{-\alpha-1}\left(\lambda + \beta y^{-2\alpha}\right) \mathbf{e}^{-\beta y^{-\alpha}} \left[1 - \frac{\beta^2 y^{-2\alpha} + 2\beta y^{-\alpha} + \beta\lambda + 2}{\beta\lambda+2}\right]^{k+i-1}. \quad (19)$$

If $k = n$ in Eq. (19) the pdf of the n^{th} order statistic $Y_{(n)}$ for the IPMC-J distribution as

$$f_{Y_{(n)}}(y,\alpha,\beta,\lambda) = \frac{n\alpha\beta^2}{\beta\lambda+2} y^{-\alpha-1}\left(\lambda + \beta y^{-2\alpha}\right) e^{-\beta y^{-\alpha}} \left[\frac{\beta^2 y^{-2\alpha} + 2\beta y^{-\alpha} + \beta\lambda + 2}{\beta\lambda+2}\right]^{k-1}. \quad (20)$$

For $k = 1$ in Eq. (20), the pdf of the first order statistic $Y_{(1)}$, for the IPMC-J distribution as

$$f_{Y_{(1)}}(y,\alpha,\beta,\lambda) \frac{n\alpha\beta^2}{\beta\lambda+2} y^{-\alpha-1}\left(\lambda + \beta y^{-2\alpha}\right) \times \left[1 - \frac{\beta^2 y^{-2\alpha} + 2\beta y^{-\alpha} + \beta\lambda + 2}{\beta\lambda+2}\right]^{n-k} e^{-\beta y^{-\alpha}}. \quad (21)$$

4.4 Rényi Entropy

A helpful method for figuring out how much information (or uncertainty) a random sample holds about the parent population is entropy. A high entropy number suggests that the data are more unpredictable. The concept of entropy is essential to many disciplines, such as economics, communication theory, physics, probability and statistics, and so forth. Among the different types of entropy are the Rényi, Shannon, and Tsallis entropies. This work took into consideration the widely used Rényi entropy. The Rényi entropy is given as follows as

$$R_\omega(y) = \frac{1}{1-\omega} \log \int_0^\infty f^\omega(y) dy. \quad (22)$$

For $\omega \to 1$, we have the special case of Shannon Entropy $R_\omega(y)$. The specific case of Shannon Entropy $R_\omega(y)$ exists for $\omega \to 1$.

Then,

$$R_\omega(y) = \frac{1}{1-\omega} \log[\frac{(\alpha\beta^2)^\omega}{(\beta\lambda+2)^\omega}$$

$$\int_0^\infty \left(y^{-\alpha-1}\left(\lambda+\beta y^{-2\alpha}\right)e^{-\beta y^{-\alpha}}\right)^\omega dy]. \quad (23)$$

By using Binomial expansion, $(a+b)^\omega = \sum_{i=0}^{\omega}\binom{\omega}{i}a^{\omega-i}b^i$.

Therefore, we obtain as

$$R_\omega(y) = \frac{1}{1-\omega}\log[\frac{\alpha^{\omega-1}\beta^{2\omega}}{(\beta\lambda+2)^\omega}\sum_{i=0}^{\omega}\binom{\omega}{i}\omega\lambda^{\omega-1}$$
$$\beta^{i-(\omega+2i)\alpha-\omega+1}\Gamma\left((\omega+2i)-\frac{\omega+1}{\alpha}\right)]. \quad (24)$$

4.5 Odds function

Odds functions are constructed for discrete lifetime distributions in lifetime data analysis, offering insights into aging features and reliability ideas [36]. As a result, odds functions are essential for many different applications, including data analysis, decision-making, and model interpretability, see [37]-[39].

