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On Cordial Labeling of Third Power of Path with Other Graphs

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Abstract: If a graph's $0-1$ labeling meets specific criteria, it is referred to as cordial. The graph that is produced from the path P_n by adding edges that connect each vertex u and v with $d(u,v)=3$ is known as the third power of paths P_n^3. In this work, we examine the cordiality of the join and the union of graphs with one path and one cycle, as well as the third power of paths.

Keywords: Cordial graph, Power of path, Join of paths, Complete graph, Eulerian graph, labelling of graphs

1 Introduction

Numerous other academic disciplines, such as chemistry, physics, biology, psychology, communication, operations research, sociology, economics, engineering, and particularly computer science, can benefit from the use of graph theory. Graph labeling is one area of graph theory that has seen a lot of recent research. Many problems that are simple to articulate but difficult to solve stem from graph labeling. They have a vast body of literature.

In a specific kind of labelling, the edges have an induced labelling that is prescribed, the vertices are assigned values from a given set, and the labelling needs to meet certain requirements. A great resource on this topic is Gallian's survey [11]. Cordial labelling is one of the most significant categories of labelling. Cahit [1] introduced cordial labelling on his own.

More precisely, cordial graphs are defined as follows. Let $G=(V,E)$ be a graph, where V (or $V(G)$) and E (or $E(G)$) are vertex set and edge set of a graph G respectively. Let $h: V \longrightarrow \{0,1\}$ be a labeling of its vertices, and let $g: E \longrightarrow \{0,1\}$ is the extension of h to the edges of G by the formula $g(vw)=h(v)+h(w)(mod 2)$. As a result, for any edge e, if both of its vertices have the same label, $g(e)=0$; otherwise, $g(e)=1$. Let e_0 and e_1 be the corresponding numbers of edges, and let v_0 and v_1 be the numbers of vertices labelled 0 and 1, respectively. When both $|v_0-v_1|\leq 1$ and $|e_0-e_1|\leq 1$, the labelling is referred to as cordial. If a graph has a cordial labelling, it is referred to as cordial. Given two disjoint graphs G and G^*, their join $G+G^*$ is derived from $G\cup G^*$ by adding all edges that bind a vertex of G to a vertex of G^*.
The union $G\cup G^*$ is just the unions of their sets of vertices and edges.

Cahit [2] proved the following: A complete graph Kn is cordial iff $n\leq 3$ and an Eulerian graph is not cordial if its size is congruent to $2(mod 4)$. Seoud, Diab, and Elsakhawi [12] have proved the following graphs are cordial: The join P_n+P_m for all m and n except for $(m,n)=(2,2)$ and Q_m+Q_n for all n and m except for $m\equiv 0(mod 4)$ and $n\equiv 2(mod 4)$ (or vice versa). Diab [3, 4, 5, 6, 7, 8, 9] proved the following graphs are cordial: Q_m+P_n iff $(m,n)\neq(3,3),(3,2),(3,1)$, $Q_n\cup P_m$ is cordial iff it is not isomorphic to $Q_n\cup P_1$ with $n\equiv 2(mod 4)$, P_m+K_1, n iff $(m,n)\equiv(1,2)$, $P_m\cup K_1$, n iff $(m,n)\neq(1,2)$, $Q_m\cup K_{1,n}$, Q_m+N_n for all m and n except $m\equiv 3(mod 4)$ and n odd, and $m\equiv 2(mod 4)$ and n even, $Q_m\cup N_n$ for all m and n except $m\equiv 2(mod 4)$, P_m+N_n, $P_m\cup N_n$, $P_m^2+P_n^2$ except for $(m,n)=(1,3),(3,1),(2,3),(3,2),(4,2),(2,4),(3,4),(4,3),(2,2)$,$(3,3)$ and $(4,4)$, $P_m^2\cup P_n^2$ except for $(m,n)=(2,2) and (3,3)$, $P_n^2+P_m$ except for $(3,1),(3,2),(2,2),(3,3)$ and $(4,2)$, $P_n^2\cup P_m$ except for $(n,m)=(2,2)$, $P_n^2+Q_m$ iff $(n,m)\neq(1,3),(2,3)$ and

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$(3,3)$, If $(n,m) \neq (3,3)$ and $P_n^2 \cup Q_m$ is not isomorphic to $P_1 \cup Q_n$ with $n \equiv 2(mod4)$, then the union $P_n^2 \cup Q_m$ of the second power of the path P_n^2 and a cycle Q_m is cordial for all n and m, $F_n + F_m$ is cordial iff $n,m \equiv 4$, $F_n \cup F_m$ is cordial if and only if $(n,m) \neq (1,1),(2,2)$, $F_n + P_m$ is cordial iff $(n,m) \neq (1,2),(2,1),(2,2),(2,3)$, or $(3,2)$, $F_n \cup P_m$ is cordial if and only if $(n,m) \neq (1,2)$, $F_n + Q_m$ is cordial iff $(n,m) \neq (1,3),(2,3)$ or $(3,3)$ and $F_n \cup Q_m$ is cordial iff $(n,m) \neq (2,3)$.

The third power of a path P_n^3 is the graph obtained from the path P_n by adding edges that join all vertices u and v with $d(u,v) = 3$. So, the order of the third power of a path P_n^3 is n and its size is $3n-6$, in particular $P_1^3 \cup P_1, P_n^3 \equiv P_2, P_3^3 \equiv Q_3$ and $P_4^3 \equiv K_4$. This means that is cordial for all $n \leq 3$ and is not cordial for $n = 4$ [2].

Seoud and Abdel Maqusoud [13] proved that P_n^3 is cordial iff $n \neq 4$ and E.A. Elsakhawi [10] used a different approach to re-prove the last result. Also, it is clear to see that from Tables 1 and 2 below that P_n^3 is cordial for all $n > 7$ and from the particular cases of P_n^3 that P_n^3 is cordial for all $1 \leq n \leq 3$ [3,12]. In the cases of $4 < n \leq 7$, we may use the following labeling 01001, 010011 and 0010111. Therefore P_n^3 is cordial for all n, where $1 \leq n \leq 3$ and $4 < n \leq 7$. In the case of $n = 4$, $P_4^3 \equiv K_4$ is not cordial from known result (see Cahit [2]). This means that we have proved the previous result, which state that the third power of a path P_n^3 is cordial iff $n \neq 4$.

The main purpose of this paper is to extend some results on paths P_n and second power of paths P_n^2 to the third power of paths P_n^3 [3,4,6]. In section 3, we show that the join $P_n^3 + P_m$ of third power of a path P_n^3 and a path P_m is cordial for all n and m iff $(n,m) \neq (2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)$. In section 4, we show that the union $P_n^3 \cup P_m$ of third power of a path P_n^3 and a path P_m is cordial for all n and m iff $(n,m) \neq (2,2)$. In section5, we show that the join $P_n^3 + Q_m$ of third power of a path P_n^3 and a cycle Q_m is cordial for all n and m iff $(n,m) \neq (1,3),(2,3),(3,3),(4,3),(4,4),(5,3)$. In section 6, we show that the union $P_n^3 \cup Q_m$ of third power of a path P_n^3 and a cycle Q_m is cordial for all n and m iff $(n,m) \neq (5,3)$ or $3+m \neq 2(mod4)$ or $P_n^3 \cup Q_m$ is not isomorphic to $P_1 \cup Q_m$ with $m \neq 2(mod4)$.

2 Notations and Terminologies

We present some vocabulary and notation for a graph with 4α vertices, we let $L_{4\alpha}$ denote the labeling 00110011...0011, $R_{4\alpha}$ denote the labeling 11001100...11 00, one exception to this is the labeling $L'_{4\alpha}$ obtain from $L_{4\alpha}$ by adding an initial 0 and deleting the last 1, that is, $L'_{4\alpha}$ is 000110011...001, Ordenotes the labeling 0000...0000 (zero repeated α-times) and, Irdenotes the labeling 111...1111 (one repeated α-times) ,i.e., O_5 is 00000, I_5 is 11111, M_α denotes the labeling 0101...(α-times)...01 (zero-one repeated α-times), and M'_α denotes the labeling 10101...(α-times)...101 (one-zero repeated α-times), this means that if α is odd then M_α is 0101...1010 and M'_α is 1010...10101, and if α is even then M_α is 1010...0101 and M'_α is 1010...1010, i.e., M_5 is 01010, M_6 is 010101, M'_5 is 10101 and M'_6 is 101010.

For the labeling of vertices of P_n^3, we have four cases:
Case (1): $n \equiv 0(mod4)$ or $n = 4\alpha$,where $\alpha > 1$. We have the following two sub-cases:

Sub-case (1). α is an even number greater than one.
The labels assigned to the vertices of $P_{4\alpha}^3$ are used by us: $E_{4\alpha}$: $O_3I_3M'_4M_4M'_4M_4 \ldots((k-1)\text{- times})\ldots M'_4M_4M'_4M_4M'_2$ for $\alpha = 2k$, it is simple to confirm that $x_0 = x_1 = 2\alpha$ and $a_0 = a_1 = 6\alpha - 3$, or $E'_{4\alpha}$: $O_3I_3M'_4M_4M'_4M_4\ldots((k-1)\text{- times})\ldots M'_4M_4M'_4M_4M_2$ for $\alpha = 2k$, it is simple to confirm that $x_0 = x_1 = 2\alpha$, $a_0 = 6\alpha - 4$ and $a_1 = 6\alpha - 2$, or

$$E''_{4\alpha} = \begin{cases} I_3O_3R_4L_4R_4L_4\ldots((k-1)-times)\ldots.R_4L_4M_2 & \alpha > 2 \\ E''_8 : I_3M_2O_3 & \alpha = 2 \end{cases}$$

for $\alpha = 2k$, it is simple to confirm that $x_0 = x_1 = 2\alpha$, $a_0 = 6\alpha - 2$ and $a_1 = 6\alpha - 4$. For example, P_{24}^3 we have the following labeling of its vertices as E_{24} : 000111101001011010010110, it is simple to confirm that $x_0 = x_1 = 12$ and $a_0 = a_1 = 33$, or E'_{24} : 000111111000011100001110, it is simple to confirm that $x_0 = x_1 = 12$, $a_0 = 32$ and $a_1 = 34$, or E''_{24} : 000111110000111100001110, it is simple to confirm that $x_0 = x_1 = 12$, $a_0 = 34$ and $a_1 = 32$.

