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Impact of Polluted Environments on Stochastic Gilpin–Ayala Population Dynamics with Dispersal

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Abstract: This research comprehensively examines the impact of incorporating pollution into a stochastic Gilpin-Ayala model with patches. The critical contribution of this study lies in expressing the conditions under which species extinction or persistence occurs based on pollution parameters. Consequently, including pollution effects in the analysis of ecological systems enables a more accurate assessment of contaminated environments. Our results emphasize the importance of considering pollution as a crucial factor in ecological systems, providing valuable insights into the complexities of some polluted environments. Finally, we present a few computational simulations to validate the results developed over the length of this article.

Keywords: Environmental pollution, Extinction, Persistence, Stationary distribution

1 Introduction

The detrimental effects of pollution on the environment are well-documented and encompass various forms, such as air, water, and soil pollution. Industrial activities release hazardous substances and emissions into the air, leading to the degradation of air quality. Similarly, agricultural practices involving pesticides, fertilizers, and other chemicals can contaminate water bodies and soil, posing severe threats to aquatic life and terrestrial ecosystems. This pressing issue has compelled scientists to undertake an in-depth analysis of population viability in contaminated environments to understand species' ability to persist or face extinction. In this context, scientific researchers are actively studying the impacts of pollution on different species and their capacity to adapt and survive in contaminated conditions. By conducting comprehensive investigations and experiments, scientists aim to unravel the intricate relationships between various environmental stressors and their consequences for populations. Assessing population survival in contaminated environments involves examining factors such as reproductive success, genetic diversity, physiological responses, and behavioral adaptations. Researchers evaluate the reproductive capabilities of species under polluted conditions to determine their ability to maintain viable population sizes. Furthermore, assessing genetic diversity provides insights into the adaptive potential of populations in polluted environments, as reduced genetic variation can limit their ability to respond to changing conditions. To comprehensively understand the impact of pollution, scientists meticulously investigate the physiological responses exhibited by various species. The main objective is to determine the tolerance thresholds of these species and uncover the mechanisms they employ to alleviate the adverse effects of contaminants effectively. Additionally, behavioral adaptations, such as altered feeding habits, migration patterns, or nesting behaviors, are studied to determine whether species can adjust their behaviors to cope with polluted environments. The ultimate goal of these scientific endeavors is to comprehensively understand the ecological consequences of pollution and its potential implications for species persistence or extinction. Such knowledge is crucial for developing effective conservation strategies, implementing pollution control measures, and advocating for sustainable development practices. The researchers in [1,2,3] have suggested deterministic population

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models with toxin effects. In reality, stochastic models hold significant advantages due to the pervasive presence of randomness and uncertainty in real-life situations. These models provide more insightful results compared to deterministic models. Consequently, numerous scholars have dedicated their efforts to investigating the impact of randomness on models (see, e.g., [4,5,6,7,8,9,10,11,12,13,14,15,16,17] and the references cited therein). As the growth of species is inevitably influenced by environmental noise, many authors have studied stochastic population models in polluted environments (see, e.g., [18,19,20,21]). For example, Z. Geng and M. Liu [21] considered a stochastic single-species Gilpin-Ayala model with a toxin effect.

$$\begin{cases} dx = \left(x(t)(r_0 - l_0 c_0(t) - k_0 x^{\theta_0}(t))\right) dt + \alpha_0 x(t) dB_1(t) \\ \quad + \beta_0 x^{1+\theta_0}(t) dB_2(t) + \gamma_0 x(t) c_0(t) dB_3(t), \\ dc_0 = (kc_e(t) - (g+m)c_0(t))\, dt, \\ dc_e = (-hc_e(t) + u(t))\, dt, \end{cases} \tag{1.1}$$

where $x(t)$ represents the population size at time t, $r_0 > 0$ and k_0 are the population's growth rate and self-competition coefficient without toxicants, θ_0 is a positive constant, $l_0 > 0$ is the response of the population to the contaminant in the organism, $k > 0$ is the net rate of uptake of toxic substances the organism from the natural world, $g > 0$ and $m > 0$ indicate the rate of toxicant egestion and detoxification of the organism, respectively. The parameter $h > 0$ signifies the rate of toxicant volatilization in the environment, $c_0(t)$ and $c_e(t)$ indicate toxicant concentrations in the organism and in the environment, respectively. The continuous positive bounded function $u(t)$ defined on $[0, +\infty)$ represents the exogenous rate of pollutant entry from the environment, α_0, β_0 and γ_0 represent the white noise intensity for r_0, k_0 and l_0, respectively, B_1, B_2 and B_3 are independent standard Brownian motions. On the other hand, dispersal frequently happens among patches in ecological ecosystems [4,22]. Therefore, we will consider the impact of dispersal phenomena in this research. To this end, we study a stochastic diffusion system containing two patches with a toxic effect.

$$\begin{cases} dx_1 = \left[x_1(r_1 - l_1 c_0(t) - k_1 x_1^{\theta_1}) + \varepsilon_{12}(x_2 - x_1)\right] dt \\ \qquad + \sum_{i=1}^{n} \left(\alpha_{1i} x_1 + \beta_{1i} x_1^{1+\theta_1} + \gamma_{1i} x_1 c_0(t)\right) dB_i, \\ dx_2 = \left[x_2(r_2 - l_2 c_0(t) - k_2 x_2^{\theta_2}) + \varepsilon_{21}(x_1 - x_2)\right] dt \\ \qquad + \sum_{i=1}^{n} \left(\alpha_{2i} x_2 + \beta_{2i} x_2^{1+\theta_2} + \gamma_{2i} x_1 c_0(t)\right) dB_i, \\ dc_0 = [kc_e(t) - (g+m)c_0(t)]\, dt, \\ dc_e = [-hc_e(t) + u(t)]\, dt, \end{cases} \tag{1.2}$$