The odds function of IPMC-J distribution is given as

$$O\ (y;\alpha,\beta,\lambda) = \frac{F(y)}{1-F(y)} = \left(\beta^2y^{-2\alpha}+2\beta y^{-\alpha}+\beta\lambda+2\right)$$
$$[(\beta\lambda+2)e^{\beta y^{-\alpha}} - \left(\beta^2y^{-2\alpha}+2\beta y^{-\alpha}+\beta\lambda+2\right)]^{-1}. \quad (25)$$

5 Stress–Strength Reliability Analysis

A system's strength is determined by its reliability. Therefore, when the system is under more stress, it breaks down and becomes untrustworthy. Assume that, there are two distinct continuous random variables, $X \sim IPMC-J(\alpha,\beta_1,\lambda_1)$ and $Y \sim IPMC-J(\alpha,\beta_2,\lambda_2)$, the stress and strength of the system. The stress-strength reliability can obtained as follows:

$$R = \int_0^\infty f_1(y)F_2(y)\,dy, \quad (26)$$

then, by using the Gamma function, obtain as

$$R = \frac{\beta_1^2}{(\lambda_1\beta_1+2)\left((\lambda_2\beta_2+2)\right)}$$
$$\left[\begin{array}{c}\left(\lambda_1\beta_2^2+\beta_2\lambda_2\beta_1+2\beta_1\right)(\beta_1+\beta_2)^{-3}\Gamma(3)\\ +(2\beta_2\lambda_1)(\beta_1+\beta_2)^{-2}\Gamma(2)\\ +(2\lambda_1+\beta_2\lambda_2\lambda_1)(\beta_1+\beta_2)^{-1}\Gamma(1)+\\ (2\beta_1\beta_2)(\beta_1+\beta_2)^{-4}\Gamma(4)+(\beta_1\beta_2^2)(\beta_1+\beta_2)^{-5}\Gamma(5)\end{array}\right]. \quad (27)$$

This completes the proof.

6 Maximum Likelihood Estimation

The maximum likelihood approach is more commonly used in estimation theory to find the parameters of statistical distributions since it has consistency, asymptotic efficiency, and invariance properties (Casella and Berger [26]).

Let $(Y_1,Y_2,Y_3.......Y_n)$ random samples of size n with joint $pdf\ f(y_1,y_2,y_3.......y_n)$, then the likelihood function of the random sample as

$$L(\alpha,\beta,\lambda) = \prod_{i=1}^{n}\frac{\alpha\beta^2}{\beta\lambda+2}y_i^{-\alpha-1}\left(\lambda+\beta y_i^{-2\alpha}\right)e^{-\beta y_i^{-\alpha}}$$
$$= \frac{(\alpha\beta^2)^n}{(\beta\lambda+2)^n}e^{-\beta\sum_{i=1}^{n}y_i^{-\alpha}}\prod_{i=1}^{n}y_i^{-\alpha-1}\left(\lambda+\beta y_i^{-2\alpha}\right). \quad (28)$$

By taking log -likelihood function on both sides,

$$\ln L = n\ln\alpha+2n\ln\beta-n\ln(\beta\lambda+2)-\beta\sum_{i=1}^{n}y_i^{-\alpha}$$
$$-(\alpha+1)\sum_{i=1}^{n}\ln y_i+\sum_{i=1}^{n}\ln\left(\lambda+\beta y_i^{-2\alpha}\right). \quad (29)$$

Taking the partial derivatives of Eq. (29) with respect $\alpha,\ \beta\ and\ \lambda$ respectively. The maximum likelihood estimates $(\hat{\alpha},\ \hat{\beta}\ and\ \lambda)$ of parameters $\alpha,\ \beta$ and λ are acquired by resolving these nonlinear equation systems $\frac{\partial}{\partial\alpha}\ln L=0,\ \frac{\partial}{\partial\beta}\ln L=0$ and $\frac{\partial}{\partial\lambda}\ln L=0$.

$$\frac{\partial\ln L}{\partial\alpha} = \frac{n}{\alpha}+\beta\sum_{i=1}^{n}y_i^{-\alpha}\ln y_i-\sum_{i=1}^{n}\ln y_i-2\sum_{i=1}^{n}\frac{\beta y_i^{-2\alpha}\ln y_i}{\lambda+\beta y_i^{-2\alpha}}, \quad (30)$$

$$\frac{\partial\ln L}{\partial\beta} = \frac{2n}{\beta}-\frac{n\beta}{\beta\lambda+2}-\sum_{i=1}^{n}y_i^{-\alpha}+\sum_{i=1}^{n}\frac{y_i^{-2\alpha}}{\lambda+\beta y_i^{-2\alpha}} \quad (31)$$

and

$$\frac{\partial\ln L}{\partial\lambda} = \frac{-n\beta}{\beta\lambda+2}+\sum_{i=1}^{n}\frac{1}{\lambda+\beta y_i^{-2\alpha}}. \quad (32)$$

