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Analyzing the Stability of Caputo Fractional Difference Equations with Variable Orders

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Abstract: This research aims to explore the complex stability of Fractional Variable Order Discrete Time Systems by introducing new stability criteria. To achieve this, we utilize the properties of Volterra convolution-type systems and the Z-transform methodology. We validate these criteria through practical numerical experiments, showing their usefulness in real-world engineering and scientific applications.

Keywords: Fractional order derivatives, variable orders, discrete-time system, stability.

1 Introduction

In recent decades, there is no doubt that non-integer calculus has emerged as a robust and versatile mathematical framework. It has not only expanded the horizons of ancient mathematical models but has also brought a new dimension to modern mathematical modeling in theoretical particle physics, biochemistry, physical chemistry, and computational science which is based on symmetrical systems [1,2,3,4,5,6]. This innovative approach has proven to be instrumental in a wide array of dynamical models and real-world applications [7,8]. One of the areas where non-integer calculus has made a significant impact is in economics, as demonstrated by the work of Machado [9]. Additionally, the field of robot manipulators has greatly benefited from non-integer calculus. By incorporating fractional-order functional operators into their models, researchers have been able to design more sophisticated and precise control systems for robots. This has led to advancements in robotics and automation, enhancing the capabilities of robotic manipulators in various industries [10]. Non-integer calculus has also been instrumental in the field of diffusion modeling [11]. Fractional-order derivatives and integrals have provided a more accurate representation of diffusion processes in various scientific and engineering applications. This has led to improved models for predicting the spread of substances, heat, or information through different mediums. In the realm of biology, non-integer calculus has paved the way for a deeper understanding of complex biological systems. By incorporating fractional calculus into biological models, researchers have been able to capture the intricate and non-linear dynamics of biological processes [12,13]. This has implications for fields such as epidemiology, ecology, and physiology. Non-integer calculus encompasses both its discrete and continuous forms, relying on fractional-order functional operators for fundamental concepts like derivatives, integrals, and differences. This framework enables the creation of sophisticated mathematical models that can accurately represent systems with exceptionally rich and complex dynamics. It has, in essence, opened up a world of possibilities for researchers and scientists, allowing them to tackle problems that were previously considered too intricate to model accurately. However, it's important to emphasize that to gain a thorough grasp of this specific realm within calculus, a detailed and thorough exploration is essential. Non-integer calculus is a rapidly evolving field, and it requires a deep dive into its principles and applications to fully appreciate its potential and impact on various scientific and engineering disciplines.

Variable-order fractional operators have become indispensable tools across a wide spectrum of disciplines, including physics, engineering, and signal processing, as underscored in [14]. Their significance lies in their ability to serve as a

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modeling solution for systems and phenomena marked by ever-changing levels of complexity and memory [15]. These operators offer an exceptionally adaptable and versatile approach for capturing the nuanced dynamics and behaviors of systems in which the fractional order dynamically adapts to shifting conditions and inputs. As a result, they have evolved into invaluable assets for the exploration and understanding of dynamic systems that exhibit variable or adaptive characteristics. Their versatile nature has opened up new horizons in research, allowing for more precise and responsive modeling in complex and dynamic scenarios.

It is widely recognized that there exist numerous definitions of fractional variable-order operators, with variations arising from the manipulation of the order through various functions and modifications. In the context of this paper, our primary focus will be on a specific operator. This particular operator was derived by simplifying the difference fractional order operator, representing it explicitly, and subsequently altering the order using a function.

In this investigation, we explore the stability analysis of Fractional Variable-order Dynamic Systems. We undertake this study due to the recent proliferation of models falling under this category. Given that stability stands as one of the paramount dynamic behaviors, it has assumed great significance to develop theoretical foundations for its examination in an attempt to generalize the results found in the constant order cases [16,17,18]. In pursuit of this goal, we utilize both the Z-transform technique and the final-value theorem as integral components of our investigation. The results acquired are then validated through numerical simulations conducted with a specialized MATLAB program.