where x_i is the population density of a species in the ith patch, r_i and k_i are the population growth rate and self-competition coefficient in the ith patch, respectively, θ_i is a positive constant in the ith patch, l_i is the response of the population to the contaminant in the organism in the ith patch, $\varepsilon_{i,j} > 0$ is a positive dispersal rate for the species from the jth patch to the ith patch $(i \neq j)$. This coefficient represents the net migration rate from the jth patch to the ith patch, which is proportional to the difference in population densities $(x_i - x_j)$ in each patch (see, e.g., [23,24] and the references cited therein). The vectors $\alpha_i = (\alpha_{i1}, \alpha_{i2}, ..., \alpha_{in})$, $\beta_i = (\beta_{i1}, \beta_{i2}, ..., \beta_{in})$ and $\gamma_i = (\gamma_{i1}, \gamma_{i2}, ..., \gamma_{in})$ stand for the white noise intensity on r_i, k_i and l_i. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the standard conditions and let $B(t) = (B_1(t), B_2(t), ..., B_n(t))^T$ be a Brownian motion in n-dimensions used for modeling the inter-correlation between the noises on r_i, k_i and l_i.

2 Persistence

Lemma 1([18]). *If $0 < k \leq g + m$ and $\lim_{t\to\infty} \sup u(t) \leq h$, then $0 \leq c_0(t) < 1$, $0 \leq c_e(t) < 1$ for all $t \geq 0$.*

To begin with, we assume that $0 < k \leq g + m$ and $\lim_{t\to\infty} \sup u(t) \leq h$. Since the last two equations of the model (1.2) are linear with respect to $c_0(t)$ and $c_e(t)$, we will only study the first two equations of model (1.2).

$$\begin{cases} dx_1 = \left[x_1(r_1 - l_1 c_0(t) - k_1 x_1^{\theta_1}) + \varepsilon_{12}(x_2 - x_1)\right] dt \\ \qquad + \sum_{i=1}^{n} \left(\alpha_{1i} x_1 + \beta_{1i} x_1^{1+\theta_1} + \gamma_{1i} x_1 c_0(t)\right) dB_i, \\ dx_2 = \left[x_2(r_2 - l_2 c_0(t) - k_2 x_2^{\theta_2}) + \varepsilon_{21}(x_1 - x_2)\right] dt \\ \qquad + \sum_{i=1}^{n} \left(\alpha_{2i} x_2 + \beta_{2i} x_2^{1+\theta_2} + \gamma_{2i} x_1 c_0(t)\right) dB_i. \end{cases} \tag{2.1}$$

By the same procedure as in the proof of [24], we obtain the existence and the positivity of x_1 and x_2.

Theorem 1. *For any $(x_1(0), x_2(0)) \in \mathbb{R}_+^2$, there exists a unique solution (x_1, x_2) to the model (2.1) in $\mathbb{R}_+^2$.*

Lemma 2([25]). *Let t, a, b and c be non-negative constants, then for any M_t, $t \geq 0$ local martingale vanishing at time 0, we have*

$$\mathbb{P}\left[\sup_{0\leqslant t\leqslant a}\left(M_t - \frac{b}{2}[M_t, M_t]\right) > c\right] \leqslant \exp(-bc),$$

where $[M_t, M_t]$ is the quadratic variation of M_t.

Let us denote

$$\xi_1 = \frac{r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1+\gamma_1\|^2}{k_1 + <\alpha_1+\gamma_1, \beta_1> + \frac{1}{2}\|\beta_1\|^2},$$

and

$$\xi_2 = \frac{r_2 - l_2 - \varepsilon_{21} - \frac{1}{2}\|\alpha_2+\gamma_2\|^2}{k_2 + <\alpha_2+\gamma_2, \beta_2> + \frac{1}{2}\|\beta_2\|^2}.$$

Theorem 2. *For any* $(x_1(0), x_2(0)) \in \mathbb{R}^2_+$,

(i) If $\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1+\gamma_1+\beta_1\|^2 - k_1\right) > 0$, *then*

$$\limsup_{t\to\infty} x_1(t) \geq \xi_1^{\frac{1}{2\theta_1}}. \tag{2.2}$$

(ii) If $\left(r_2 - l_2 - \varepsilon_{21} - \frac{1}{2}\|\alpha_1+\gamma_2+\beta_2\|^2 - k_2\right) > 0$, *then*

$$\limsup_{t\to\infty} x_2(t) \geq \xi_2^{\frac{1}{2\theta_2}}.$$

Proof. It is sufficient to demonstrate (i).
For $p \in \mathbb{N}^*$, let

$$A_p = \left\{ \limsup_{t\to\infty} x_1(t) < \xi_1^{\frac{p}{2p\theta_1+2}} \right\},$$

and

$$A = \left\{ \limsup_{t\to\infty} x_1(t) < \xi_1^{\frac{1}{2\theta_1}} \right\}.$$

Since

$$\xi_1 > 1, \tag{2.3}$$

thus

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{p\in\mathbb{N}^*} A_p\right) = \lim_{p\to\infty} \mathbb{P}(A_p). \tag{2.4}$$

Suppose that (2.2) is not true.
So, from (2.4), we get $\mathbb{P}(A) > 0$ and then there exists $p_0 \in \mathbb{N}^*$ such that, for $p \geq p_0$, we have $\mathbb{P}(A_p) > 0$.
Hence, for $p \geq p_0$ and for every $\zeta \in A_p$, there is a $T(\zeta) > 0$ such that

$$x_1(t) < \xi_1^{\frac{p}{2p\theta_1+1}} \quad \text{for} \quad t \geq T(\zeta). \tag{2.5}$$