The maximum likelihood estimates of α, β and λ are acquired by the nonlinear system's Eqs. (30) - (32). To numerically maximize the log-likelihood function, nonlinear optimization procedures like the quasi-Newton algorithm are typically more practical.

The $(1-\zeta)100\%$ confidence interval for the parameters α, β and λ can be presented as

$$(\hat{\alpha}_L,\hat{\alpha}_U) = \hat{\alpha}\pm Z_{1-\frac{\zeta}{2}}\sqrt{var(\hat{\alpha})},$$

$$\left(\hat{\beta}_L,\hat{\beta}_U\right) = \hat{\beta}\pm Z_{1-\frac{\zeta}{2}}\sqrt{var\left(\hat{\beta}\right)}$$

and $\left(\hat{\lambda}_L,\hat{\lambda}_U\right) = \hat{\lambda}\pm Z_{1-\frac{\zeta}{2}}\sqrt{var\left(\hat{\lambda}\right)},$

where $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ are the maximum likelihood estimates of α, β and λ, $Z_{1-\frac{\zeta}{2}}$ is the percent of the standard normal distribution and $var(\hat{\alpha})$, $var\left(\hat{\beta}\right)$ and $var\left(\hat{\lambda}\right)$ are the asymptotic variances of maximum likelihood estimates computed utilizing the inverse of the information matrix as follows as

$$I^{-1} = \begin{bmatrix} \frac{-\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{-\partial^2 \ln L}{\partial \beta^2} & \frac{\partial^2 \ln L}{\partial \beta \partial \lambda} \\ \frac{\partial^2 \ln L}{\partial \lambda \partial \beta} & \frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} & \frac{-\partial^2 \ln L}{\partial \lambda^2} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} var(\hat{\alpha}) & cov\left(\hat{\alpha},\hat{\beta}\right) & cov\left(\hat{\alpha},\hat{\lambda}\right) \\ cov\left(\hat{\beta},\hat{\alpha}\right) & var\left(\hat{\beta}\right) & cov\left(\hat{\beta},\hat{\lambda}\right) \\ cov\left(\hat{\lambda},\hat{\beta}\right) & cov\left(\hat{\lambda},\hat{\alpha}\right) & var\left(\hat{\lambda}\right) \end{bmatrix}^{-1}. \quad (33)$$

6.1 Confidence intervals

A crucial statistical tool for estimating unknown quantities with a given degree of certainty is a confidence interval. They support the interpretation of study findings by offering a range of tenable values for the parameter of interest, see [40].

The log-likelihood formulae listed in Eqs. (30)-(32) do not solve closed form. Consequently, this subsection examines the estimations of the unknown parameters α, β and λ. Explicit construction of their matching confidence intervals is not possible. Finding the approximate confidence intervals for α, β and λ is obtained necessary. To find the Fisher information matrix, we must first acquire the second-order partial derivatives in this regard. Consequently,

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n}{\alpha^2}$$

$$-2\sum_{i=1}^{n} \frac{\beta y_i^{-2\alpha} (\ln y_i)^2 \left[2y_i^{-2\alpha} - \left(\lambda + \beta y_i^{-2\alpha}\right)\right]}{\left(\lambda + \beta y_i^{-2\alpha}\right)^2}, \quad (34)$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} y_i^{-\alpha} \ln y_i$$

$$-2\sum_{i=1}^{n} \frac{\left(\lambda + \beta y_i^{-2\alpha}\right) y_i^{-2\alpha} \ln y_i - \beta \left(y_i^{-2\alpha}\right)^2 \ln y_i}{\left(\lambda + \beta y_i^{-2\alpha}\right)^2}, \quad (35)$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n} \frac{\beta y_i^{-2\alpha} \ln (y_i)}{\left(\lambda + \beta y_i^{-2\alpha}\right)^2}, \quad (36)$$