Sub-case (2). α is an odd number greater than one.
The following labels for the vertices of $P_{4\alpha}^3$ are used by us: $D_{4\alpha}$: $O_3I_3M'_4M_4M'_4M_4 \ldots((k-1)\text{- times})\ldots M'_4M_4M'_4M_4M'_2$ for $\alpha = 2k+1$, it is simple to confirm that $x_0 = x_1 = 2\alpha$ and $a_0 = a_1 = 6\alpha–3$, or $D'_{4\alpha}$: $O_3I_3R_4L_4R_4L_4\ldots((k-1)\text{- times})\ldots R_4L_4R_4L_4M'_2$ for $\alpha = 2k+1$, it is simple to confirm that $x_0 = x_1 = 2\alpha$, $a_0 = 6\alpha - 2$ and $a_1 = 6\alpha–4$, for example, P_{20}^3 we have the following labeling of its vertices as D_{20} : 00011110100101010 01, it is simple to confirm that $x_0 = x_1 = 10$ and $a_0 = a_1 = 27$, or D'_{20} : 00011111000011110010, it is simple to confirm that $x_0 = x_1 = 10$, $a_0 = 28$ and $a_1 = 26$.

Case (2): $n \equiv 1(mod4)$ or $n = 4\alpha + 1$, where $\alpha > 1$. We have the following two sub-cases:

Sub-case (1). α is an even number greater than one.
The labels assigned to the vertices of $P_{4\alpha+1}^3$ are used by us: $E_{4\alpha}$: $O_3I_3R_4L_4R_4L_4 \ldots((k-1)\text{- times})\ldots R_4L_4R_4L_4R_4$ for $\alpha = 2k_1$, it is simple to confirm that $x_0 = 2\alpha + 1, x_1 = 2\alpha$ and $a_0 = 6\alpha - 2, a_1 = 6\alpha–1$, or $E'_{4\alpha+1}$: $O_3I_3L_4R_4L_4R_4\ldots((k-1)\text{- times})\ldots R_4L_4R_4L_4$ for $\alpha = 2k+1$, it is simple to confirm that $x_0 = 2\alpha, x_1 = 2\alpha + 1$, $a_0 = 6\alpha - 2$ and $a_1 = 6\alpha–1$, or

$$E''_{4\alpha+1} = \begin{cases} I_3O_3R_4L_4R_4L_4\ldots((k-1)-times)\ldots.R_4L_4101 & \alpha > 2 \\ E''_9 : I_3O_3 10 & \alpha = 2 \end{cases}$$

where $\alpha = 2k+1$, it is simple for $\alpha \leq 2$ that $x_0 = 2\alpha$, $x_1 = 2\alpha+1$ and $a_0 = 6\alpha-1$ and $a_1 = 6\alpha$–2, or

$$E''_{4\alpha+1} = \begin{cases} O_3I_3R_4L_4R_4L_4\ldots((k-1)-times)\ldots.R_4L_4010 & \alpha > 2 \\ E''_9 : O_4I_301 & \alpha = 2 \end{cases}$$

where $\alpha = 2k+1$, it is simple for $\alpha \leq 2$ that $x_0 = 2\alpha+1$, $x_1 = 2\alpha$ and $a_0 = 6\alpha-1$ and $a_1 = 6\alpha$–2. For example, P^3_{17} we have the following labeling of its vertices as E_{17} : 00011001111001100, it is simple to confirm that $x_0 = 9, x_1 = 8, a_0 = 22$ and $a_1 = 23$, or E'_{17} : 11100001111000011, it is simple to confirm that $x_0 = 9, x_1 = 8$, $a_0 = 22 and = a_1 = 23$, or E''_{17} : 11100000111100101, it is simple to confirm that $x_0 = 9, x_1 = 8$, $a_0 = 23$ and $a_1 = 22$. In case of $\alpha = 1$, we may use the following labeling P^3_5 : 00011 (i.e., $x_0 = 3, x_1 = 2, a_0 = 4$ and $a_1 = 5$).

Sub-case (2). α is an odd number greater than one.
The labels assigned to the vertices of $P^3_{4\alpha+1}$ are used by us: $D_{4\alpha+1}$: $O_3I_3R_4L_4R_4$ $L_4\ldots((k-1)$- times)$\ldots$ R_4L_4 for $\alpha = 2k+1$, it is simple to confirm that $x_0 = 2\alpha+1, x_1 = 2\alpha$ and $a_0 = 6\alpha-2$ and $a_1 = 6\alpha$–1, or $D'_{4\alpha+1}$: $O_3I_3L_4R_4L_4R_4\ldots((k-1)$- times)$\ldots$ L_4R_4 for $\alpha = 2k+1$, it is simple to confirm that $x_0 = 2\alpha, x_1 = 2\alpha+1$ and $a_0 = 6\alpha-2$ and $a_1 = 6\alpha$–1, or $D''_{4\alpha+1} = I_3O_3L_4R_4L_4R_4\ldots((k-1)$-times)$\ldots$ $L_4R_4L_4101$ for $\alpha = 2k+1$, it is simple to confirm that $x_0 = 2\alpha, x_1 = 2\alpha+1$ and $a_0 = 6\alpha-1$ and $a_1 = 6\alpha$–2. For example, P^3_{21} we have the following labeling of its vertices as D_{21} : 000111100001111000011, it is simple to confirm that $x_0 = 11, x_1 = 10$, $a_0 = 22$ and $a_1 = 23$, or D'_{21} : 111000011110000111100, it is simple to confirm that $x_0 = 10, x_1 = 11$, $a_0 = 22$ and $a_1 = 23$, or D''_{21} : 111000001111000011101, it is simple to confirm that $x_0 = 10, x_1 = 11$, $a_0 = 23$ and $a_1 = 22$.

Case (3): $n \equiv 0(mod4)$ or $n = 4\alpha+2$,where $\alpha > 1$. We have the following two sub-cases:

Sub-case (1). α is an even number greater than one.
The labels assigned to the vertices of $P^3_{4\alpha+2}$ are used by us: $E_{4\alpha+2} : O_3I_3M'_4M_4$ $M'_4M_4\ldots((k-1)$- times)$\ldots$ $M'_4M_4M'_4M_4M'_4$ for $\alpha = 2k$, it is simple to confirm that $x_0 = x_1 = 2\alpha+1$ and $a_0 = a_1 = 6\alpha$, or $E'_{4\alpha+2} : O_3I_3R_4L_4R_4L_4\ldots((k-1)$- times)$\ldots$ $R_4L_4R_4L_4R_4$ for $\alpha = 2k$, it is simple to confirm that $x_0 = x_1 = 2\alpha+1$, $a_0 = 6\alpha+1$ and $a_1 = 6\alpha$–1. For example, P^3_{18}, we have the following labeling of its vertices as E_{18} : 000111101001011010, it is simple to confirm that $x_0 = x_1 = 9$ and $a_0 = a_1 = 24$, or E'_{18} : 000111110000111100, it is simple to confirm that $x_0 = x_1 = 9, a_0 = 25$ and $a_1 = 23$.

Sub-case (2). α is an odd number greater than one.
The labels assigned to the vertices of $P^3_{4\alpha+2}$ are used by us: $D_{4\alpha+2} : O_3I_3M'_4M_4$ $M'_4M_4\ldots((k-1)$- times)$\ldots$ $M'_4M_4M'_4M_4M'_4$ for $\alpha = 2k+1$, it is simple to confirm that $x_0 = x_1 = 2\alpha+1$ and $a_0 = a_1 = 6\alpha$, or $D'_{4\alpha+2} : O_3I_3R_4L_4R_4L_4\ldots((k-1)$- times)$\ldots$ R_4L_4 for $\alpha = 2k+1$, it is simple to confirm that $x_0 = x_1 = 2\alpha+1$, $a_0 = 6\alpha+1$ and $a_1 = 6\alpha$–1. For example, P^3_{22}, we have the following labeling of its vertices as D_{22} : 0001111010010110100101, it is simple to confirm that $x_0 = x_1 = 11$ and $a_0 = a_1 = 24$, or D'_{22} : 0001111100001111000011, it is simple to confirm that $x_0 = x_1 = 11, a_0 = 31$ and $a_1 = 29$.

Case (4): $n \equiv 3(mod4)$ or $n = 4\alpha+3$, where $\alpha > 1$. We have the following two sub-cases:

Sub-case (1). α is an even number greater than one.
The labels assigned to the vertices of $P^3_{4\alpha+3}$ are used by us: $E_{4\alpha+3}$: $O_3I_2R_4L_4R_4$ $L_4\ldots((k-1)$- times)$\ldots$ $R_4L_4R_4M'_2$ for $\alpha = 2k+1$, it is simple to confirm that $x_0 = 2\alpha+2, x_1 = 2\alpha+1$ and $a_0 = 6\alpha+1$ and $a_1 = 6\alpha+2$, or $E'_{4\alpha+3}$: $I_3O_2L_4R_4L_4R_4\ldots((k-1)$- times)$\ldots$ $L_4R_4L_4R_4L_4M_2$ for $\alpha = 2k+1$, it is simple to confirm that $x_0 = 2\alpha+1, x_1 = 2\alpha+2$ and $a_0 = 6\alpha+1$ and $a_1 = 6\alpha+2$, or $E''_{4\alpha+3}$: $I_3O_3L_4R_4L_4R_4\ldots((k-1)$- times)$\ldots$ $L_4R_4L_4M_41$ for $\alpha = 2k+1$, it is simple to confirm that $x_0 = 2\alpha+1, x_1 = 2\alpha+2$ and $a_0 = 6\alpha+2$ and $a_1 = 6\alpha+1$, or $E'''_{4\alpha+3}$: $O_3I_3R_4L_4R_4L_4\ldots((k-1)$- times)$\ldots$ $R_4L_4M'_40$, it is simple to confirm that $x_0 = 2\alpha+2, x_1 = 2\alpha+1$ and $a_0 = 6\alpha+2$ and $a_1 = 6\alpha+1$. For example, we have the following labeling P^3_{19} of its vertices as E_{19} : 0001111000011110010, it is simple to confirm that $x_0 = 10, x_1 = 9, a_0 = 25$ and $a_1 = 26$, or E'_{19} : 1110000111100001101, it is simple to confirm that $x_0 = 10, x_1 = 9, a_0 = 25$ and $a_1 = 26$, or E''_{19} : 1110000011110001011, it is simple to confirm that $x_0 = 9, x_1 = 10, a_0 = 26$ and $a_1 = 25$, or E'''_{19} : 0001111100001110100, it is simple to confirm that $x_0 = 10, x_1 = 9, a_0 = 26$ and $a_1 = 25$. In case of $\alpha = 1$, we may use the following labeling P^3_7: 0001101 (i.e., $x_0 = 4, x_1 = 3, a_0 = 7$ and $a_1 = 8$) or 0010111 (i.e., $x_0 = 3, x_1 = 4, a_0 = 8$ and $a_1 = 7$).