Now, by Itô formula, we get

$$\begin{aligned} \log(x_1(t)) = \log(x_1(0)) + M_t + \int_0^t \Bigg[(r_1 - l_1 c_0(s) \\ - k_1 x_1^{\theta_1}(s)\Big) + \frac{\varepsilon_{12}(x_2(s) - x_1(s))}{x_1(s)} \\ - \frac{1}{2}\sum_{i=1}^n \Big(\alpha_{1i} + \beta_{1i} x_1^{\theta_1}(s) + \gamma_{1i} c_0(s)\Big)^2 \Bigg] ds, \end{aligned} \tag{2.6}$$

where M_t is a local martingale vanishing at $t = 0$, defined by

$$M_t = \int_0^t \sum_{i=1}^n \Big(\alpha_{1i} + \beta_{1i} x_1^{\theta_1}(s) + \gamma_{1i} c_0(s)\Big) dB_i(s).$$

Applying Lemma 2, we have for any ε sufficiently small and any integer $q \geq 1$

$$\mathbb{P}\left[\sup_{0\leq t\leq q} \Big(-M_t - \frac{\varepsilon}{2}[M_t, M_t]\Big) > \frac{2}{\varepsilon}\log q \right] \leq \frac{1}{q^2},$$

where

$$\begin{aligned} [M_t, M_t] = \int_0^t \sum_{i=1}^n \Big(\alpha_{1i} + \beta_{1i} x_1^{\theta_1}(s) \\ + \gamma_{1i} c_0(s))^2 ds \end{aligned} \tag{2.7}$$

Since $\sum_{q=1}^{\infty} \frac{1}{q^2}$ converges, the Borel-Cantelli lemma implies that there is a $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that for all $\zeta \in \Omega_1$, there exists an integer $q_1(\zeta)$ verifying

$$\begin{aligned} M_t \geq -\frac{2}{\varepsilon}\log q - \frac{\varepsilon}{2}[M_t, M_t], \\ \text{for} \quad q \geq q_1(\zeta), \quad 0 \leq t \leq q. \end{aligned} \tag{2.8}$$

Thus, it follows from (2.6), (2.7), and (2.8) that for $\zeta \in \Omega_1$, $q \geq q_1(\zeta)$ and $0 \leq t \leq q$

$$\begin{aligned} \log(x_1(t)) \geq \log(x_1(0)) - \frac{2}{\varepsilon}\log(q) \\ + \int_0^t \Bigg[r_1 - l_1 - \varepsilon_{12} - \frac{1+\varepsilon}{2}\|\alpha_1+\gamma_1\|^2 \\ - \big(k_1 + (1+\varepsilon)\langle \alpha_1+\gamma_1, \beta_1\rangle\big) x_1^{\theta_1}(s) \\ - \frac{1+\varepsilon}{2}\|\beta_1\|^2 x_1^{2\theta_1}(s) \Bigg] ds. \end{aligned}$$

Using (2.3) and (2.5), we obtain for $p \geq p_0$, $\zeta \in A_p \cap \Omega_1$, $q \geq q_1(\zeta) \wedge T(\zeta)$ and $T(\zeta) \leq t \leq q$

$$\begin{aligned}\log(x_1(t)) \geq{}& \log(x_1(0)) - \frac{2}{\varepsilon}\log(q)\\&+\int_0^t \left(r_1 - l_1 - \varepsilon_{12} - \frac{1+\varepsilon}{2}\|\alpha_1+\gamma_1\|^2\right) ds\\&-\int_0^T \left[k_1 + (1+\varepsilon)\langle\alpha_1+\gamma_1,\beta_1\rangle\right.\\&\left.+\frac{1+\varepsilon}{2}\|\beta_1\|^2 x_1^{\theta_1}(s)\right] x_1^{\theta_1}(s)ds\\&-\int_T^t \left[k_1 + (1+\varepsilon)\langle\alpha_1+\gamma_1,\beta_1\rangle\right.\\&\left.+\frac{1+\varepsilon}{2}\|\beta_1\|^2\right] \xi_1^{\frac{2p\theta_1}{2p\theta_1+1}} ds. \qquad (2.9)\end{aligned}$$

From (2.9), one can easily verify for $p \geq p_0$, $\zeta \in A_p \cap \Omega_1$, and t large enough such that $[t] \geq q_1(\zeta)$, where $[t]$ is the biggest integer smaller than t, that we have

$$\begin{aligned}\frac{1}{t}\log(x_1(t)) \geq{}& \frac{1}{t}\log(x_1(0)) - \frac{2}{\varepsilon[t]}\log([t]+1)\\&+\left(r_1 - l_1 - \varepsilon_{12} - \frac{1+\varepsilon}{2}\|\alpha_1+\gamma_1\|^2\right)\\&-\frac{1}{t}\int_0^T \left[k_1 + (1+\varepsilon)\langle\alpha_1+\gamma_1,\beta_1\rangle\right.\\&\left.+\frac{1+\varepsilon}{2}\|\beta_1\|^2 x_1^{\theta_1}(s)\right] x_1^{\theta_1}(s)ds\\&-\frac{t-T}{t}\left[k_1 + (1+\varepsilon)\langle\alpha_1+\gamma_1,\beta_1\rangle\right.\\&\left.+\frac{1+\varepsilon}{2}\|\beta_1\|^2\right] \xi_1^{\frac{2p\theta_1}{2p\theta_1+1}}. \qquad (2.10)\end{aligned}$$

Letting $t \to \infty$ and $\epsilon \to 0$, and using the following inequality

$$y^\rho < 1 + \rho(y-1), \quad y \geq 0,\ 0 \leq \rho \leq 1,$$

with

$$y = \xi_1, \quad \rho = \frac{2p\theta_1}{2p\theta_1+1}.$$

For $p \geq p_0$, $\zeta \in A_p \cap \Omega_1$ and $[t] \geq q_1(\zeta)$, we get