$$\frac{\partial \ln L}{\partial \beta^2} = \frac{-2n}{\beta^2} + \frac{\lambda^2 n}{(\beta\lambda + 2)^2} - \sum_{i=1}^{n} \frac{y_i^{-2\alpha}}{\left(\lambda + \beta y_i^{-2\alpha}\right)^2}, \quad (37)$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} = \frac{-2n}{(\beta\lambda + 2)^2} - \sum_{i=1}^{n} \frac{y_i^{-2\alpha}}{\left(\lambda + \beta y_i^{-2\alpha}\right)^2} \quad (38)$$

and

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = \frac{\beta^2 n}{(\beta\lambda + 2)^2} - \sum_{i=1}^{n} \left(\lambda + \beta y_i^{-2\alpha}\right)^2. \quad (39)$$

The expressions defined in Eqs. (34)-(39) are used to find the Fisher information.

7 Simulation

The simulation research is carried out by producing random samples that follow the IPMC-J distribution using the relation $F(y) = u$, where u is an observation from the uniform $(0,1)$ and $F(y)$ is the cumulative distribution function of the IPMC-J distribution. For $(\alpha, \beta, \lambda) = (0.50, 0.15, 1.75), (0.05, 0.50, 1.07), (0.50, 0.20, 1.57)$ and $(0.15, 0.75, 1.25)$, 1000 replications of the simulation experiment were conducted using sample sizes of $30, 50, 70, 90$ and 100. The following indicators are calculated: the $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ bias values of the parameters α, β and λ are given as

$bias(\alpha) = \frac{1}{M} \sum_{i=1}^{M} (\hat{\alpha}_i - \alpha)$,

$bias(\beta) = \frac{1}{M} \sum_{i=1}^{M} \left(\hat{\beta}_i - \beta\right)$,

and

$bias(\lambda) = \frac{1}{M} \sum_{i=1}^{M} \left(\hat{\lambda}_i - \hat{\lambda}\right)$.

The MSE of $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ of the parameters α, β and λ are given as

$MSE(\hat{\alpha}) = \frac{1}{M} \sum_{i=1}^{M} (\hat{\alpha}_i - \alpha)^2$,

$MSE\left(\hat{\beta}\right) = \frac{1}{M} \sum_{i=1}^{M} \left(\hat{\beta}_i - \beta\right)^2$,

and

$MSE\left(\hat{\lambda}\right) = \frac{1}{M} \sum_{i=1}^{M} \left(\hat{\lambda}_i - \hat{\lambda}\right)^2$.

Table 1 displays the average estimators of the parameters α, β and λ, the mean square error (MSE), and the bias values of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ of the parameters α, β and λ. The results demonstrate that the estimated parameters values are close to the assumed parameter values, demonstrating the consistency feature.

The bias values go smaller and the MSE of the parameters reduces with increasing sample size.

8 Practical Data Examples

In this section, we investigate the capability of the IPMC-J distribution by fitting distributions, namely, the

Burr, Gamma, Weibull, Exponentiated Inverse Exponential (EIE), Lomax, Chris-Jerry distribution (C-J) and two-parameter Chris -Jerry $(TPC-J)$ distributions. The APMC-J distribution explains much more flexibility than the corresponding distributions. By making use of a real data set, we illustrate the applicability of the IPMC-J distribution among a set of classical and recent distributions, based on a set of goodness-of-fit statistics. We estimate the model parameters by using the maximum likelihood method. We compare the goodness-of-fit of the models with the Akaike Information Criterion (AIC), consistent Akaike information criterion (CAIC) and Bayesian Information Criterion (BIC) goodness -of-fit statistics.

Further, we get the Kolmogorov-Smirnov (K-S) statistic with its corresponding P-value. In general, the model has the smaller values of these statistics and the largest value of the P-value is the best model to fit the data.

8.1 Infant mortality rate data

This subsection suggests and compares the IPMC-J distribution's goodness of fit to real-world data with several one- and two-parameter distributions. This data set provides an overview of the infant mortality rate per 1000 live births for a select few countries in 2021 as reported by https://data.worldbank.org/indicator/SP.DYN.IMRT.IN.