Sub-case (2). α is an odd number greater than one.
The labels assigned to the vertices of P^3_{4r+3} are used by us: $D_{4\alpha+3} := I_3O_2L_4R_4$ $L_4R_4\ldots(k$- times)$\ldots$ $L_4R_4L_4R_4M'_2$ for $\alpha = 2k+1$, it is simple to confirm that $x_0 = 2\alpha+1, x_1 = 2\alpha+2$ and $a_0 = 6\alpha+1$ and $a_1 = 6\alpha+2$. For example, we label the vertices of P^3_{15} as $D_{15} : I_3O_2L_4R_4M'_2 = 111000011110010$, it is simple to confirm that $x_0 = 7, x_1 = 8, a_0 = 19$ and $a_1 = 20$, or $D'_{4\alpha+3}$: $O_3I_2R_4L_4R_4L_4\ldots(k$-times)$\ldots$ $R_4L_4R_4L_4M_2$ for $\alpha = 2k+1$, it is simple to confirm that $x_0 = 2\alpha+2, x_1 = 2\alpha+1, a_0 = 6\alpha+1$ and $a_1 = 6\alpha+2$. For example, we label the vertices of P^3_{15} as $D'_{15} : O_3I_2R_4L_4M_2 = 000111100001101$, it is simple to confirm that $x_0 = 8, x_1 = 7, a_0 = 19$ and $a_1 = 20$, or $D''_{4\alpha+3} : I_3O_3L_4R_4L_4R_4\ldots(k$-times)$\ldots$ L_4R_41 for $\alpha = 2k+1$, it is simple to confirm that $x_0 = 2\alpha+1, x_1 = 2\alpha+2, a_0 = 6\alpha+2$ and $a_1 = 6r+1$. For example, P^3_{23}, we label the vertices of as $D''_{15} := I_3O_3L_4R_41 = 111000001111001$, it is simple to confirm that $x_0 = 7, x_1 = 8, a_0 = 20$ and $a_1 = 19$. Also, for example, we have the following labeling of its vertices as D_{23} : 11100001111000011110010, it is simple to confirm that $x_0 = 12, x_1 = 11, a_0 = 31$ and $a_1 = 32$, or $D'_{23} :=$ 00011110000111100001101, it is simple to confirm that $x_0 = 11, x_1 = 12, a_0 = 31$ and $a_1 = 32$, or $D''_{23} :=$ 11100000111100001111001, it is simple to confirm that $x_0 = 11, x_1 = 12, a_0 = 32$ and $a_1 = 31$.

We let $[L;M]$ signify the joint labeling for specific labeling L and M of $G+G^*$ or $G\cup G^*$, where G is the third power of a path and G^* is a path or cycle. A bijection between the vertex sets of G and G^* such that any two vertices of G, u and v, are adjacent in G if and only if $h(u)$ and $h(v)$ are adjacent in G^* is an isomorphism of graphs G and G^*, $h: V(G) \longrightarrow V(G^*)$. When two graphs can be transformed into each other, they are said to be isomorphic and are represented by the notation $G \equiv G^*$.

3 Join Between Two Third-Power of Paths

In this section, we show that the join $P_n^3 + P_m$ of third power of a path P_n^3 and a path P_m is cordial for all n and m iff $(n,m) \neq (2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)$.

Lemma 3.1. The join $P_n^3 + P_m$ of third power of a path P_n^3 and P_m is cordial for all $n > 7$ and $m > 3$.

Proof. For specified i and j values, where $n = 4\alpha + i$, $m = 4\beta + j$ with $0 \leq i, j \leq 3$, we have two cases:

Case (1): α even.
We use the labeling E_i or E_i''' for the third power of a path P_n^3 and B_j for the path P_m as given in Table 1 and Table 3. Using Tables 1 and 3 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, the values displayed in Table 4's final two columns can be calculated. The lemma follows as these are all $0, 1$, or -1.

Case (2): α odd.
We use the labeling D_i, D_i' or D_i'' for the third power of a path P_n^3 and B_j or B_j' for the path P_m as given in Table 1 and Table 3. Using Tables 1 and 3 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, the values displayed in Table 5's final two columns can be calculated. The lemma follows as these are all $0, 1$, or -1. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

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Table 1: Labeling of P_n^3 where $n = 4\alpha + i, i = 0,1,2,3$ and α is even

$n = 4\alpha+$ $ii = 0,1,2,3$	labeling of P_n^3	x_0	x_1	a_0	a_1
$i=0$	$E_{4\alpha}$	2α	2α	$6\alpha-3$	$6\alpha-3$
	$E'_{4\alpha}$	2α	2α	$6\alpha-4$	$6\alpha-2$
$i=1$	$E_{4\alpha+1}$	$2\alpha+1$	2α	$6\alpha-2$	$6\alpha-1$
	$E'_{4\alpha+1}$	2α	$2\alpha+1$	$6\alpha-2$	$6\alpha-1$
	$E''_{4\alpha+1}$	2α	$2\alpha+1$	$6\alpha-1$	$6\alpha-2$
	$E'''_{4\alpha+1}$	$2\alpha+1$	2α	$6\alpha-1$	$6\alpha-2$
$i=2$	$E_{4\alpha+2}$	$2\alpha+1$	$2\alpha+1$	6α	6α
	$E'_{4\alpha+2}$	$2\alpha+1$	$2\alpha+1$	$6\alpha+1$	$6\alpha-1$
$i=3$	$E_{4\alpha+3}$	$2\alpha+2$	$2\alpha+1$	$6\alpha+1$	$6\alpha+2$
	$E'_{4\alpha+3}$	$2\alpha+1$	$2\alpha+2$	$6\alpha+1$	$6\alpha+2$
	$E''_{4\alpha+3}$	$2\alpha+1$	$2\alpha+2$	$6\alpha+2$	$6\alpha+1$
	$E'''_{4\alpha+3}$	$2\alpha+2$	$2\alpha+1$	$6\alpha+2$	$6\alpha+1$

Table 2: Labeling of P_n^3 where $n = 4\alpha + i, i = 0,1,2,3$ and α is odd

$n = 4\alpha+$ $i, i = 0,1,2,3$	labeling of P_n^3	x_0	x_1	a_0	a_1
$i=0$	$D_{4\alpha}$	2α	2α	$6\alpha-3$	$6\alpha-3$
	$D'_{4\alpha}$	2α	2α	$6\alpha-2$	$6\alpha-4$
$i=1$	$D_{4\alpha+1}$	$2\alpha+1$	2α	$6\alpha-2$	$6\alpha-1$
	$D'_{4\alpha+1}$	2α	$2\alpha+1$	$6\alpha-2$	$6\alpha-1$
	$D''_{4\alpha+1}$	2α	$2\alpha+1$	$6\alpha-1$	$6\alpha-2$
	$D'''_{4\alpha+1}$	$2\alpha+1$	2α	$6\alpha-1$	$6\alpha-2$
$i=2$	$D_{4\alpha+2}$	$2\alpha+1$	$2\alpha+1$	6α	6α
	$D'_{4\alpha+2}$	$2\alpha+1$	$2\alpha+1$	$6\alpha+1$	$6\alpha-1$
$i=3$	$D_{4\alpha+3}$	$2\alpha+2$	$2\alpha+1$	$6v+1$	$6\alpha+2$
	$D'_{4\alpha+3}$	$2\alpha+1$	$2\alpha+2$	$6\alpha+1$	$6\alpha+2$
	$D''_{4\alpha+3}$	$2\alpha+1$	$2\alpha+2$	$6\alpha+2$	$6\alpha+1$

Table 3: Labeling of P_m

$m = 4\beta + j,$ $j = 0,1,2,3$	labeling of P_m^2	y_0	y_1	b_0	b_1
$j=0$	$B_0 = L_{4\beta}$	2β	2β	2β	$2\beta-1$
	$B'_0 = L''_{4\beta}$	2β	2β	$2\beta-1$	2β
$j=1$	$B_1 = R_{4\beta}1$	2β	$2\beta+1$	2β	2β
	$B'_1 = L'_{4\beta}0$	$2\beta+1$	2β	2β	2β
$j=2$	$B_2 = L_{4\beta}10$	$2\beta+1$	$2\beta+1$	$2\beta+1$	2β
$j=3$	$B_3 = L_{4\beta}011$	$2\beta+1$	$2\beta+2$	$2\beta+1$	$2\beta+1$
	$B'_3 = L'_{4\beta}001$	$2\beta+2$	$2\beta+1$	$2\beta+1$	$2\beta+1$
	$B''_3 = R_{4\beta}011$	$2\beta+1$	$2\beta+2$	$2\beta+2$	2β

Remark 3.1. It is clear to see from Tables 1 and 2 below that P_n^3 is cordial for all $n > 4$ and from the particular cases of P_n^3 that P_n^3 is cordial for all $1 \leq n \leq 3$ [3,12]. This means that P_n^3 is cordial for all n, where $1 \leq n \leq 3$ and $n > 4$. In the case of