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log(x_1(t)),\\&\geq \left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1+\gamma_1\|^2\right)\\&\times\left(k_1 + <\alpha_1+\gamma_1,\beta_1> + \frac{1}{2}\|\beta_1\|^2\right)\\&\times\left[1 + \frac{2p\theta_1}{2p\theta_1+1}(\xi_1 - 1)\right],\\&\geq \left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1+\gamma_1\|^2\right)\\&-\left(k_1 + <\alpha_1+\gamma_1,\beta_1> + \frac{1}{2}\|\beta_1\|^2\right)\\&-\frac{2p\theta_1}{2p\theta_1+1}\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1+\gamma_1\|^2\right)\\&+\frac{2p\theta_1}{2p\theta_1+1}\left(k_1 + <\alpha_1+\gamma_1,\beta_1> + \frac{1}{2}\|\beta_1\|^2\right),\\&\geq \left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1+\gamma_1+\beta_1\|^2 - k_1\right)\\&\times\left(1 - \frac{2p\theta_1}{2p\theta_1+1}\right),\\&> 0.\end{aligned}$$

Hence, $\lim_{t\to\infty} x_1(t) = \infty$. But this contradicts (2.5).
(ii) It is identical to (i).

3 Extinction

Theorem 3. *For every $(x_1(0), x_2(0)) \in \mathbb{R}_+^2$, the solution of system (2.1) obey*

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\frac{x_1(t)}{\varepsilon_{12}} + \frac{x_2(t)}{\varepsilon_{21}}\right) \leqslant M - \frac{1}{2}m^2 \quad a.s., \qquad (3.1)$$

where

$$M = \max\left\{r_1 - l_1 \inf_{t\geq 0} c_0(t), r_2 - l_2 \inf_{t\geq 0} c_0(t)\right\},$$

$$m = \left[\min\left(\alpha_{1i} + \gamma_{1i}\inf_{t\geq 0} c_0(t), \alpha_{2i} + \gamma_{2i}\inf_{t\geq 0} c_0(t)\right)\right]_{1\leq i\leq n}.$$

Moreover, if $M - \frac{1}{2}m^2 < 0$, then the extinction of the species in (2.1).

Proof. Using Itô's formula, we obtain

$$d\log\left(\frac{x_1(t)}{\varepsilon_{12}}+\frac{x_2(t)}{\varepsilon_{21}}\right),$$
$$=\frac{1}{\frac{x_1(t)}{\varepsilon_{12}}+\frac{x_2(t)}{\varepsilon_{21}}}\left(\frac{x_1(t)}{\varepsilon_{12}}\left(r_1-l_1c_0(t)-k_1x_1^{\theta_1}(t)\right)\right.$$
$$\left.+\frac{x_2(t)}{\varepsilon_{21}}\left(r_2-l_2c_0(t)-k_2x_2^{\theta_2}(t)\right)\right)dt$$
$$-\frac{1}{2\left(\frac{x_1(t)}{\varepsilon_{12}}+\frac{x_2(t)}{\varepsilon_{21}}\right)^2}\sum_{i=1}^{n}\left(\frac{x_1(t)}{\varepsilon_{12}}(\alpha_{1i}\right.$$
$$\left.+\beta_{1i}x_1^{\theta_1}(t)+\gamma_{1i}c_0(t)\right)+\frac{x_2(t)}{\varepsilon_{21}}\left(\alpha_{2i}+\beta_{2i}x_2^{\theta_2}(t)\right.$$
$$+\gamma_{2i}c_0(t)))^2\,dt+\frac{1}{\frac{x_1(t)}{\varepsilon_{12}}+\frac{x_2(t)}{\varepsilon_{21}}}\sum_{i=1}^{n}\left(\frac{x_1(t)}{\varepsilon_{12}}(\alpha_{1i}\right.$$
$$\left.+\beta_{1i}x_1^{\theta_1}(t)+\gamma_{1i}c_0(t)\right)+\frac{x_2(t)}{\varepsilon_{21}}(\alpha_{2i}$$
$$\left.\left.+\beta_{2i}x_2^{\theta_2}(t)+\gamma_{2i}c_0(t)\right)\right)dB_i. \tag{3.2}$$

Integrating we get

$$\log\left(\frac{x_1(t)}{\varepsilon_{12}}+\frac{x_2(t)}{\varepsilon_{21}}\right)-\log\left(\frac{x_1(0)}{\varepsilon_{12}}+\frac{x_2(0)}{\varepsilon_{21}}\right),$$
$$=\int_0^t\frac{1}{\frac{x_1(s)}{\varepsilon_{12}}+\frac{x_2(s)}{\varepsilon_{21}}}\left(\frac{x_1(s)}{\varepsilon_{12}}(r_1-l_1c_0(s)\right.$$
$$\left.\left.-k_1x_1^{\theta_1}(s)\right)+\frac{x_2(s)}{\varepsilon_{21}}\left(r_2-l_2c_0(s)-k_2x_2^{\theta_2}(s)\right)\right)ds$$
$$-\int_0^t\frac{1}{2\left(\frac{x_1(s)}{\varepsilon_{12}}+\frac{x_2(s)}{\varepsilon_{21}}\right)^2}\sum_{i=1}^{n}\left(\frac{x_1(s)}{\varepsilon_{12}}(\alpha_{1i}\right.$$
$$\left.+\beta_{1i}x_1^{\theta_1}(s)+\gamma_{1i}c_0(s)\right)+\frac{x_2(s)}{\varepsilon_{21}}\left(\alpha_{2i}+\beta_{2i}x_2^{\theta_2}(s)\right.$$
$$+\gamma_{2i}c_0(s)))^2\,ds+\ M_t\ , \tag{3.3}$$

with the local martingale

$$M_t=\int_0^t\frac{1}{\frac{x_1(s)}{\varepsilon_{12}}+\frac{x_2(s)}{\varepsilon_{21}}}\sum_{i=1}^{n}\left(\frac{x_1(s)}{\varepsilon_{12}}(\alpha_{1i}\right.$$
$$+\beta_{1i}x_1^{\theta_1}(s)+\gamma_{1i}c_0(s))+\frac{x_2(s)}{\varepsilon_{21}}(\alpha_{2i}$$
$$\left.+\beta_{2i}x_2^{\theta_2}(s)+\gamma_{2i}c_0(s))\right)dB_i(s),$$