The following sections of this paper will be organized in the following manner: Section 2 delivers an elucidation of the core principles and concepts associated with the fractional difference systems with variable order. Section 3 delves into the meticulous examination of the stability of linear systems. Section 4 encompasses the thorough validation of our results through numerical means. Lastly, Section 5 encapsulates the key conclusions that emerge from this study.

2 Preliminaries

In this section, we'll begin by providing a foundational overview of discrete fractional calculus, setting the stage for a deeper exploration of the topic. Discrete fractional calculus deals with discrete analogs of fractional derivatives and integrals. It plays a crucial role in understanding complex dynamic systems with irregularities or variable characteristics.

To delve into this subject, we will elucidate the concept of the fractional operator with a variable order, which represents a fundamental aspect of our investigation. This operator assumes a key role in our study, allowing us to model and analyze systems where the order of differentiation varies with time or other parameters. By defining this operator in its explicit form, we lay the groundwork for a more in-depth examination of its properties and applications within the context of fractional discrete-time systems with variable order. This foundational step is pivotal in unraveling the intricate dynamics of such systems.

Definition 1.[19] *Assume* $\eta > 0$. *In this case, the* η *fractional sum of* $u : \mathbb{N}_a \to \mathbb{R}$ *with respect to* t*:*

$$\Delta_a^{-\eta} u(t) = \frac{1}{\Gamma(\eta)} \sum_{s=a}^{t-\eta} (t-s-1)^{(\eta-1)} u(s), \text{ for } t \in \mathbb{N}_{a+\eta}, \tag{1}$$

where $\mathbb{N}_a = \{a, a+1, a+2, \cdots\}$ *and* $t^{(\eta)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\eta)}$.

Definition 2.[19] *When* $0 < \eta \leq 1$, *the Caputo fractional discrete-time operator of* η*-order is defined as follows:*

$$^{C}\Delta_a^{\eta} u(t) = \begin{cases} \Delta_a^{-(1-\eta)} \Delta u(t) = \frac{1}{\Gamma(1-\eta)} \sum\limits_{s=a}^{t-(1-\eta)} (t-s-1)^{(-\eta)} \Delta u(s), & 0 < \eta < 1 \\ \Delta u(t), & \eta = 1 \end{cases} \quad \forall t \in \mathbb{N}_{a+1-\eta}, \tag{2}$$

In our analysis, we consider the case where the parameter "a" is set to zero, a choice made for the sake of simplicity and without any loss of generality. This particular setting simplifies the mathematical expressions while still preserving the generality and relevance of our findings. With "a" set to zero, we transition to a notation where we represent the fractional discrete time operator as $^{C}\Delta_0^{\eta}$ by $^{C}\Delta^{\eta}$. By further simplifying the expressions and making use of Pascal's rule, we can express this fractional discrete time operator in a more concise and structured form as in [16]:

$$\begin{aligned}({}^{C}\Delta_0^{\eta} u)(t+1-\eta) &= u(t+1) + \sum_{s=0}^{t} (-1)^{t-s+1} \binom{\eta}{t-s+1} u(s) + (-1)^t \binom{\eta-1}{t+1} u(0), \\ &= u(t+1) + (-1)^{t+1} \binom{\eta}{t+1} * u(t) + (-1)^t \binom{\eta-1}{t+1} u(0).\end{aligned}$$

This simplification is a crucial step in our analysis, as it brings us to a point where we can express the fractional discrete time operator in an explicit form. This explicit representation is pivotal for our research, as it enables us to formulate a precise and well-defined definition for the variable-order discrete time operator, which is a central concept in our study.

Definition 3.*Let $\eta : Z \to (0,1]$. Then, the Caputo fractional variable-order discrete time operator with order function $\eta(.)$ is defined by:*

$$\begin{aligned} {}^{C}\Delta^{\eta(t)}u(t+1-\eta(t)) &= u(t+1) + (-1)^{t+1}\tbinom{\eta(t)}{t+1} * u(t) + (-1)^{t}\tbinom{\eta(t)-1}{t+1}u(0), \\ &= u(t+1) + \sum_{s=0}^{t}(-1)^{t-s+1}\tbinom{\eta(t-s)}{t-s+1}u(s) + (-1)^{t}\tbinom{\eta(t)-1}{t+1}u(0), \end{aligned} \tag{3}$$