Table 4: Combination of Labeling

$n=4\alpha+i$, where α even, $\alpha>1$ and $i=0,1,2,3$	$m=4\beta+j$, $j=0,1,2,3$	P_n^3	P_m	v_0-v_1	e_0-e_1
0	0	$E_{4\alpha}$	B_0	0	1
0	1	$E_{4\alpha}$	B_1	-1	0
0	2	$E_{4\alpha}$	B_2	0	1
0	3	$E_{4\alpha}$	B_3	-1	0
1	0	$E_{4\alpha+1}$	B_0	1	0
1	1	$E_{4\alpha+1}$	B_1	1	-1
1	2	$E_{4\alpha+1}$	B_2	1	0
1	3	$E'''_{4\alpha+1}$	B_3	0	0
2	0	$E_{4\alpha+2}$	B_0	0	1
2	1	$E_{4\alpha+2}$	B_1	-1	0
2	2	$E_{4\alpha+2}$	B_2	0	1
2	3	$E_{4\alpha+2}$	B_3	-1	0
3	0	$E_{4\alpha+3}$	B_0	1	0
3	1	$E'''_{4\alpha+3}$	B_1	0	0
3	2	$E_{4\alpha+3}$	B_2	1	0
3	3	$E'''_{4\alpha+3}$	B_3	0	0

Table 5: Combination of Labeling

$n=4\alpha+i$, where α odd, $\alpha>1$ and $i=0,1,2,3$	$m=4\beta+j$, $j=0,1,2,3$	P_n^3	P_m	v_0-v_1	e_0-e_1
0	0	$D_{4\alpha}$	B_0	0	-1
0	1	$D_{4\alpha}$	B_1	-1	0
0	2	$D_{4\alpha}$	B_2	0	1
0	3	$D_{4\alpha}$	B_3	-1	0
1	0	$D_{4\alpha+1}$	B_0	1	0
1	1	$D''_{4\alpha+1}$	B'_1	0	0
1	2	$D_{4\alpha+1}$	B_2	1	0
1	3	$D''_{4\alpha+1}$	B'_3	0	0
2	0	$D_{4\alpha+2}$	B_0	0	1
2	1	$D_{4\alpha+2}$	B_1	-1	0
2	2	$D_{4\alpha+2}$	B_2	0	1
2	3	$D_{4\alpha+2}$	B_3	-1	0
3	0	$D'_{4\alpha+3}$	B_0	1	0
3	1	$D''_{4\alpha+3}$	B'_1	0	0
3	2	$D'_{4\alpha+3}$	B_2	1	0
3	3	$D''_{4\alpha+3}$	B'_3	0	0

$n=4$ and $P_4^3 \equiv K_4$, we see that P_4^3 is not cordial from known result (see Cahit [2]). Therefore, we have proved the fact that has been proved by E.A. Elsakhawi [10], and Seoud and Abdel Maqusoud [13], which state that P_n^3 is cordial iff $n \neq 4$.

Lemma 3.2. The join $P_n^3 + P_m$ of third power of a path P_n^3 and a path P_m is cordial for all $4 > 7$ and m, where $1 \le m \le 3$.

Proof. Let $n = 4\alpha + i$, where $\alpha > 1$ and $0 \le i \le 3$, then we have two cases:

Case (1): α even.
For given values of m, where $0 \le i \le 3$, we use the labeling E_n or E''_n for the third power of a path P_n^3 as given in Table 1 and A_m or A'_m for the path P_m as given in Table 6 and Table 1. Using Tables 1 and 6 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, the values displayed in Table 7's final two columns can be calculated. The lemma follows as these are all $0, 1$, or -1.

Case (2): α odd.
For given values of m, where $0 \le i \le 3$, we use the labeling D_n or D''_n for the third power of a path P_n^3 as given in Table 1 and A_m or A'_m for the path P_m as given in Table 6 and Table 1. Using Table 6 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, the values displayed in Table 8's final two columns can be calculated. The lemma follows as these are all $0, 1$, or -1.

Table 6: Labeling of P_m where $1 \le m \le 3$

$1 \le m \le 3$	labeling of P_m	x_0	x_1	a_0	a_1
$m=1$	$A_1 = 0$	1	0	0	0
	$A'_1 = 1$	0	1	0	0
$m=2$	$A_2 = 01$	1	1	0	1
	$A'_2 = 00$	2	0	1	0
	$A''_2 = 11$	0	2	1	0
$m=3$	$A_3 = 001$	2	1	1	1

Table 7: Combination of Labeling

$n=4\alpha+i$, where α even, $\alpha>1$ and $i=0,1,2,3$	$1 \le m \le 3$	P_n^3	P_m	v_0-v_1	e_0-e_1
0	1	$E_{4\alpha}$	A'_1	-1	0
0	2	$E_{4\alpha}$	A_2	0	-1
0	3	$E_{4\alpha}$	B_3	1	0
1	1	$E''_{4\alpha+1}$	A_1	0	0
1	2	$E''_{4\alpha+1}$	A_2	-1	0
1	3	$E''_{4\alpha+1}$	A_3	0	0
2	1	$E_{4\alpha+2}$	A_1	1	0
2	2	$E_{4\alpha+2}$	A_2	0	-1
2	3	$E_{4\alpha+2}$	A_3	1	0
3	1	$E''_{4\alpha+3}$	A_1	0	0
3	2	$E''_{4\alpha+3}$	A'_2	1	0
3	3	$E''_{4\alpha+3}$	A_3	0	0

Table 8: Combination of Labeling

$n=4\alpha+i$, where α even, $\alpha>1$ and $i=0,1,2,3$	$1 \le m \le 3$	P_n^3	P_m	v_0-v_1	e_0-e_1
0	1	$D_{4\alpha}$	A'_1	-1	0
0	2	$D_{4\alpha}$	A_2	0	-1
0	3	$D_{4\alpha}$	A_3	1	0
1	1	$D''_{4\alpha+1}$	A_1	0	0
1	2	$D''_{4\alpha+1}$	A_2	-1	0
1	3	$D''_{4\alpha+1}$	A_3	0	0
2	1	$D_{4\alpha+2}$	A_1	1	0
2	2	$D_{4\alpha+2}$	A_2	0	-1
2	3	$D_{4\alpha+2}$	A_3	1	0
3	1	$D''_{4\alpha+2}$	A_1	0	0
3	2	$D''_{4\alpha+2}$	A'_2	1	0
3	3	$D''_{4\alpha+2}$	A_3	0	0

Lemma 3.3. The join $P_7^3 + P_m$ of third power of a path P_7^3 and a path P_m is cordial for all m.

Proof. We consider the cases of m separately.

Case (1): $1 \leq m \leq 7$, the following labeling suffice:
$P_7^3 + P_1$:[0010111:0],
$P_7^3 + P_2$:[0010111:01] and
$P_7^3 + P_3$:[0010111:001].

Case (2): $m > 3$. Let $m = 4\beta + j$, where $\beta > 1$ and $j = 0,1,2,3$, then the following labeling suffice:
$P_7^3 + P_{4\beta}$:[0001101; $L_{4\beta}$],
$P_7^3 + P_{4\beta+1}$:[0010111; $L_{4\beta+1}$0],
$P_7^3 + P_{4\beta+2}$:[0010111; $L_{4\beta+2}$01], and
$P_7^3 + P_{4\beta+3}$:[0010111; L_{4s+3}001], the lemma follows.

Lemma 3.4. The join $P_6^3 + P_m$ of third power of a path P_6^3 and a path P_m is cordial for all m.

Proof. We consider the cases of m separately.

Case (1): $1 \leq m \leq 3$, the following labeling suffice:
$P_6^3 + P_1$:[000111:0],
$P_6^3 + P_2$:[000111:01] and
$P_6^3 + P_3$:[000111:001].

Case (2): $m > 3$. Let $m = 4\beta + j$, where $\beta > 1$ and $j = 0,1,2,3$, then the following labeling suffice:
$P_6^3 + P_{4\beta}$:[000111; $L_{4\beta}$],
$P_6^3 + P_{4\beta+1}$:[000111; $L_{4\beta+1}$0],
$P_6^3 + P_{4\beta+2}$:[000111; $L_{4\beta+2}$01], and
$P_6^3 + P_{4\beta+3}$:[000111; $L_{4\beta+3}$001], the lemma follows.

Lemma 3.5. The join $P_5^3 + P_m$ of third power of a path P_5^3 and a path P_m is cordial for all m iff $m \neq 1,2,3$.

Proof. We consider the cases of m separately.

Case (1): $1 \leq m \leq 3$, it is easy to verify that by investigating all possible labeling of vertices of P_5^3 and P_k, where $1 \leq k \leq 3$ that $P_5^3 + P_1, P_5^3 + P_2$ and $P_5^3 + P_3$ does not have a cordial labeling.

Case (2): $m > 3$. Let $m = 4\beta + j$, where $\beta > 1$ and $j = 0,1,2,3$, then the following labeling suffice:
$P_5^3 + P_{4\beta}$:[00011; $L_{4\beta}$],
$P_5^3 + P_{4\beta+1}$:[00011; $L_{4\beta+1}$1],
$P_5^3 + P_{4\beta+2}$:[00011; $L_{\beta+2}$10], and
$P_5^3 + P_{4\beta+3}$:[00011; $L_{4\beta+3}$011], the lemma follows.

Lemma 3.6. The join $P_4^3 + P_m$ of third power of a path P_4^3 and a path P_m is cordial for all m iff $m \neq 1,2,3$.

Proof. We consider the cases of m separately.

Case (1): $1 \leq m \leq 3$, then from the facts that $P_4^3 \equiv K_4$ and consequently $P_4^3 + P_1 \equiv K_4 + P_1 \equiv K_5$ and $P_4^3 + P_2 \equiv K_4 + P_2 \equiv K6$, we obtain that the graphs $P_4^3 + P_1$ and $P_4^3 + P_2$ are not cordial (see Cahit [2]). Also, by investigating all possible labelings of K_4 and P_3, it is easy to see that $K_4 + P_3$ does not has a cordial labeling.