Now, for ϵ sufficiently small, according to Lemma 2 we have for each $k\geqslant 1$

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant k}\left[M_t-\frac{\epsilon}{2}[M_t,M_t]\right]>\frac{2\log(k)}{\epsilon}\right)\leqslant\frac{1}{k^2},$$

where

$$[M_t,M_t]=\int_0^t\frac{1}{(\frac{x_1(s)}{\varepsilon_{12}}+\frac{x_2(s)}{\varepsilon_{21}})^2}\sum_{i=1}^{n}\left(\frac{x_1(s)}{\varepsilon_{12}}(\alpha_{1i}\right.$$
$$\left.+\beta_{1i}x_1^{\theta_1}(s)+\gamma_{1i}c_0(s)\right)$$
$$\left.+\frac{x_2(s)}{\varepsilon_{21}}(\alpha_{2i}+\beta_{2i}x_2^{\theta_2}(s)+\gamma_{2i}c_0(s))\right)^2ds.$$

Using the Borel-Cantelli lemma, there is a $\Omega_1\subset\Omega$ with $\mathbb{P}(\Omega_1)=1$ such that for all $\zeta\in\Omega_1$ an integer $k_1(\zeta)$ such that

$$M_t\leqslant\frac{2\log(k)}{\epsilon}+\frac{\epsilon}{2}[M_t,M_t],\quad \text{for } 0\leqslant t\leqslant k\ ,\ k\geqslant k_1(\zeta). \tag{3.4}$$

Hence, it derives from (3.4) and (3.3) that for $\zeta\in\Omega_1$, $k\geqslant k_1(\zeta)$ and $0\leqslant t\leqslant k$

$$\log\left(\frac{x_1(t)}{\varepsilon_{12}}+\frac{x_2(t)}{\varepsilon_{21}}\right),$$
$$\leqslant\int_0^t\frac{1}{\frac{x_1(s)}{\varepsilon_{12}}+\frac{x_2(s)}{\varepsilon_{21}}}\left(\frac{x_1(s)}{\varepsilon_{12}}\left(r_1-l_1c_0(s)-k_1x_1^{\theta_1}(s)\right)\right.$$
$$\left.+\frac{x_2(s)}{\varepsilon_{21}}\left(r_2-l_2c_0(s)-k_2x_2^{\theta_2}(s)\right)\right)ds$$
$$-\int_0^t\frac{1-\epsilon}{2(\frac{x_1(s)}{\varepsilon_{12}}+\frac{x_2(s)}{\varepsilon_{21}})^2}\sum_{i=1}^{n}(\frac{x_1(s)}{\varepsilon_{12}}(\alpha_{1i}+\beta_{1i}x_1^{\theta_1}(s)$$
$$+\gamma_{1i}c_0(s))+\frac{x_2(s)}{\varepsilon_{21}}(\alpha_{2i}+\beta_{2i}x_2^{\theta_2}(s)+\gamma_{2i}c_0(s)))^2ds$$
$$+\log\left(\frac{x_1(0)}{\varepsilon_{12}}+\frac{x_2(0)}{\varepsilon_{21}}\right)+\frac{2\log(k)}{\epsilon}, \tag{3.5}$$

which implies

$$\log\left(\frac{x_1(t)}{\varepsilon_{12}}+\frac{x_2(t)}{\varepsilon_{21}}\right),$$
$$\leqslant\int_0^t\frac{1}{\frac{x_1(s)}{\varepsilon_{12}}+\frac{x_2(s)}{\varepsilon_{21}}}\left(\frac{x_1(s)}{\varepsilon_{12}}(r_1-l_1c_0(s))\right.$$
$$\left.+\frac{x_2(s)}{\varepsilon_{21}}(r_2-l_2c_0(s))\right)ds$$
$$-\int_0^t\frac{1-\epsilon}{2(\frac{x_1(s)}{\varepsilon_{12}}+\frac{x_2(s)}{\varepsilon_{21}}t)^2}\sum_{i=1}^{n}\left(\frac{x_1(s)}{\varepsilon_{12}}(\alpha_{1i}\right.$$
$$\left.+\gamma_{1i}c_0(s))+\frac{x_2(s)}{\varepsilon_{21}}(\alpha_{2i}+\gamma_{2i}c_0(s))\right)^2ds$$
$$+\log\left(\frac{x_1(0)}{\varepsilon_{12}}+\frac{x_2(0)}{\varepsilon_{21}}\right)+\frac{2\log(k)}{\epsilon}. \tag{3.6}$$

Therefore, it is simple to conclude from (3.6) that

$$\begin{aligned}&\log\left(\frac{x_1(t)}{\varepsilon_{12}}+\frac{x_2(t)}{\varepsilon_{21}}\right),\\&\leqslant\left(M-\frac{1-\epsilon}{2}m^2\right)t+\log\left(\frac{x_1(0)}{\varepsilon_{12}}+\frac{x_2(0)}{\varepsilon_{21}}\right)\\&+\frac{2\log(k)}{\epsilon}.\end{aligned}$$