In this work, our focus turns to the examination of the stability characteristics of discrete systems featuring variable orders. To accomplish this, we will embark on a detailed analysis of a specific system. To set the stage for our investigation, let us consider the following system as our subject of study:

$$ {}^{C}\Delta^{\eta(t)}u(t+1-\eta(t)) = h(u(t)), t \in \mathbb{N}_0, \tag{4}$$

with initial condition $u(0) = u_0 \in \mathbb{R}^n$, where $\eta(.) : \mathbb{N}_0 \to [0,1]$, $h : \mathbb{N}_0 \to \mathbb{R}$ is a state function and $h = (h_1, h_2, ..., h_n)^t : \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable function, and suppose $h(0) = 0$ (all cases can be transferred to be 0 the equilibrium point).

This system will serve as the foundation upon which we will build our exploration of the intricate dynamics and stability properties associated with variable-order discrete systems. By delving into the stability of this particular system, we aim to unravel essential insights that can be applied to a broader understanding of systems with varying orders. This pursuit is crucial in our efforts to attain a thorough comprehension of the dynamics and behaviors exhibited by these systems.

Using 3 and taylor devlopment we get

$$u(t+1) = Ju(k) + \sum_{s=0}^{t} B(t-s)u(s) + g(t) + o(\|u(t)\|), \quad t = 0, 1, \cdots, \tag{5}$$

Here, J represents the Jacobian matrix of h evaluated at 0, $B(t) = (-1)^t\tbinom{\eta(t)}{t+1}I_n$ and $g(t) = (-1)^{t+1}\tbinom{\eta(t)-1}{t+1}u(0)$. so $u(t)$ is a solution of 4 if and only if it is a solution of 5. first analyze its homogeneous part

$$u(t+1) = Ju(t) + \sum_{s=0}^{t} B(t-s)u(s), \quad t = 0, 1, \cdots. \tag{6}$$

When $(B(k))_{k\in\mathbb{N}} \in \ell^1(\mathbb{N}^{n\times n})$, the resolvent matrix $R(k)$ of 6 is defined as:

$$R(t+1) = JR(t) + \sum_{s=0}^{t} B(t-s)R(s), \quad R(0) = I_n, t \in \mathbb{N}.$$

Applying the variation of constants formula, we derive:

$$u(t) = R(t)u(0) + \sum_{s=0}^{t-1} R(t-s-1)\left(g(s) + o(\|u(t)\|)\right).$$

In light of our investigation, it is pertinent to introduce a theorem that holds significant importance for our forthcoming study. This theorem serves as a foundational building block for our analysis and plays a pivotal role in shaping the direction of our research.

Theorem 1.*[20] If $(B(t))_{t\in\mathbb{N}} \in \ell^1(\mathbb{N}^{n\times n})$, for equation 6, the following assertions are equivalent: (1)* $\det(zI - A - \tilde{B}(z)) \neq 0$ *for* $|z| \geq 1$.
(2) $(R(t))_{t\in\mathbb{N}} \in \ell^1(\mathbb{N}^{n\times n})$.
(3) The zero solution of equation 6 exhibits uniform asymptotic stability.

In preparation for applying the previous theorem, we establish the requisite notation concerning the Z-transform of $b(t) = (-1)^t\tbinom{\eta(t)}{t+1}$, with a radius of convergence $R = 1$. We have the following notation:

$$\tilde{b}(z) = \sum_{k=0}^{\infty}(-1)^k\binom{\eta(k)}{k+1}z^{-k}. \tag{7}$$

3 Stability in Linear Variable Order Systems

In this section, our focus centers on the examination of the stability characteristics inherent to discrete linear systems with variable order. To facilitate our analysis, we shall delve into a specific system model featuring a variable-order structure. This chosen system takes on the following form:

$$^{C}\Delta^{\eta(t)}u\left(t+1-\eta\left(t\right)\right)=Au\left(t\right),t\in\mathbb{N}_0, \tag{8}$$

$u(0)=u_0\in\mathbb{R}^n$, $\eta\left(.\right):\mathbb{N}_0\to[0,1]$ and $A\in\mathbb{R}^{n\times n}$.