This actual collection of data is displayed in Table 2.

Here, we contrast the fit goodness of the IPMC-JD with the Exponentiated Inverse Exponential (EIE) distribution by [4], Weibull distribution[3], Burr III distribution by [1], Gamma distribution[2], the C-JD by [7], TPC-J distribution by [6] , and Lomax distribution[5], as shown in Table 3. The negative log-likelihood is one of the fitness indicators taken into account. (-L), the corrected AIC (CAIC), the Akaike information criterion (AIC), and the Bayesian information criterion (BIC).

From Table 3, Estimates of parameters for the distributions fitted to Infant mortality data. From Table 4, the smallest values of the $K-S$, AIC, BIC and $CAIC$ and the largest value of p-value are obtained for the IPMC-J distribution. So, we conclude that the IPMC-J distribution provides the best fit among the compared distributions.

Figure 5 confirms this result where the estimated densities function for the compared distributions of the data set are plotted based on the density function of each distribution. Figure 5 shows the empirical pdf for the simulated data. Distributions listed in Table 3 also show that the IPMC-J distribution is the best fit for the real data.

Figure 6 shows the Kaplan-Meier curve for the simulated data and the survival functions of the distributions mentioned in Table 3, also shows that the IPMC-J distribution is the best fit for the data.

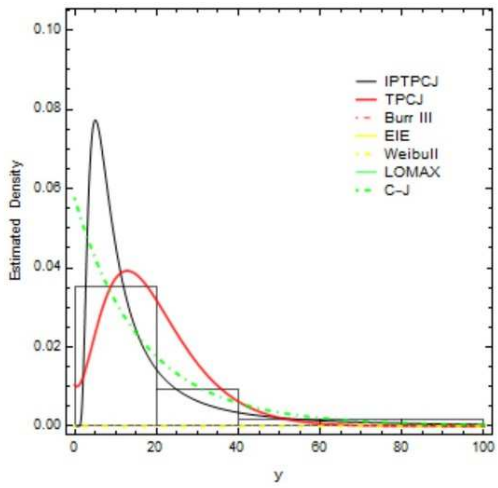

Fig. 5: Plots of the estimated IPMC-J distribution for Infant mortality data.

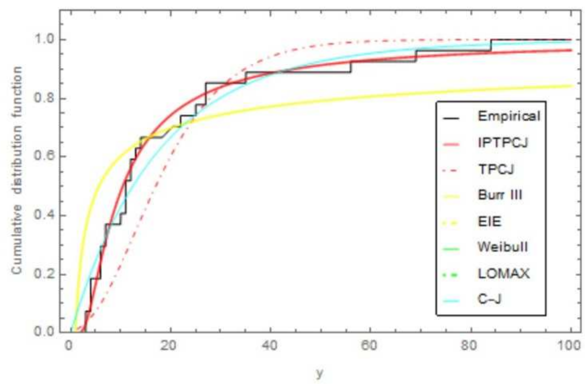

Fig. 6: Plots of Kaplan Meier curve for the simulated data and the survival functions of some distributions for Infant mortality data.

9 Conclusion

In this article, we introduce a more flexible extension of the two-parameter Chris-Jerry distribution called the Inverse-Powe Modifed Chris-Jerry (IPMC-J) distribution that provides more accuracy and flexibility in fitting medicine data. The new model was generated based on the inverse-power transformation technique. The hazard rate function of the IPMC-J distribution can have the following forms depending on its shape parameters, monotonically increasing, decreasing, and upside-down bat hub-shaped hazard rates. Therefore, it can be used quite effectively in analyzing lifetime data. Some of its basic mathematical properties are derived. The three parameters of the IPMC-J distribution are estimated using maximum likelihood estimation. The behavior and

Table 1 : Estimators, MSEs and bias values for the parameteres α, β and λ.