Case (2): $m > 3$. Let $m = 4\beta + j$, where $\beta > 1$ and $j = 0,1,2,3$, then the following labeling suffice:
$P_4^3 + P_{4\beta}$:[0011; $L_{4\beta}$],
$P_4^3 + P_{4\beta+1}$:[0011; $L_{4\beta+1}$1],
$P_4^3 + P_{4\beta+2}$:[0011; $L_{\beta+2}$10], and
$P_4^3 + P_{4\beta+3}$:[0011; $L_{4\beta+3}$001], the lemma follows.

Corollary 3.1. The join $P_3^3 + P_m$ of third power of a path P_n^3 and a path P_m is cordial for all m iff $m \neq 1,2,3$.

Proof. Since $P_3^3 \equiv Q_3$, then the proof follows directly from the fact that $Q_n + P_m$ is cordial for all n and m iff $(n,m) \neq (3,1),(3,2),(3,3)$ [3], the corollary follows.

Corollary 3.2. The graphs $P_2^3 + P_m$ is cordial for all m and $P_1^3 + P_m$ is cordial for all m iff $m \neq 2$.

Proof. Since $P_1^3 \equiv P_1$ and $P_2^3 \equiv P_2$, then the proof follows directly from the fact that the join $P_n + P_m$ of two paths P_n and P_m is cordial if and only if $(n,m) \neq (2,2)$ [8], the corollary follows.

We can construct the following Theorem based on the final facts.

Theorem 3.1. The join $P_n^3 + P_m$ of third power of paths P_n^3 and P_m is cordial iff $(n,m) \neq (2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)$.

Proof. The proof follows directly from Lemma 3.1, Lemma 3.2, Lemma 3.3, Lemma 3.4, Lemma 3.5 , Lemma 3.6, Corollary 3.2 and Corollary 3.3, the theorem follows.

4 Union between Third Power of Paths and Paths

In this section, we show that the union $P_n^3 \cup P_m$ of third power of a path P_n^3 and a path P_m is cordial for all n and m iff $(n,m) \neq (2,2)$.

Lemma 4.1. The union $P_n^3 \cup P_m$ of third power of paths P_n^3 and a path P_m is cordial for all $n > 7$ and $m > 3$.

Proof. For given values of i and j, where $n = 4\alpha + i, m = 4\beta + j$ with $i = 0,1,2,3$ and $j = 0,1,2,3$, we have two cases:

Case (1): α even.
We use the labeling E_i or E_i''' for the third power of a path P_n^3 and B_j for the path P_m as given in Table 1 and Table 3. Using Tables 1 and 3 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, the values displayed in Table 9's final two columns can be calculated. The lemma follows as

these are all $0, 1$, or -1.

Case (2): α odd.
We use the labeling D_i, D_i' or D_i'' for the third power of a path P_n^3 and B_j or B_j' for the path P_m as given in Table 1 and Table 3. Using Tables 1 and 3 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, the values displayed in Table 10's final two columns can be calculated. The lemma follows as these are all $0, 1$, or -1.

Table 9: Combination of Labeling

$n = 4\alpha + i$, where α even, $\alpha > 1$ and $i = 0,1,2,3$	$m = 4\beta + j$, $j = 0,1,2,3$	P_n^3	P_m	$v_0 - v_1$	$e_0 - e_1$
0	0	$E_{4\alpha}$	B_0	0	1
0	1	$E_{4\alpha}$	B_1	-1	0
0	2	$E_{4\alpha}$	B_2	0	1
0	3	$E_{4\alpha}$	B_3	-1	0
1	0	$E_{4\alpha+1}$	B_0	1	0
1	1	$E_{4\alpha+1}$	B_1	1	-1
1	2	$E_{4\alpha+1}$	B_2	1	0
1	3	$E'''_{4\alpha+1}$	B_3	0	0
2	0	$E_{4\alpha+2}$	B_0	0	1
2	1	$E_{4\alpha+2}$	B_1	-1	0
2	2	$E_{4\alpha+2}$	B_2	0	1
2	3	$E_{4\alpha+2}$	B_3	-1	0
3	0	$E_{4\alpha+3}$	B_0	1	0
3	1	$E'''_{4\alpha+3}$	B_1	0	0
3	2	$E_{4\alpha+3}$	B_2	1	0
3	3	$E'''_{4\alpha+3}$	B_3	0	0

Table 10: Combination of Labeling

$n = 4\alpha + i$, where α odd, $\alpha > 1$ and $i = 0,1,2,3$	$m = 4\beta + j$, $j = 0,1,2,3$	P_n^3	P_m	$v_0 - v_1$	$e_0 - e_1$
0	0	$D_{4\alpha}$	B_0	0	-1
0	1	$D_{4\alpha}$	B_1	-1	0
0	2	$D_{4\alpha}$	B_2	0	1
0	3	$D_{4\alpha}$	B_3	-1	0
1	0	$D_{4\alpha+1}$	B_0	1	0
1	1	$D''_{4\alpha+1}$	B_1'	0	0
1	2	$D_{4\alpha+1}$	B_2	1	0
1	3	$D''_{4\alpha+1}$	B_3'	0	0
2	0	$D_{4\alpha+2}$	B_0	0	1
2	1	$D_{4\alpha+2}$	B_1	-1	0
2	2	$D_{4\alpha+2}$	B_2	0	1
2	3	$D_{4\alpha+2}$	B_3	-1	0
3	0	$D'_{4\alpha+3}$	B_0	1	0
3	1	$D''_{4\alpha+3}$	B_1'	0	0
3	2	$D'_{4\alpha+3}$	B_2	1	0
3	3	$D''_{4\alpha+3}$	B_3'	0	0

Lemma 4.2. The union $P_n^3 \cup P_m$ of third power of paths P_n^3 and a path P_m is cordial for all $n > 7$ and $1 \leq m \leq 3$.

Proof. For given $n = 4\alpha + i$ and $\alpha > 1$ with $i = 0, 1, 2, 3$, we have two cases:

Case (1): α even.
For given values of m, where $1 \leq m \leq 3$, we use the labeling E_n or E_n'' for the third power of a path P_n^3 as given in Table 1 and A_m or A_m' or A_m'' for the path P_m as given in Table 6. Using Tables 1 and 6 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, the values displayed in Table 11's final two columns can be calculated. The lemma follows as these are all $0, 1$, or -1.

Case (2): α odd.
For given values of m, where $1 \leq m \leq 3$, we use the labeling D_n or D_n'' for the third power of a path P_n^3 as given in Table 1 and A_m or A_m' or A_m'' for the path P_m as given in Table 6. Using Tables 1 and 6 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, the values displayed in Table 12's final two columns can be calculated. The lemma follows as these are all $0, 1$, or -1.

Table 11: Combination of Labeling

$n = 4\alpha + i$, where α even, $\alpha > 1$ and $i = 0,1,2,3$	$1 \leq m \leq 3$	P_n^3	P_m	$v_0 - v_1$	$e_0 - e_1$
0	1	$E_{4\alpha}$	A_1'	-1	0
0	2	$E_{4\alpha}$	A_2	0	-1
0	3	$E_{4\alpha}$	B_3	1	0
1	1	$E''_{4\alpha+1}$	A_1	0	1
1	2	$E''_{4\alpha+1}$	A_2	-1	0
1	3	$E''_{4\alpha+1}$	A_3	0	1
2	1	$E_{4\alpha+2}$	A_1	1	0
2	2	$E_{4\alpha+2}$	A_2	0	-1
2	3	$E_{4\alpha+2}$	A_3	1	0
3	1	$E''_{4\alpha+3}$	A_1	0	1
3	2	$E_{4\alpha+3}$	A_2''	-1	0
3	3	$E''_{4\alpha+3}$	A_3	0	1

Table 12: Combination of Labeling

$n = 4\alpha + i$, where α even, $\alpha > 1$ and $i = 0,1,2,3$	$1 \leq m \leq 3$	P_n^3	P_m	$v_0 - v_1$	$e_0 - e_1$
0	1	$D_{4\alpha}$	A_1'	-1	0
0	2	$D_{4\alpha}$	A_2	0	-1
0	3	$D_{4\alpha}$	A_3	1	0
1	1	$D''_{4\alpha+1}$	A_1	0	1
1	2	$D''_{4\alpha+1}$	A_2	-1	0
1	3	$D''_{4\alpha+1}$	A_3	0	-1
2	1	$D_{4\alpha+2}$	A_1	1	0
2	2	$D_{4\alpha+2}$	A_2	0	-1
2	3	$D_{4\alpha+2}$	A_3	1	0
3	1	$D''_{4\alpha+2}$	A_1	0	1
3	2	$D_{4\alpha+2}$	A_2''	-1	0
3	3	$D''_{4\alpha+2}$	A_3	0	1

Lemma 4.3. The graph $P_7^3 \cup P_m$ of third power of paths P_7^3 and a path P_m is cordial for all m.

Proof. We consider the cases of m separately.

Case (1): $1 \leq m \leq 3$. Appropriate labeling are the following: $P_7^3 \cup P_3$:[0010111:001], $P_7^3 \cup P_2$:[0010111:01] and $P_7^3 \cup P_1$:[0010111:0].

Case (2): $m > 3$. Let $m = 4\beta + j$, where $\beta > 1$ and $j = 0, 1, 2, 3$, then the following labeling suffice:
$P_7^3 \cup P_{4\beta}$:[0001101; $L_{4\beta}$],
$P_7^3 \cup P_{4\beta+1}$:[0010111; $L_{4\beta+1}$0],
$P_7^3 \cup P_{4\beta+2}$:[0010111; $L_{4\beta+2}$01] and
$P_7^3 \cup P_{4\beta+3}$:[0010111; $L_{4\beta+3}$001], the lemma follows.

Lemma 4.4. The graph $P_6^3 \cup P_m$ of third power of paths P_6^3 and a path P_m is cordial for all m.

Proof. We consider the cases of m separately.

Case (1): $1 \leq m \leq 3$. Appropriate labeling are the following: $P_6^3 \cup P_3$:[000111:001], $P_6^3 \cup P_2$:[000111:01] and $P_6^3 \cup P_1$:[000111:0].