By dissecting and scrutinizing this system, uur objective is to achieve a thorough understanding of stability considerations within the framework of variable-order systems. This exploration will shed light on the behavior of such systems under different conditions, providing insights into their dynamics and potential applications.

Theorem 2.*System 8 exhibits asymptotic stability solutions of:*

$$\det\left(zI_n-A-\tilde{B}(z)\right)=0, \tag{9}$$

are inside the unit circle. Furthermore, if ther is a solution of 9 located outside the unit circle then the solution of 8 is not stable.

*Proof.*If we assume that the solutions of equation 9 lie within the unit circle, applying the variation of constants formula yields:

$$u(t)=\sum_{s=0}^{t}R(t-s)(-1)^s\binom{\eta\left(s\right)-1}{s}u(0).$$

We have

$$\begin{aligned}\|u(t)\| &= \left\|\sum_{s=0}^{t}R(t-s)(-1)^s\binom{\eta(s)-1}{s}u(0)\right\|\\ &\leq \left\|\sum_{s=0}^{t}R(t-s)(-1)^s\binom{\eta(s)-1}{s}\right\|\|u(0)\|\\ &\leq \zeta_1\sum_{s=0}^{t}\frac{1}{(s+1)^{\eta_{\min}}}\|R(t-s)\|\\ &= \zeta_1\left(\sum_{s=0}^{\lfloor t/2\rfloor}\frac{1}{(s+1)^{\eta_{\min}}}\|R(t-s)\|+\sum_{s=\lfloor t/2\rfloor+1}^{t}\frac{1}{(s+1)^{\eta_{\min}}}\|R(t-s)\|\right),\end{aligned}$$

where $\zeta_1>0$ is a real constant and $\lfloor.\rfloor$ is the floor function. Since the components of R belongs to $\ell^1(\mathbb{N}_0)$, we have $\|R(t)\|=O(t^{-1})$ as $t\to\infty$ and there exist $\zeta_2,\zeta_3>0$ such that

$$\sum_{s=0}^{\lfloor t/2\rfloor}\frac{1}{(s+1)^{\eta_{\min}}}\|R(t-s)\|\leq\frac{\zeta_2}{t+1}\sum_{s=0}^{\lfloor t/2\rfloor}\frac{1}{(s+1)^{\eta_{\min}}}\leq\frac{\zeta_3}{(t+1)^{\eta_{\min}}},$$

where we have used the inequality $\sum_{s=1}^{t}(s+1)^{-\eta}\leq\int_0^t(u+1)^{-\eta}du$. Similarly, the second sum can be estimated as

$$\sum_{s=\lfloor t/2\rfloor+1}^{t}\frac{1}{(s+1)^{\eta_{\min}}}\|R(t-s)\|\leq\frac{\zeta_4}{(t+1)^{\eta_{\min}}}\sum_{s=\lfloor t/2\rfloor+1}^{t}\|R(t-s)\|\leq\frac{\zeta_5}{(t+1)^{\eta}},$$

for suitable $\zeta_4,\zeta_5>0$. In summary, we have $\|u(t)\|\leq\zeta_5(t+1)^{-\eta}$, hence $\|u(t)\|=O(t^{-\eta})$ as $t\to\infty$.
Hence, if there exists a zero z with $|z|>1$, then the radius of convergence of at least one component u_i of u exceeds 1. Consequently,

$$r=\lim_{t\to\infty}\sup\sqrt[k]{|u_i(t)|}>1,$$

according Cauchy-Hadamard theorem, consequently, $\lim\limits_{k\to\infty}\sup|u_i(k)|=+\infty$ which proves that u is not bounded and thus 8 is not stable.