Parameter	n	MSE($\hat{\alpha}$)	MSE($\hat{\beta}$)	MSE($\hat{\lambda}$)	Bias($\hat{\alpha}$)	Bias($\hat{\beta}$)	Bias($\hat{\lambda}$)
$\alpha = 0.50$	30	0.0003557	0.0003084	8.146550	0.0047202	0.0042445	7.3695600
$\beta = 0.15$	50	0.0003060	0.0002396	6.132770	0.0126014	0.0091972	6.6029900
$\lambda = 1.75$	70	0.0001853	0.0001545	0.775334	−0.000845	−0.0009980	−0.040887
	90	0.0001000	0.0000840	1.423040	0.0045374	0.00412950	0.1170710
	100	0.0000868	0.0000790	1.729790	−0.0017976	−0.0011575	−0.0919157
$\alpha = 0.05$	30	0.0001215	0.0232254	1.148600	0.0109744	0.14828900	0.5559340
$\beta = 0.50$	50	0.0001413	0.0235758	2.348340	0.0118177	0.14930900	−3.992540
$\lambda = 1.07$	70	0.0001288	0.0209457	0.195803	0.0113149	0.14281800	0.1928090
	90	0.0001354	0.0232107	0.366303	0.0115994	0.15083700	0.3150840
	100	0.0001309	0.0223750	0.387787	0.0114153	0.14755800	0.2782180
$\alpha = 0.50$	30	0.0004379	0.0005573	4.378540	0.0053517	0.00568020	5.4028000
$\beta = 0.20$	50	0.0003872	0.0004346	2.854780	0.0143894	0.01247000	1.3795600
$\lambda = 1.57$	70	0.0002234	0.0002775	0.476575	−0.0008067	−0.00131076	−0.0352325
	90	0.0001242	0.0001517	1.232360	0.0050389	0.00554890	0.0369444
	100	0.0001047	0.0001412	0.585013	−0.0020347	−0.0015614	0.0540577
$\alpha = 0.15$	30	0.0001489	0.0091008	0.645818	0.0042791	0.02403890	0.2389210
$\beta = 0.75$	50	0.0001612	0.0074070	0.643710	0.0100127	0.05441500	0.0974150
$\lambda = 1.25$	70	0.0000655	0.0043887	0.123471	0.0002705	−0.0039742	−0.0229941
	90	0.0000306	0.0018752	0.181995	0.0016959	0.01826740	0.1328190
	100	0.0000334	0.0021133	0.240623	−0.0012389	−0.0067700	0.0463389

Table 2: Infant mortality data.

56	10	22	3	69
6	7	11	4	4
19	13	7	27	12
3	4	11	84	27
25	6	35	14	11
12	6			

Table 3: Estimates of parameters for the distributions fitted to Infant mortality data

	Estimates		
Model	α	β	λ
IPMC-J(α,β,λ)	1.31	16.07	2.39
TPCJ(β,λ)		399.51	0.06
Burr III(β,λ)		9.50	0.04
EIE(β,λ)		0.42	6.66
Weibull (β,λ)		0.90	8.90
Gamma(β,λ)		1.80	9.74
Lomax(β,λ)		232.81	13.37
C-J(β)		0.15	-

performance of these estimators are explored using simulation results. The flexibility and practical importance of the IPMC-J distribution were explored empirically using real-life datasets. It is shown that the IPMC-J distribution has a fit compared to the two-parameter Chris-Jerry distribution and other competing models.

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Table 4: The statistics of model selection criteria for the distributions fitted to Infant mortality data

Model	-L	AIC	BIC	$CAIC$	W*	A*	$K-S$	$P-value$
IPMC-J(α,β,λ)	102.66	207.33	208.62	207.49	0.047	0.30	0.11	0.8477
TPCJ(β,λ)	106.16	216.31	218.90	216.81	0.11	0.75	0.16	0.5345
Burr III(β,λ)	119.08	242.16	244.75	242.66	0.04	0.26	0.36	0.0021
EIE(β,λ)	103.88	211.76	214.36	212.26	0.08	0.50	0.17	0.4187
Weibull (β,λ)	106.11	231.36	233.95	231.86	0.13	0.82	0.32	0.0084
Gamma(β,λ)	105.76	217.90	220.49	218.40	0.13	0.82	0.18	0.3436
Lomax(β,λ)	106.17	216.33	218.92	216.83	0.11	0.71	0.16	0.5158
C-J(β)	112.39	226.77	228.07	226.93	0.17	1.10	0.28	0.0260

References

[1] Papadopoulos, A. S. The Burr distribution as a failure model from a Bayesian approach. IEEE Transactions on Reliability, 27(5), 369-371, 1978.