Let $\zeta\in\Omega_1$ and t large enough that the biggest integer smaller than t proves that $[t]\geqslant k_1(\zeta)$. We have from (3.7) that

$$\begin{aligned}&\frac{1}{t}\log\left(\frac{x_1(t)}{\varepsilon_{12}}+\frac{x_2(t)}{\varepsilon_{21}}\right),\\&\leqslant M-\frac{1-\epsilon}{2}m^2+\frac{1}{[t]}\left(\log\left(\frac{x_1(0)}{\varepsilon_{12}}+\frac{x_2(0)}{\varepsilon_{21}}\right)\right.\\&\left.+\frac{2\log([t]+1)}{\epsilon}\right).\end{aligned}$$

This yields

$$\lim_{t\to\infty}\sup\ \frac{1}{t}\log\left(\frac{x_1(t)}{\varepsilon_{12}}+\frac{x_2(t)}{\varepsilon_{21}}\right)\leqslant M-\frac{1-\epsilon}{2}m^2.$$

Letting $\epsilon\longrightarrow 0$ gives

$$\lim_{t\to\infty}\sup\ \frac{1}{t}\log\left(\frac{x_1(t)}{\varepsilon_{12}}+\frac{x_2(t)}{\varepsilon_{21}}\right)\leqslant M-\frac{1}{2}m^2.$$

4 Stationary distribution

The following theorem establishes a sufficient condition for a stationary distribution.

Theorem 4. *Let $n\geq 4$. If α_1, α_2, β_1, β_2, γ_1 and γ_2 are linearly independent,* $\left(r_1-l_1-\varepsilon_{12}-\frac{1}{2}\|\alpha_1+\gamma_1\|^2\right)>0$, *and* $\left(r_2-l_2-\varepsilon_{21}-\frac{1}{2}\|\alpha_2+\gamma_2\|^2\right)>0$, *then, the solution $(x_1(t),x_2(t))$ admits a unique ergodic stationary distribution.*

Proof. Consider the open-bounded subset

$$D=\left(\frac{1}{\mu},\mu\right)\times\left(\frac{1}{\mu},\mu\right)\subset\mathbb{R}^2_+, \tag{4.1}$$

where μ is a positive constant. Since α_1, α_2, β_1, β_2, γ_1 and γ_2 are linearly independent, then $w_1\triangleq\left[\alpha_{1r}x_1(t)+\beta_{1r}x_1^{1+\theta_1}(t)+\gamma_{1r}x_1c_0(t)\right]_{1\leq r\leq n}$, and $w_2\triangleq\left[\alpha_{2r}x_2(t)+\beta_{2r}x_2^{1+\theta_2}(t)+\gamma_{2r}x_2c_0(t)\right]_{1\leq r\leq n}$, are also linearly independent.
Hence, the diffusion matrix Γ, namely $(\Gamma_{ij})_{1\leq i,j\leq 2}=(<w_i,w_j>)_{1\leq i,j\leq 2}$ is positive definite. Thus, the ellipticity condition in [26] is verified (see Chapter 3 of [26]). Now, consider the following positive functions

$$\begin{aligned}&\psi_1(x_1)=\frac{1}{2}\log^2(x_1),\quad \psi_2(x_2)=\frac{1}{2}\log^2(x_2),\\&\psi_3(x_1,x_2)=\varepsilon_{21}x_1+\varepsilon_{12}x_2,\end{aligned}$$

and

$$\psi(x_1,x_2)=\psi_1(x_1)+\psi_2(x_2)+\psi_3(x_1,x_2).$$

According to the Itô formula, we have

$$\begin{aligned}\mathcal{L}\psi_1(x_1)&=\log(x_1)\left[r_1-l_1c_0-k_1x_1^{\theta_1}+\varepsilon_{12}\left(\frac{x_2}{x_1}-1\right)\right]\\&+\frac{1}{2}(1-\log(x_1))\sum_{r=1}^{n}\left(\alpha_{1r}+\beta_{1r}x_1^{\theta_1}+\gamma_{1r}c_0\right)^2.\end{aligned}$$

Using $\log(x_1)\leq x_1$ and rearranging yields

$$\begin{aligned}\mathcal{L}\psi_1(x_1)\leqslant&\left(r_1-l_1c_0-\varepsilon_{12}-\frac{1}{2}\|\alpha_1+\gamma_1c_0\|^2\right)\log(x_1)\\&+\varepsilon_{12}x_2-k_1x_1^{\theta_1}\log(x_1)+\frac{1}{2}\|\alpha_1+\gamma_1c_0\|^2\\&+\frac{1}{2}\left(2<(\alpha_1+\gamma_1c_0),\beta_1>+\|\beta_1\|^2x_1^{\theta_1}\right)x_1^{\theta_1}\\&\times(1-\log(x_1)).\end{aligned}\tag{4.2}$$

Similarly, we have

$$\begin{aligned}\mathcal{L}\psi_2(x_2)\leqslant&\left(r_2-l_2c_0-\varepsilon_{21}-\frac{1}{2}\|\alpha_2+\gamma_2c_0\|^2\right)\log(x_2)\\&+\varepsilon_{21}x_1-k_2x_2^{\theta_2}\log(x_2)+\frac{1}{2}\|\alpha_2+\gamma_2c_0\|^2\\&+\frac{1}{2}\left(2<(\alpha_2+\gamma_2c_0),\beta_2>+\|\beta_2\|^2x_2^{\theta_2}\right)x_2^{\theta_2}\\&\times(1-\log(x_2)),\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}\mathcal{L}\psi_3(x_1,x_2)=&\,\varepsilon_{21}\left(r_1x_1-l_1c_0x_1-k_1x_1^{1+\theta_1}\right)\\&+\varepsilon_{12}\left(r_2x_2-l_2c_0x_2-k_2x_2^{1+\theta_2}\right).\end{aligned}\tag{4.4}$$