Now, we proceed to unveil a practical result that emerges from our analysis, a result that serves as a valuable addition to the understanding and application of the concepts under examination. To do this, we introduce a fundamental set, a concept that will play a pivotal role in our ongoing exploration:

$$\begin{aligned} S^{\eta(0)} &= \left\{ z - \sum_{k=0}^{\infty} (-1)^k \binom{\eta(0)}{k+1} z^{-k}, z \in \mathbb{C}, |z| < 1 \right\} \\ &= \left\{ z \in \mathbb{C} : |z| < \left(2\cos \frac{|\arg z| - \pi}{2-\eta(0)} \right)^{\eta(0)} \text{ and } |\arg z| > \frac{\eta(0)\pi}{2} \right\}. \end{aligned} \tag{10}$$

This introduced set encapsulates key elements that underpin the practical implications of our study. It paves the way for the application of our theoretical insights in real-world scenarios and problem-solving. Through a closer examination of this set, we aim to highlight its importance and its role in bridging the gap between theoretical concepts and their practical relevance, ultimately contributing to the advancement of knowledge and the solution of real-world challenges.

Theorem 3. *Let*

$$\rho := \max \left\{ \left(1 - \frac{\eta_{\min}}{\eta_{\max}} + \eta_{\min} - \eta(0) \right), \left(\frac{\eta_{\max}}{\eta_{\min}} - \eta_{\max} - 1 + \eta(0) \right) \right\},$$

where

$$\eta_{\min} = \inf \eta(t), \eta_{\max} = \sup \eta(t),$$

and

$$d\left(\lambda, \mathbb{C} \backslash S^{\eta(0)}\right) = \inf \left\{ |\lambda - z|, z \in \mathbb{C} \backslash S^{\eta(0)} \right\}.$$

If

$$d\left(\lambda, \mathbb{C} \backslash S^{\eta(0)}\right) > \rho, \tag{11}$$

for any eigenvalues λ of A, then system 8 is asymptotically stable.

Proof. We have

$$|z - \lambda - \tilde{b}(z)| \geq \left| \left| z - \lambda - \sum_{k=0}^{\infty} (-1)^k \binom{\eta(0)}{k+1} z^{-k} \right| - \left| \sum_{k=0}^{\infty} (-1)^k \binom{\eta(0)}{k+1} z^{-k} - \sum_{k=0}^{\infty} (-1)^k \binom{\eta(k)}{k+1} z^{-k} \right| \right|$$

This means if

$$\left| z - \lambda - \sum_{k=0}^{\infty} (-1)^k \binom{\eta(0)}{k+1} z^{-k} \right| > \left| \sum_{k=0}^{\infty} (-1)^k \binom{\eta(0)}{k+1} z^{-k} - \sum_{k=0}^{\infty} (-1)^k \binom{\eta(k)}{k+1} z^{-k} \right|,$$

we get for any eigenvalues λ of A :

$$z - \lambda - \tilde{b}(z) \neq 0, \forall z \in \mathbb{C}, |z| \geq 1. \tag{12}$$

We have

$$\begin{aligned} &\left| \sum_{k=0}^{\infty} (-1)^k \binom{\eta(0)}{k+1} z^{-k} - \sum_{k=0}^{\infty} (-1)^k \binom{\eta(k)}{k+1} z^{-k} \right| = \\ &\qquad \left| \sum_{k=1}^{\infty} \left(\frac{(\eta(0))(1-\eta(0))\cdots(k-\eta(0))}{\Gamma(k+2)} - \frac{(\eta(k))(1-\eta(k))\cdots(k-\eta(k))}{\Gamma(k+2)} \right) z^{-k} \right|. \end{aligned}$$