[2] Thom, H. C. A note on the gamma distribution. Monthly weather review, 86(4), 117-122, 1958.

[3] Lai, C. D., Xie, M. and Murthy, D. N. P. A modified Weibull distribution. IEEE Transactions on reliability, 52(1), 33-37, 2003.

[4] OGUNTUNDE, Pelumi E.; ADEJUMO, Adebowale O.; OWOLOKO, Enahoro A. On the exponentiated generalized inverse exponential distribution. In: World Congress on Engineering, London. 2017.

[5] Lemonte, A. J. and Cordeiro, G. M. An extended Lomax distribution. Statistics, 47(4), 800-816, 2013.

[6] Chinedu, E. Q., Chukwudum, Q. C., Alsadat, N., Obulezi, O. J., Almetwally, E. M., and Tolba, A. H. New lifetime distribution with applications to single acceptance sampling plan and scenarios of increasing hazard rates. Symmetry, 15(10), 1881, 2023.

[7] Onyekwere, C. K., and Obulezi, O. J. Chris-Jerry distribution and its applications. Asian Journal of Probability and Statistics, 20(1), 16-30, 2022.

[8] Odom, C.C. and Ijomah, M.A. Odoma distribution and its application. Asian J. Probab. Stat., 4, 1–11, 2019.

[9] Rényi, A. On measures of entropy and information. In Proceedings of the fourth Berkeley symposium on mathematical statistics and probability, volume 1: contributions to the theory of statistics (Vol. 4, pp. 547-562). University of California Press, 1961.

[10] Buzaridah, M.M.; Ramadan, D.A.; El-Desouky, B.S. Estimation of some lifetime parameters of flexible reduced logarithmic-inverse Lomax distribution under progressive type-II censored data. J. Math. 2022, 2022, 1690458.

[11] Farhat, A. T., Ramadan, D. A., and El-Desouky, B. S. Statistical Inference of Modified Frechetâ€“Exponential Distribution with Applications to Real-Life Data. Appl. Math. Inf. Sci, 17(1), 109-124, 2023.

[12] Khalifa, Eman H., Dina A. Ramadan, Hana N. Alqifari, and Beih S. El-Desouky. Bayesian Inference for Inverse Power Exponentiated Pareto Distribution Using Progressive Type-II Censoring with Application to Flood-Level Data Analysis. Symmetry 16, 3: 309, 2024.

[13] Elnagar, Kariema A., Dina A. Ramadan, and Beih S. El-Desouky. Statistical Inference to the Parameter of the Inverse Power Ishita Distribution under Progressive Type-II Censored Data with Application to COVID-19 Data. Journal of Mathematics 2022, 1 (2022), 7704167.

[14] Khalifa, E.H.; Ramadan, D.A.; El-Desouky, B.S. A new three-Parameters Inverse Power Exponentiated Pareto Distribution: Properties and its Applications. J. Fac. Sci. 2022, 36, 2022.

[15] Obulezi, O.J.; Anabike, I.C.; Oyo, O.G. and Igbokwe, C.P. Marshall-Olkin Chris-Jerry Distribution and its Applications. Int. J. Innov.Sci. Res. Technol., 8, 522–433, 2023.

[16] Shukla, K.K. Pranav distribution with properties and its applications. Biom. Biostat. Int. J., 7, 244–254, 2018.

[17] Shanker, R.; Shukla, K.K.; Shanker, R. and Pratap, A. A generalized Akash distribution. Biom. Biostat. Int. J., 7, 18–26, 2018.

[18] Shanker, R. Sujatha distribution and its Applications. Stat. Transit. New Ser., 17, 391–410, 2016.