From (4.2), (4.3) and (4.4), we have

$$\mathcal{L}\psi(x_1,x_2)\leqslant\chi_1(x_1)+\chi_2(x_2), \tag{4.5}$$

where

$$\begin{aligned}\chi_1(x_1)=&\left(r_1-l_1c_0-\varepsilon_{12}-\frac{1}{2}\|\alpha_1+\gamma_1c_0\|^2\right)\log(x_1)\\&+\frac{1}{2}\left(2<(\alpha_1+\gamma_1c_0),\beta_1>+\|\beta_1\|^2x_1^{\theta_1}\right)x_1^{\theta_1}\\&\times(1-\log(x_1))-k_1x_1^{\theta_1}\log(x_1)+(\varepsilon_{21}r_1+\varepsilon_{21}\\&-\varepsilon_{21}l_2c_0)x_1k_1\varepsilon_{21}x_1^{1+\theta_1}+\frac{1}{2}\|\alpha_1+\gamma_1c_0\|^2,\end{aligned}$$

and

$$\chi_2(x_2) = \left(r_2 - l_2c_0 - \varepsilon_{21} - \frac{1}{2}\|\alpha_2+\gamma_2c_0\|^2\right)\log(x_2) + \frac{1}{2}\left(2 < (\alpha_2+\gamma_2c_0), \beta_2 > +\|\beta_2\|^2x_2^{\theta_2}\right)x_2^{\theta_2} \times(1-\log(x_2)) - k_2x_2^{\theta_2}\log(x_2) + (\varepsilon_{12}r_2+\varepsilon_{12} - \varepsilon_{12}l_1c_0)x_2 - k_2\varepsilon_{12}x_2^{1+\theta_2} + \frac{1}{2}\|\alpha_2+\gamma_2c_0\|^2.$$

Hence

$$\chi_1(x_1) \underset{x_1\longrightarrow\infty}{\sim} -\frac{1}{2}\|\beta_1\|^2x_1^{2\theta_1}\log(x_1) - k_1\varepsilon_{21}x_1^{1+\theta_1},$$

$$\chi_1(x_1) \underset{x_1\longrightarrow 0}{\sim} \left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1+\gamma_1\|^2\right)\log(x_1),$$

$$\chi_2(x_2) \underset{x_2\longrightarrow\infty}{\sim} -\frac{1}{2}\|\beta_2\|^2x_2^{2\theta_2}\log(x_2) - k_2\varepsilon_{12}x_2^{1+\theta_2},$$

and

$$\chi_2(x_2) \underset{x_2\longrightarrow 0}{\sim} \left(r_2 - l_2 - \varepsilon_{21} - \frac{1}{2}\|\alpha_1+\gamma_2\|^2\right)\log(x_2).$$

Since $[r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1+\gamma_1\|^2] > 0$,
and
$[r_2 - l_2 - \varepsilon_{21} - \frac{1}{2}\|\alpha_1+\gamma_2\|^2] > 0$,
then

$$\lim_{x_1\to+\infty}\chi_1(x_1) = \lim_{x_1\to 0}\chi_1(x_1) = \lim_{x_2\to+\infty}\chi_2(x_2) = \lim_{x_2\to 0}\chi_2(x_2) = -\infty.$$

Thus, from (4.5), (4.1) and for μ large enough, we get
$\mathcal{L}\psi(x_1,x_2) \leq -1$ for all $(x_1,x_2) \in D^c$.
Hence, the proof is completed.

5 Simulations

We have the following discrete system using the Euler classical scheme developed in [27].

$$\begin{cases} x_1(k+1) = x_1(k) + \Big(x_1(k)\big(r_1 - l_1c_0(k) - k_1x_1^{\theta_1}(k)\big) + \varepsilon_{12}(x_2(k)-x_1(k))\Big)h \\ \quad + \sum_{i=1}^{n}\Big(\alpha_{1i}x_1(k) + \beta_{1i}x_1^{1+\theta_1}(k) + \gamma_{1i}x_1(k)c_0(k)\Big)\sqrt{h}\eta_i, \\ x_2(k+1) = x_2(k) + \Big(x_2(k)\big(r_2 - l_2c_0(k) - k_2x_2^{\theta_2}(k)\big) + \varepsilon_{21}(x_1(k)-x_2(k))\Big)h \\ \quad + \sum_{i=1}^{n}\Big(\alpha_{2i}x_2(k) + \beta_{2i}x_2^{1+\theta_2}(k) + \gamma_{2i}x_2(k)c_0(k)\Big)\sqrt{h}\eta_i, \end{cases}$$

where η_i $(i = 1,2,...)$ are independent random variables distributed on $\mathcal{N}(0,1)$. So, we take
$c_0(t) = 0.1 + 0.05\sin(t)$.

Example 1. *Set* $x_1(0) = 0.7$, $x_2(0) = 0.8$, $r_1 = 0.7$, $r_2 = 0.7$, $l_1 = 0.2$, $l_2 = 0.1$, $k_1 = 0.2$, $k_2 = 0.3$, $\alpha_1 = 0.12$, $\alpha_2 = 0.15$, $\gamma_1 = 0.08$, $\gamma_2 = 0.09$, $\beta_1 = 0.05$, $\beta_2 = 0.06$, $\theta_1 = 1$, $\theta_2 = 1$, $\varepsilon_{12} = 0.35$ *and* $\varepsilon_{21} = 0.4$. *This gives*

$$r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1+\gamma_1\beta_1\|^2 - k_1 > 0,$$

and

$$r_2 - l_2 - \varepsilon_{21} - \frac{1}{2}\|\alpha_2+\gamma_2+\beta_2\|^2 - k_2 > 0.$$

The persistence condition of Theorem 2 is satisfied. The simulations in Fig.1 well support these findings.