So

$$\begin{aligned} \left| \sum_{k=1}^{\infty} (-1)^k \binom{\eta(0)}{k+1} z^{-k} - \sum_{k=1}^{\infty} (-1)^k \binom{\eta(k)}{k+1} z^{-k} \right| < \max &\left\{ \sum_{k=1}^{\infty} \left(\frac{(\eta(0))(1-\eta(0))\cdots(k-\eta(0))}{\Gamma(k+2)} - \frac{(\eta_{\min})(1-\eta_{\max})\cdots(k-\eta_{\max})}{\Gamma(k+2)} \right), \right. \\ &\left. \sum_{k=1}^{\infty} \left(\frac{(\eta_{\max})(1-\eta_{\min})\cdots(k-\eta_{\min})}{\Gamma(k+2)} - \frac{(\eta(0))(1-\eta(0))\cdots(k-\eta(0))}{\Gamma(k+2)} \right) \right\}, \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} &\left| \sum_{k=1}^{\infty} (-1)^k \binom{\eta(0)}{k+1} z^{-k} - \sum_{k=1}^{\infty} (-1)^k \binom{\eta(k)}{k+1} z^{-k} \right| < \\ &\max \left\{ \sum_{k=1}^{\infty} \left(\frac{(\eta(0))(1-\eta(0))\cdots(k-\eta(0))}{\Gamma(k+2)} - \frac{(\eta_{\min})}{(\eta_{\max})} \frac{(\eta_{\max})(1-\eta_{\max})\cdots(k-\eta_{\max})}{\Gamma(k+2)} \right), \right. \\ &\left. \sum_{k=1}^{\infty} \left(\frac{(\eta_{\max})}{(\eta_{\min})} \frac{(\eta_{\min})(1-\eta_{\min})\cdots(k-\eta_{\min})}{\Gamma(k+2)} - \frac{(\eta(0))(1-\eta(0))\cdots(k-\eta(0))}{\Gamma(k+2)} \right) \right\}, \end{aligned}$$

$\Leftrightarrow$

$$\left|\sum_{k=1}^{\infty}(-1)^k\binom{\eta(0)}{k+1}z^{-k}-\sum_{k=1}^{\infty}(-1)^k\binom{\eta(k)}{k+1}z^{-k}\right|<$$
$$\max\left\{\sum_{k=1}^{\infty}(-1)^k\binom{\eta(0)}{k+1}-\frac{(\eta_{\min})}{(\eta_{\max})}\sum_{k=1}^{\infty}(-1)^k\binom{\eta_{\max}}{k+1},\right.$$
$$\left.\frac{(\eta_{\max})}{(\eta_{\min})}\sum_{k=1}^{\infty}(-1)^k\binom{\eta_{\min}}{k+1}-\sum_{k=1}^{\infty}(-1)^k\binom{\eta(0)}{k+1}\right\},$$

$\Leftrightarrow$

$$\left|\sum_{k=1}^{\infty}(-1)^k\binom{\eta(0)}{k+1}z^{-k}-\sum_{k=1}^{\infty}(-1)^k\binom{\eta(k)}{k+1}z^{-k}\right|<$$
$$\max\left\{\left(1-\frac{\eta_{\min}}{\eta_{\max}}+\eta_{\min}-\eta(0)\right),\left(\frac{\eta_{\max}}{\eta_{\min}}-\eta_{\max}-1+\eta(0)\right)\right\},$$

If $\left|z-\lambda-\sum_{k=0}^{\infty}(-1)^k\binom{\eta(0)}{k+1}z^{-k}\right|>\max\left\{\left(1-\frac{\eta_{\min}}{\eta_{\max}}+\eta_{\min}-\eta(0)\right),\left(\frac{\eta_{\max}}{\eta_{\min}}-\eta_{\max}-1+\eta(0)\right)\right\}$ then $\left|z-\lambda-\tilde{b}(z)\right|>0$ therefore the condition 9 is fulfilled.

3.1 Stability in Non-Linear Variable Order Systems

Now, we turn our attention to the examination of the stability properties of the nonlinear system 4 in a more comprehensive context. To achieve this, we introduce a critical component to our analysis, a Lemma that will aid in our understanding of the system's stability under various conditions and scenarios:

Theorem 4.*System 4 exhibits local asymptotic stability if all solutions of:*

$$\det\left(zI_n-J-\tilde{B}(z)\right)=0, \tag{13}$$

Are contained within the unit disk.