[19] Bjerkedal, T. (1960). Acquisition of Resistance in Guinea Pies infected with Different Doses of Virulent Tubercle Bacilli. Am. J. Hyg. 1960,72, 130–148.

[20] Yahaya, A. and Abdullahi, J. Theoretical Study of Four-Parameter Odd-Generalized Exponential-Pareto Distribution. Ann. Stat.Theory Appl. ASTA., 2, 103–114, 2019.

[21] Enogwe, S.U.; Nwosu, D.F.; Ngome, E.C.; Onyekwere, C.K. and Omeje, I.L. Two-parameter Odoma distribution with applications. J.Xidian Univ., 14, 740–764, 2020.

[22] Abouammoh, A.M. and Alshingiti, A.M. Reliability estimation of generalized inverted exponential distribution. J. Stat. Comput.Simul., 79, 1301–1315, 2009.

[23] Shukla, K.K. and Shanker, R. Shukla distribution and its Application. Reliab. Theory Appl., 14, 46–55, 2019.

[24] Umeh, E. and Ibenegbu, A. A Two-Parameter Pranav Distribution with Properties and Its Application. J. Biostat. Epidemiol., 5,74–90, 2019.

[25] Shukla, K.K. Inverse Ishita Distribution: Properties and Applications. Reliab. Theory Appl., 16, 98–108, 2021.

[26] Casella, G. and Berger, R. L. Statistical Inference, Brooks/Cole Publishing Company, Cal ifornia, 1990.

[27] Singh, S.K., Singh, U. and Kumar, M. Bayesian inference for exponentiated Pareto model with application to Bladder cancer remission time. Statistics in Transition, 15, 3, 403-426, 2014.

[28] Ali, M. M., Pal, M., and Woo, J. S. Some exponentiated distributions. Communications for Statistical Applications and Methods, 14, 1, 93-109, 2007.

[29] D., Desai., V., Mariappan., Milind, Sakhardande. Nature of Reversed Hazard Rate: An Investigation. International journal of performability engineering, 7(2):165, 2011.
[30] Paduthol, Godan, Sankaran., V., L., Gleeja., T., M., Jacob. Nonparametric Estimation of Reversed Hazard Rate. Calcutta Statistical Association Bulletin, 59:55-68, 2007.
[31] Tirthankar, Ghosh., Dilip, Roy., Nimai, Kumar, Chandra. Reliability Approximation through the Discretization of Random Variables using Reversed Hazard Rate Function. 7(4):735-739, 2013.
[32] Asok, K., Nanda., Prasanta, Paul. Tests for Reversed Hazard Rate Function. Calcutta Statistical Association Bulletin, 54:181-194, 2003.
[33] Block, H. W., Savits, T. H., and Singh, H. The reversed hazard rate function. Probability in the Engineering and informational Sciences, 12(1), 69-90, 1998.
[34] Satyanshu, K., Upadhyay. Hazard Rate Function. Wiley Encyclopedia of Operations Research and Management Science, 2010.
[35] Henry, W., Block., Thomas, H., Savits., Harshinder, Singh. The Reversed Hazard Rate Function. Probability in the Engineering and Informational Sciences, 12(01):69-90, 1998.
[36] N., Unnikrishnan, Nair., Paduthol, Godan, Sankaran. Odds Function and Odds Rate for Discrete Lifetime Distributions. Communications in Statistics-theory and Methods, 44(19):4185-4202, 2015.
[37] Carlos, Brás-Geraldes., Carlos, Brás-Geraldes., Ana, Luísa, Papoila., Ana, Luísa, Papoila., Patrícia, Xufre. Odds ratio function estimation using a generalized additive neural network. Neural Computing and Applications, 32(8):3459-3474, 2020.
[38] F, Thomas, Bruss. Odds -theorem and monotonicity. Mathematica Applicanda, 47(1):25-43, 2019.
[39] F, Thomas, Bruss., Guy, Louchard. The Odds-algorithm Based on Sequential Updating and its Performance. Advances in Applied Probability, 41(1):131-153, 2009.
[40] Charles, R., Marshall. Confidence intervals on stratigraphic ranges. Paleobiology, 16(1):1-10, 1990.

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