Case (2): $m > 3$. Let $m = 4\beta + j$, where $\beta > 1$ and $j = 0, 1, 2, 3$, then the following labeling suffice:
$P_6^3 \cup P_{4\beta}$:[000111; $L_{4\beta}$],
$P_6^3 \cup P_{4\beta+1}$:[000111; $L_{4\beta+1}$0],
$P_6^3 \cup P_{4\beta+2}$:[000111; $L_{4\beta+2}$01] and
$P_6^3 \cup P_{4\beta+3}$:[000111; $L_{4\beta+3}$001], the lemma follows.

Lemma 4.5. The graph $P_5^3 \cup P_m$ of third power of paths P_5^3 and a path P_m is cordial for all m.

Proof. We consider the cases of m separately.

Case (1): $1 \leq m \leq 3$. Appropriate labeling are the following: $P_5^3 \cup P_3$:[00011:011], $P_5^3 \cup P_2$:[00011:11] and $P_5^3 \cup P_1$:[00011:1].

Case (2): $m > 3$. Let $m = 4\beta + j$, where $\beta > 1$ and $j = 0, 1, 2, 3$, then the following labeling suffice:
$P_5^3 \cup P_{4\beta}$:[00011; $L_{4\beta}$],
$P_5^3 \cup P_{4\beta+1}$:[00011; $L_{4\beta+1}$1],
$P_5^3 \cup P_{4\beta+2}$:[00011; $L_{4\beta+2}$11] and
$P_5^3 \cup P_{4\beta+3}$:[00011; $L_{4\beta+3}$110], the lemma follows.

Lemma 4.6. The graph $P_4^3 \cup P_m$ of third power of paths P_4^3 and a path P_m is cordial for all m.

Proof. We consider the cases of m separately.

Case (1): $1 \leq m \leq 3$. Appropriate labeling are the following: $P_4^3 \cup P_3$:[0001111:001], $P_4^3 \cup P_2$:[0001:11] and $P_4^3 \cup P_1$:[0001:1].

Case (2): $m > 3$. Let $m = 4\beta + j$, where $\beta > 1$ and $j = 0, 1, 2, 3$, then the following labeling suffice:
$P_4^3 \cup P_{4\beta}$:[0011; $L_{4\beta}$],
$P_4^3 \cup P_{4\beta+1}$:[0011; $L_{4\beta+1}$1],
$P_4^3 \cup P_{4\beta+2}$:[0011; $L_{4\beta+2}$10] and
$P_4^3 \cup P_{4\beta+3}$:[0011; $L_{4\beta+3}$001], the lemma follows.

Lemma 4.7. The graph $P_3^3 \cup P_m$ of third power of paths P_3^3 and a path P_m is cordial for all m.

Proof. We consider the cases of m separately.

Case (1): $1 \leq m \leq 3$. Appropriate labeling are the following: $P_3^3 \cup P_3$:[001:110], $P_3^3 \cup P_2$:[001:11] and $P_3^3 \cup P_1$:[001:1].

Case (2): $m > 3$. Let $m = 4\beta + j$, where $\beta > 1$ and $j = 0, 1, 2, 3$, then the following labeling suffice:
$P_3^3 \cup P_{4\beta}$:[001; $L_{4\beta}$],
$P_3^3 \cup P_{4\beta+1}$:[011; $L_{4\beta+1}$0],
$P_3^3 \cup P_{4\beta+2}$:[001; $L_{4\beta+2}$10] and
$P_3^3 \cup P_{4\beta+3}$:[011; $L_{4\beta+3}$001], the lemma follows.

Lemma 4.8. The graph $P_2^3 \cup P_m$ of third power of paths P_2^3 and a path P_m is cordial for all m iff $m \neq 2$.

Proof. We consider the cases of m separately.

Case (1): $1 \leq m \leq 3$ and $m \neq 2$. Appropriate labeling are the following: $P_2^3 \cup P_3$:[01:011] and $P_2^3 \cup P_1$:[01:0].

Case (2): $m > 3$. Let $m = 4\beta + j$, where $\beta > 1$ and $j = 0, 1, 2, 3$, then the following labeling suffice:
$P_2^3 \cup P_{4\beta}$:[01; $L_{4\beta}$],
$P_2^3 \cup P_{4\beta+1}$:[01; $L_{4\beta+1}$0],
$P_2^3 \cup P_{4\beta+2}$:[01; $L_{4\beta+2}$10] and
$P_2^3 \cup P_{4\beta+3}$:[01; $L_{4\beta+3}$001], the lemma follows.

Lemma 4.9. The graph $P_1^3 \cup P_m$ of third power of paths P_1^3 and a path P_m is cordial for all m.

Proof. Since $P_1^3 \equiv P_1$, then we consider the cases of m separately.

Case (1): $1 \leq m \leq 3$. Appropriate labeling are the following: $P_1 \cup P_3$:[1:001] $P_1 \cup P_2$:[0:01] and $P_1 \cup P_1$:[1:0].

Case (2): $m > 3$. Let $m = 4\beta + j$, where $\beta > 1$ and $j = 0, 1, 2, 3$, then the following labeling suffice:
$P_1^3 \cup P_{4\beta}$:[0; $L_{4\beta}$],
$P_1^3 \cup P_{4\beta+1}$:[1; $L_{4\beta+1}$0],
$P_1^3 \cup P_{4\beta+2}$:[1; $L_{4\beta+2}$01] and
$P_1^3 \cup P_{4\beta+3}$:[1; $L_{4\beta+3}$001], the lemma follows.

Theorem 4.1. The union $P_n^3 \cup P_m$ of third power of paths P_n^3 and a path P_m is cordial iff $(n,m) \neq (2,2)$.

Proof. The evidence is clear from the above Lemmas, the theorem follows.

5 Joins of Third Power of Paths and Cycles

In this section, we show that the join $P_n^3 + Q_m$ of third power of a path P_n^3 and a cycle Q_m is cordial for all n and m iff $(n,m) \neq (1,3),(2,3),(3,3),(4,3),(4,4),(5,3)$.

Table 13: Labeling of a cycle Q_m

$m=4\beta+j$, $j=0,1,2,3$	labeling of a cycle Q_m	x_0	x_1	a_0	a_1
$j=0$	$A_0=L_{4\beta}$	2β	2β	2β	2β
$j=1$	$A_1=L_{4\beta}1$	2β	$2\beta+1$	2β	2β
$j=2$	$A_2=0L_{4\beta}1$	$2\beta+1$	$2\beta+1$	$2\beta+2$	2β
	$A_2'=01L_{4\beta}$	$2\beta+1$	$2\beta+1$	2β	$2\beta+2$
$j=3$	$A_3=L_{4\beta}011$	$2\beta+1$	$2\beta+2$	$2\beta+1$	$2\beta+2$
	$A_3'=L_{4\beta}110$	$2\beta+1$	$2\beta+2$	$2\beta+3$	2β
	$A_3''=L_{4\beta}001$	$2\beta+2$	$2\beta+1$	$2\beta+1$	$2\beta+2$

Lemma 5.1. The join $P_n^3 + Q_m$ of third power of a path P_n^3 and a cycle P_m^2 is cordial for all $n > 7$ and $m > 6$.

Proof. For given values of i and j, where $n = 4\alpha + i$ with $0 \leq i \leq 3$ and $\alpha > 1$, and $m = 4\beta + j$ with $0 \leq j \leq 3$ and $\beta > 1$, we have two cases:

Case (1): α is an even number. We use the labeling E_i for the third power of a path P_n^3 and A_j or A_j' or A_j'' for the cycle Q_m as given in Table 1 and Table 13. Using Tables 1 and 13 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)_(x_0 - x_1)(y_0 - y_1)$, the values displayed in Table 14's final two columns can be calculated. The lemma follows as these are all $0, 1$, or -1.

Case (2): $alpha$ is an odd number. We use the labeling D_i or D_i' for the third power of path P_n^3 and A_j or A_j' or A_j'' for the cycle Q_m as given in Table 1 and Table 13. Using Tables 1 and 13 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)_(x_0 - x_1)(y_0 - y_1)$, the values displayed in Table 15's final two columns can be calculated. The lemma follows as these are all $0, 1$, or -1.

Table 14: Combination of Labeling

$n=4\alpha+i$, where α even, $i=0,1,2,3$	$m=4\beta+j$, $j=0,1,2,3$	P_n^3	Q_m	v_0–v_1	e_0–e_1
0	0	$E_{4\alpha}$	A_0	0	0
0	1	$E_{4\alpha}$	A_1	-1	1
0	2	$E'_{4\alpha}$	A_2	0	0
0	3	$E_{4\alpha}$	A_3	-1	-1
1	0	$E_{4\alpha+1}$	A_0	1	-1
1	1	$E_{4\alpha+1}$	A_1	0	-1
1	2	$E_{4\alpha+1}$	A_2	1	1
1	3	$E''_{4\alpha+1}$	A''_3	0	-1
2	0	$E_{4\alpha+2}$	A_0	0	0
2	1	$E_{4\alpha+2}$	A_1	-1	1
2	2	$E'_{4\alpha+2}$	A'_2	0	0
2	3	$E_{4\alpha+2}$	A_3	-1	-1
3	0	$E_{4\alpha+3}$	A_0	1	-1
3	1	$E_{4\alpha+3}$	A_1	0	-1
3	2	$E_{4\alpha+3}$	A_2	1	1
3	3	$E'''_{4\alpha+3}$	A_3	0	-1

Table 15: Combination of Labeling

$n=4\alpha+i$, where α odd, $i=0,1,2,3$	$m=4\beta+j$, $j=0,1,2,3$	P_n^3	Q_m	v_0–v_1	e_0–e_1
0	0	$D_{4\alpha}$	A_0	0	0
0	1	$D_{4\alpha}$	A_1	-1	1
0	2	$D'_{4\alpha}$	A_2	0	0
0	3	$D_{4\alpha}$	A_3	-1	-1
1	0	$D_{4\alpha+1}$	A_0	1	-1
1	1	$D_{4\alpha+1}$	A_1	0	-1
1	2	$D_{4\alpha+1}$	A_2	1	1
1	3	$D'''_{4\alpha+1}$	A_3	0	-1
2	0	$D_{4\alpha+2}$	A_0	0	0
2	1	$D_{4\alpha+2}$	A_1	-1	1
2	2	$D'_{4\alpha+2}$	A'_2	0	0
2	3	$D_{4\alpha+2}$	A_3	-1	-1
3	0	$D'_{4\alpha+3}$	A_0	1	-1
3	1	$D'_{4\alpha+3}$	A_1	0	-1
3	2	$D'_{4\alpha+3}$	A_2	1	1
3	3	$D''_{4\alpha+3}$	A''_3	0	-1

Lemma 5.2. The join $P_n^3 + Q_3$ of third power of a path P_n^3 and a cycle Q_3 is cordial for all $n > 7$.