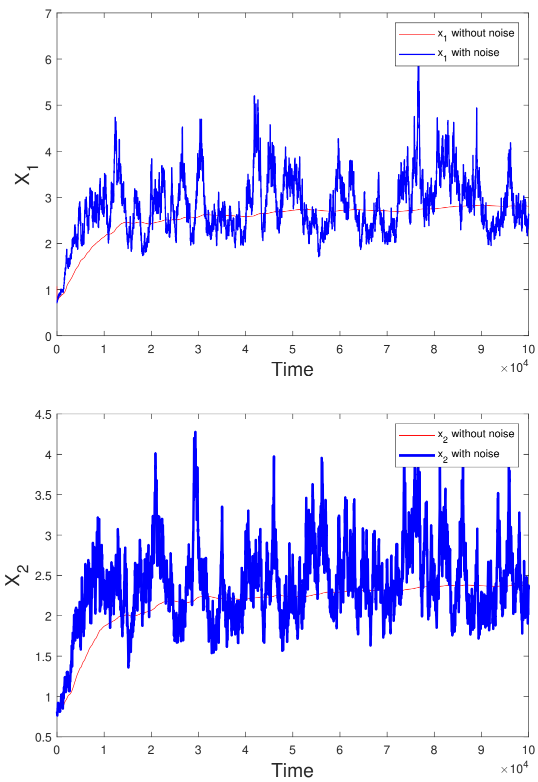

Fig. 1: Trajectories of x_1 and x_2 of system (2.1) with parameter values in Example 1.

Example 2. *Set* $x_1(0) = 0.1$, $x_2(0) = 0.2$, $r_1 = 0.06$, $r_2 = 0.05$, $l_1 = l_2 = 1$, $k_1 = 0.7$, $k_2 = 0.8$, $\alpha_1 = 0.5$, $\alpha_2 = 0.51$, $\gamma_1 = 0.05$, $\gamma_2 = 0.1$, $\beta_1 = 0.95$, $\beta_2 = 0.85$, $\theta_1 = 0.5$, $\theta_2 = 0.6$, $\varepsilon_{12} = 0.9$ *and* $\varepsilon_{21} = 0.8$. *This gives*

$$M - \frac{1}{2}m^2 = -0.09125 < 0.$$

As a result, the extinction condition of Theorem 3 is verified. Simulations in Fig.2 confirm these findings.

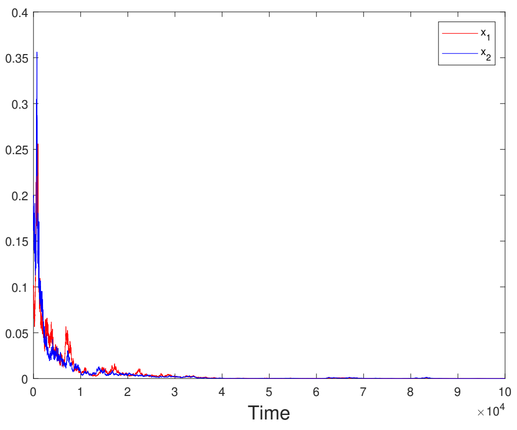

Fig. 2: Trajectories of x_1 and x_2 of system (2.1) with parameter values in Example 2.

Example 3. *Set* $x_1(0) = 0.7$, $x_2(0) = 0.8$, $r_1 = 0.4$, $r_2 = 0.5$, $l_1 = 0.1$, $l_2 = 0.15$, $k_1 = 0.4$, $k_2 = 0.3$, $\alpha_1 = 0.15$, $\alpha_2 = 0.2$, $\gamma_1 = 0.05$, $\gamma_2 = 0.1$, $\beta_1 = 0.15$, $\beta_2 = 0.3$, $\theta_1 = 0.85$, $\theta_2 = 0.95$, $\varepsilon_{12} = 0.1$ *and* $\varepsilon_{21} = 0.15$.
This gives
$r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^1 = 0.18 > 0,$
and
$r_2 - l_2 - \varepsilon_{21} - \frac{1}{2}\|\alpha_2 + \gamma_2\|^2 = 0.155 > 0$. *Consequently, the stationary distribution condition of Theorem 4 is verified.*

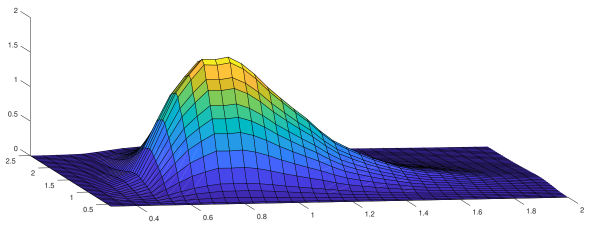

Fig. 3: Estimation Kernel density of (x_1, x_2) with parameter values in Example 3.

6 Conclusion

This study comprehensively integrated pollution into a stochastic Gilpin-Ayala model with patches, providing novel insights into species dynamics within contaminated environments. We identified the conditions under which species extinction or persistence occurs, highlighting the critical role of pollution in shaping ecological outcomes. Finally, computational simulations establish the theoretical results, further validating the model's efficacy. These findings underscore the necessity of incorporating pollution as a central element in ecological frameworks. This paves the way for more accurate predictions and effective conservation strategies in a world increasingly impacted by environmental contaminants.

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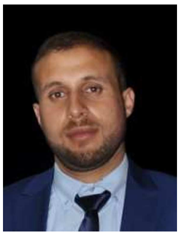

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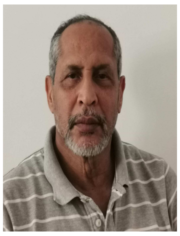

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