Proof. We consider the cases of n separately.

Case (1): $n \equiv 0(mod4)$ or $n = 4\alpha$ and $\alpha > 1$. The following labelings suffice.
$P_{4\alpha}^3 + Q_3 : [E_{4\alpha}; 001]$ if α is even, and
$P_{4\alpha}^3 + Q_3 : [D4\alpha; 001]$ if α is odd.

Case (2): $n \equiv 0(mod4)$ or $n = 4\alpha + 1$ and $\alpha > 1$. The following labelings suffice.
$P_{4\alpha+1}^3 + Q_3 : [E''_{4\alpha+1}; 001]$ if α is even, and
$P_{4\alpha+1}^3 + Q_3 : [D''_{4\alpha+1}; 001]$ if α is odd.

Case (3): $n \equiv 0(mod4)$ or $n = 4\alpha + 2$ and $\alpha > 1$. The following labelings suffice.
$P_{4\alpha+2}^3 + Q_3 : [E_{4\alpha+2}; 001]$ if α is even, and
$P_{4\alpha+2}^3 + Q_3 : [D_{4\alpha+2}; 001]$ if α is odd.

Case (4): $n \equiv 0(mod4)$ or $n = 4\alpha+3$ and $\alpha > 1$. The following labelings suffice.
$P^3_{4\alpha+3} + Q_3 : [E''4\alpha+3; 001]$ if α is even, and
$P^3_{4\alpha+3} + Q_3 : [D''4\alpha+3; 001]$ if α is odd, the lemma follows.

Lemma 5.3. The join $P^3_7 + Q_m$ of third power of a path P^3_n and a cycle Q_m is cordial for all m.

Proof.

Case (1): Let $m = 3$, the following labeling suffice:
$P^3_7 + Q_3$:[0010111:001].

Case (2): Let $m > 3$ and $m = 4\beta + j$, where $j = 0,1,2,3$, then the following labelings suffice:
$P^3_7 + Q_{4\beta} : [0001101; L_{4\beta}]$,
$P^3_7 + Q_{4\beta+1} : [0001101; L_{4\beta}1]$,
$P^3_7 + Q_{4\beta+2} : [0001101; 0L_{4\beta}1]$ and
$P^3_7 + Q_{4\beta+3} : [0010111; L_{4\beta}001]$, the lemma follows.

Lemma 5.4. The join $P^3_6 + Q_m$ of third power of a path P^3_n and a cycle Q_m is cordial for all m.

Proof.

Case (1): Let $m = 3$, the following labeling suffice:
$P^3_6 + Q_3$:[000111:110]..

Case (2): Let $m > 3$ and $m = 4\beta + j$, where $j = 0,1,2,3$, then the following labelings suffice:
$P^3_6 + Q_{4\beta} : [000111; L_{4\beta}]$,
$P^3_6 + Q_{4\beta+1} : [000111; L_{4\beta}1]$,
$P^3_6 + Q_{4\beta+2} : [000111; 0L_{4\beta}1]$ and
$P^3_6 + Q_{4\beta+3} : [000111; L_{4\beta}001]$, the lemma follows.

Lemma 5.5. The join $P^3_5 + Q_m$ of third power of a path P^3_n and a cycle Q_m is cordial for all m iff $m \neq 3$.

Proof.

Case (1): Let $m = 3$, then it is simple to confirm that the graph $P^3_5 + Q_3$ does not have a cordial labeling.

Case (2): Let $m > 3$ and $m = 4\beta + j$, where $j = 0,1,2,3$, then the following labelings suffice:
$P^3_5 + Q_{4\beta} : [00011; L_{4\beta}]$,
$P^3_5 + Q_{4\beta+1} : [00011; L_{4\beta}1]$,
$P^3_5 + Q_{4\beta+2} : [00011; 0L_{4\beta}1]$ and
$P^3_5 + Q_{4\beta+3} : [00011; L_{4\beta}110]$, the lemma follows.

Lemma 5.6. The join $P^3_4 + Q_m$ of third power of a path P^3_n and a cycle Q_m is cordial for all m iff $m \neq 3,4$.

Proof.

Case (1): Let $3 \leq m \leq 4$, then from the fact that $P^3_4 + Q_3 \equiv K_4 + K_3 \equiv K_7$, which is not cordial (see Cahit [2]). Also, by investigation all possible labelings of K_4 and Q_4 it is easy to verify that $P^3_4 + Q_4 \equiv K_4 + C_4$ does not have a cordial labeling.

Case (2): Let $m > 4$ and $m = 4\beta + j$, where $j = 0,1,2,3$, then the following labelings suffice:
$P^3_4 + Q_{4\beta+1} : [0011; L_{4\beta}0]$,
$P^3_4 + Q_{4\beta+2} : [0011; 0L_{4\beta}1]$ and
$P^3_4 + Q_{4\beta+3} : [0011; L_{4\beta}110]$,
Now, for $j = 0$, we labeling the vertices of P^3_4 as 0001 (i.e., $x_0 = 3, x_1 = 1, a_0 = 3$ and $a_1 = 3$) and $Q_{4\beta}$ as $I_5O_3R_{4\beta-8} = I_5O_301010101\ldots(\beta-2)$–times $\ldots$ 01010101 for $\beta > 2$ (i.e., $y_0 = 2\beta - 1, y_1 = 2\beta + 1, b_0 = 2\beta + 2$ and $b_1 = 2\beta - 2$), therefore v_0–$v_1 = 0$ and e_0–$e_1 = 0$. For $\beta = 2$, we may use the labeling of vertices of Q_8 as $O_5I_3 = 00000111$ (i.e., $y_0 = 5, y_1 = 3, b_0 = 6$ and $b_1 = 2$), hence we obtain that v_0–$v_1 = 0$ and e_0–$e_1 = 0$, the lemma follows.

Lemma 5.7. The join $P^3_3 + Q_m$ of third power of a path P^3_3 and a cycle Q_m is cordial for all m iff $m \neq 3$.

Proof.

Case (1): Let $m = 3$, then the graph $P^3_3 + Q_3 \equiv Q_3 + Q_3 \equiv K_6$ is not cordial [2].

Case (2): Let $m \neq 3$, The evidence is clear from the facts that $P^3_3 \equiv Q_3$ and the join of two cycles Q_n and Q_m is cordial except for $n \equiv 0(mod4)$ and $m \equiv 2(mod4)$ (or vice versa) [12], the lemma follows.

Example 5.1. If $n \leq 3$, then $P^3_n + Q_3$ are not cordial.

Solution. The solution follows directly from the fact that $P^3_1 + Q_3 \equiv P_1 + Q_3 \equiv K_1 + K_3 \equiv K_4$, $P^3_2 + Q_3 \equiv P_2 + Q_3 \equiv K_2 + K_3 \equiv K_5$ and $P^3_3 + Q_3 \equiv Q_3 + Q_3 \equiv K_6$, which are not cordial (see Cahit [2]).

From the above Lemmas and Example 5.1, we can introduce the following theorem.

Theorem 5.1. The join $P^3_n + Q_m$ of third power of a path P^3_n and a cycle Q_m is cordial for all n and m iff $(n,m) \neq (1,3),(2,3),(3,3),(4,3),(4,4),(5,3)$.

Proof. The proof follows of the above Lemmas and Examples.

6 Unions of Third Power of Paths and Cycles

In this section, the union $P^3_n \cup Q_m$ of third power of a path P^3_n and a cycle Q_m is cordial for all n and m iff $(n,m) \neq (5,3)$ or $3+m \neq 2(mod4)$ or $P^3_n \cup Q_m$ is not isomorphic to $P_1 \cup Q_m$ with $m \neq 2(mod4)$.

Lemma 6.1. The union $P^3_n \cup Q_m$ of third power of a path P^3_n and a cycle Q_m is cordial for all $n > 7$ and $m > 3$.

Proof. For given values of i and j, where $n = 4\alpha + i$ with $0 \leq i \leq 3$ and $\alpha > 1$, and $m = 4\beta + j$ with $0 \leq j \leq 3$ and $\beta > 1$, we have two cases:

Case(1): α is an even number. We use the labeling E_i or E_i'' or E_i''' for the third power of a path P_n^3 and A_j or A_j'' for the cycle Q_m as given in Table 1 and Table 13. Using Tables 1 and 13 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we may calculate the values displayed in Table 16's final two columns. These are all $0, 1$, or -1, therefore the lemma is evident.

Case (2): α is an odd number. We use the labeling D_i or D_i' or D_i'' or $D_i'''$for the third power of path P_n^3 and A_j or A_j' for the cycle Q_m as given in Tables 2 and 13. Using Tables 2 and 13 and fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we may calculate the values displayed in Table 17's final two columns. These are all $0, 1$, or -1, therefore the lemma is evident.

Table 16: Combination of Labeling

$n=4\alpha+i$, where α even, $i=0,1,2,3$	$m=4\beta+j$, $j=0,1,2,3$	P_n^3	Q_m	v_0-v_1	e_0-e_1
0	0	$E_{4\alpha}$	A_0	0	0
0	1	$E_{4\alpha}$	A_1	-1	1
0	2	$E'_{4\alpha}$	A_2	0	0
0	3	$E_{4\alpha}$	A_3	-1	-1
1	0	$E_{4\alpha+1}$	A_0	1	-1
1	1	$E_{4\alpha+1}$	A_1	0	0
1	2	$E_{4\alpha+1}$	A_2	1	1
1	3	$E''_{4\alpha+1}$	A''_3	0	0
2	0	$E_{4\alpha+2}$	A_0	0	0
2	1	$E_{4\alpha+2}$	A_1	-1	1
2	2	$E'_{4\alpha+2}$	A'_2	0	0
2	3	$E_{4\alpha+2}$	A_3	-1	-1
3	0	$E_{4\alpha+3}$	A_0	1	-1
3	1	$E_{4\alpha+3}$	A_1	0	0
3	2	$E_{4\alpha+3}$	A_2	1	1
3	3	$E'''_{4\alpha+3}$	A_3	0	0

Table 17: Combination of Labeling

$n=4\alpha+i$, where α odd, $i=0,1,2,3$	$m=4\beta+j$, $j=0,1,2,3$	P_n^3	Q_m	v_0-v_1	e_0-e_1
0	0	$D_{4\alpha}$	A_0	0	0
0	1	$D_{4\alpha}$	A_1	-1	1
0	2	$D'_{4\alpha}$	A'_2	0	0
0	3	$D_{4\alpha}$	A_3	-1	-1
1	0	$D_{4\alpha+1}$	A_0	1	-1
1	1	$D_{4\alpha+1}$	A_1	0	0
1	2	$D_{4\alpha+1}$	A_2	1	1
1	3	$D''_{4\alpha+1}$	A_3	0	0
2	0	$D_{4\alpha+2}$	A_0	0	0
2	1	$D_{4\alpha+2}$	A_1	-1	1
2	2	$D'_{4\alpha+2}$	A'_2	0	0
2	3	$D_{4\alpha+2}$	A_3	-1	-1
3	0	$D'_{4\alpha+3}$	A_0	1	-1
3	1	$D'_{4\alpha+3}$	A_1	0	0
3	2	$D'_{4\alpha+3}$	A_2	1	1
3	3	$D''_{4\alpha+3}$	A''_3	0	0

Lemma 6.2. The union $P_n^3 \cup Q_3$ of third power of a path P_n^3 and a cycle Q_3 is cordial for all $n > 7$.

Proof. We consider the cases of n separately.

Case (1): $n \equiv 0(mod4)$ or $n = 4\alpha$ and $\alpha > 1$. The following labelings suffice.
$P_{4\alpha}^3 \cup Q_3 : [E_{4\alpha};001]$ if α is even, and
$P_{4\alpha}^3 \cup Q_3 : [D4\alpha;001]$ if α is odd.

Case (2): $n \equiv 0(mod4)$ or $n = 4\alpha + 1$ and $\alpha > 1$. The following labelings suffice.
$P_{4\alpha+1}^3 \cup Q_3 : [E''_{4\alpha+1};001]$ if α is even, and
$P_{4\alpha+1}^3 \cup Q_3 : [D''_{4\alpha+1};001]$ if α is odd.

Case (3): $n \equiv 0(mod4)$ or $n = 4\alpha + 2$ and $\alpha > 1$. The following labelings suffice.
$P_{4\alpha+2}^3 \cup Q_3 : [E_{4\alpha+2};001]$ if α is even, and
$P_{4\alpha+2}^3 \cup Q_3 : [D_{4\alpha+2};001]$ if α is odd.

Case (4): $n \equiv 0(mod4)$ or $n = 4\alpha + 3$ and $\alpha > 1$. The following labelings suffice.
$P_{4\alpha+3}^3 \cup Q_3 : [E''4\alpha+3;001]$ if α is even, and
$P_{4\alpha+3}^3 \cup Q_3 : [D''4\alpha+3;001]$ if α is odd, the lemma follows.

Lemma 6.3. The union $P_7^3 \cup Q_m$ of third power of a path P_n^3 and a cycle Q_m is cordial for all m.

Proof.

Case (1): Let $m = 3$, the following labeling suffice:
$P_7^3 \cup Q_3$:[0010111:001].

Case (2): Let $m > 3$ and $m = 4\beta + j$, where $j = 0, 1, 2, 3$, then the following labelings suffice:
$P_7^3 \cup Q_{4\beta} : [0001101; L_{4\beta}]$,
$P_7^3 \cup Q_{4\beta+1} : [0001101; L_{4\beta}1]$,
$P_7^3 \cup Q_{4\beta+2} : [0001101; 0L_{4\beta}1]$ and
$P_7^3 \cup Q_{4\beta+3} : [0010111; L_{4\beta}001]$, the lemma follows.

Lemma 6.4. The union $P_6^3 \cup Q_m$ of third power of a path P_n^3 and a cycle Q_m is cordial for all m.

Proof.

Case (1): Let $m = 3$, the following labeling suffice:
$P_6^3 \cup Q_3$:[000111:110]..

Case (2): Let $m > 3$ and $m = 4\beta + j$, where $j = 0, 1, 2, 3$, then the following labelings suffice:
$P_6^3 \cup Q_{4\beta} : [000111; L_{4\beta}]$,
$P_6^3 \cup Q_{4\beta+1} : [000111; L_{4\beta}1]$,
$P_6^3 \cup Q_{4\beta+2} : [000111; 0L_{4\beta}1]$ and
$P_6^3 \cup Q_{4\beta+3} : [000111; L_{4\beta}001]$, the lemma follows.

Lemma 6.5. The union $P_5^3 \cup Q_m$ of third power of a path P_n^3 and a cycle Q_m is cordial for all m iff $m \neq 3$.

Proof.

Case (1): Let $m = 3$, then it is simple to confirm that the graph $P_5^3 \cup Q_3$ does not have a cordial labeling.

Case (2): Let $m > 3$ and $m = 4\beta + j$, where $j = 0, 1, 2, 3$, then the following labelings suffice:
$P_5^3 \cup Q_{4\beta} : [00011; L_{4\beta}]$,
$P_5^3 \cup Q_{4\beta+1} : [00011; L_{4\beta}1]$,
$P_5^3 \cup Q_{4\beta+2} : [00011; 0L_{4\beta}1]$ and
$P_5^3 \cup Q_{4\beta+3} : [00010; I_3M_4R_{4\beta-4}]$ for $\beta > 1$ and for $\beta = 1$, $P_5^3 \cup Q_7 : [00010; 1110101]$, the lemma follows.

Lemma 6.6. The union $P_4^3 \cup Q_m$ of third power of a path P_n^3 and a cycle Q_m is cordial for all m.

Proof.

Case (1): For $m = 3$, we may use the following labeling:
$P_4^3 \cup Q_3 \equiv K_4 \cup K_3 : [1110; 001]$.

Case (2): Let $m > 3$ and $m = 4\beta + j$, where $j = 0, 1, 2, 3$, then the following labelings suffice:
$P_4^3 \cup Q_{4\beta+1} : [0011; L_{4\beta}0]$,
$P_4^3 \cup Q_{4\beta+2} : [0011; 0L_{4\beta}1]$ and
$P_4^3 \cup Q_{4\beta+3} : [0011; L_{4\beta}110]$,
Now, for $j = 0$, $P_4^3 \cup Q_{4\beta} : [1110; L'_{4\beta}]$ (i.e., $P_4^3 \equiv K_4 \equiv 1110$, we have $x_0 = 1, x_1 = 3, a_0 = 3$ and $a_1 = 3$) and $Q_{4\beta} \equiv L'_{4\beta}$, we have $y_0 = 2\beta + 11, y_1 = 2\beta - 1, b_0 = 2\beta$ and $b_1 = 2\beta$), therefore $v_0 – v_1 = 0$ and $e_0 – e_1 = 0$, the lemma follows.

Lemma 6.7. The union $P_3^3 \cup Q_m$ of third power of a path P_3^3 and a cycle Q_m is cordial for all m iff $3 + m \neq 2(mod4)$.

Proof. The evidence is clear from the facts that $P_3^3 \equiv Q_3$ and the union of two cycles Q_n and Q_m is cordial iff $n + m \neq 2(mod4)$ [8], the lemma follows.

Lemma 6.8. The union $P_2^3 \cup Q_m$ of third power of a path P_2^3 and a cycle Q_m is cordial for all m iff $3 + m \neq 2(mod4)$.

Proof.

Case (1): Let $m = 3$, then the graph $P_2^3 \cup Q_3 \equiv P_2 + Q_3 : [11; 001]$.

Case (2): Let $m > 3$ and $m = 4\beta + j$, where $j = 0, 1, 2, 3$, then the following labelings suffice:
$P_2^3 \cup Q_{4\beta} \equiv P_2 \cup Q_{4\beta} : [01; L_{4\beta}]$,
$P_2^3 \cup Q_{4\beta+1} \equiv P_2 \cup Q_{4\beta+1} : [01; L_{4\beta}0]$,
$P_2^3 \cup Q_{4\beta+2} \equiv P_2 \cup Q_{4\beta+2} : [01; 0L_{4\beta}1]$ and
$P_2^3 \cup Q_{4\beta+3} \equiv P_2 \cup Q_{4\beta+3} : [01; L_{4\beta}011]$, the lemma follows.

Lemma 6.9. The union $P_1^3 \cup Q_m$ of third power of a path P_1^3 and a cycle Q_m is cordial iff $P_1^3 \cup Q_m$ is not isomorphic to $P_1 \cup Q_m$ with $m \neq 2(mod4)$.

Proof. The evidence is clear from the facts that $P_1^3 \equiv P_1$ and the union $P_n \cup Q_m$ of a path P_n and a cycle Q_m is cordial iff it is not isomorphic to $P_1 \cup Q_m$ with $m \equiv 2(mod4)$ [8], the lemma follows.

Theorem 6.1. The union $P_n^3 \cup Q_m$ of third power of a path P_n^3 and a cycle Q_m is cordial for all n and m iff $(n, m) \neq (5, 3)$ or $3 + m \neq 2(mod4)$ or $P_n^3 \cup Q_m$ is not isomorphic to $P_1 \cup Q_m$ with $m \neq 2(mod4)$.

Proof. The proof follows directly from the above Lemmas, the theorem follows.

7 Conclusion

A graph is termed cordial if its $0 - 1$ labeling satisfies specific criteria. The third power of a path P_n is created by adding edges to connect each pair of vertices u and v where the distance $d(u, v) = 3$. This study explores the cordiality of the join and union of graphs involving one path and one cycle, as well as the third power of paths.

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