*Proof.*Suppose that all the solutions of 13 are contained within the unit disk, by the variation of constants, we obtain

$$u(t)=R(t)u(0)+\sum_{s=0}^{t-1}R(t-s-1)\left(g(s)+o(\|u(s)\|\right).$$

where $R(t)$ is the resolvent matrix. We have

$$\|u(t)\|\leq\|R(t)\|\,\|u(0)\|+\sum_{s=0}^{t-1}\|R(t-s-1)\|\,\|o(\|u(s)\|)\|+\sum_{s=0}^{t-1}\|R(t-s-1)\|\,\|g(s)\|, \tag{14}$$

for a given $\varepsilon>0$ there is $\delta>0$ such that $o(\|u\|)<\varepsilon\|u\|$ whenever $\|u\|<\delta$. So as long as $\|u(s)\|<\delta$, 14 becomes

$$\|u(t)\|\leq\|R(t)\|\,\|u(0)\|+\varepsilon\sum_{s=0}^{t-1}\|R(t-s-1)\|\,\|u(s)\|+\sum_{s=0}^{t-1}\|R(t-s-1)\|\,\|g(s)\|,$$

we defined $y(t)$ as follow

$$y(t)=r(t)y(0)+\varepsilon\sum_{s=0}^{t-1}r(t-s-1)y(s)+\sum_{s=0}^{t-1}r(t-s-1)h(s)$$

where

$$r(t)=\|R(t)\|,\quad h(s)=\|g(s)\|,\quad y(0)=\|u(0)\|.$$

$\Rightarrow$

$$y(t+1)=r(t+1)y(0)+\varepsilon\sum_{s=0}^{t}r(t-s)y(s)+\sum_{s=0}^{t}r(t-s)h(s)$$

$\Rightarrow$

$$y(t+1)=r(t+1)y(0)+\varepsilon r(t)*y(t)+r(t)*h(t). \tag{15}$$

We have

$$\|u(t)\| \le y(t),$$

we see that $r(t) \in \ell^1(\mathbb{N})$. Taking the $Z-$transform on 15 gives:

$$z\tilde{y}(z) - y(0)z = (z\tilde{r}(z) - z)y(0) + \varepsilon\tilde{r}(z)\tilde{y}(z) + \tilde{r}(z)\tilde{h}(z),$$

with $R_r \le 1$, and $R_h = 1$, where R_r is the convergence radius of $\tilde{r}(z)$ and R_r is the convergence radius of $\tilde{h}(z) \Rightarrow$

$$\tilde{y}(z) = (z - \varepsilon\tilde{r}(z))^{-1}\left(z\tilde{r}(z)y(0) + \tilde{r}(z)\tilde{h}(z)\right),$$

for $|z| > \max\{R_r, 1, \varepsilon\tilde{r}(1)\}$.
Choose $\varepsilon < \frac{1}{\tilde{r}(1)}$, we get $\max\{R_r, 1, \varepsilon\tilde{r}(1)\} = 1$, by final value theorem

$$\lim_{t\to\infty} y(t) = \lim_{z\to 1}(z-1)\tilde{y}(z) = \lim_{z\to 1}(z-1)((z-\varepsilon\tilde{r}(z))^{-1}z\tilde{r}(z)y(0) + \tilde{r}(z)\tilde{h}(z))$$

we have

$$\begin{aligned}\sum_{s=0}^{t}\|R(t-s)\|\,\|H(s)\| &\le \zeta_1\sum_{s=0}^{t}\frac{1}{(s+1)^{\eta}}\|R(t-s)\|\,\|u(0)\| \\ &\le \zeta_1\left(\sum_{s=0}^{\lfloor t/2\rfloor}\frac{1}{(s+1)^{\eta}}\|R(t-s)\| + \sum_{s=\lfloor t/2\rfloor+1}^{t}\frac{1}{(s+1)^{\eta}}\|R(t-s)\|\right)\end{aligned}$$

where $\zeta_1 > 0$ is a suitable real constant and the symbol $\lfloor . \rfloor$ stands for the floor function and $\eta = \min_{1\le i\le n}\{\eta_i\}$. Since $\|R(t)\|$ belongs to $\ell^1(\mathbb{N})$, we have $\|R(t)\| = O(t^{-1})$ as $t\to\infty$ and there exist $\zeta_2, \zeta_3 > 0$ such that

$$\sum_{s=0}^{\lfloor t/2\rfloor}\frac{1}{(s+1)^{\eta}}\|R(t-s)\| \le \frac{\zeta_2}{t+1}\sum_{s=0}^{\lfloor t/2\rfloor}\frac{1}{(s+1)^{\eta}} \le \frac{\zeta_3}{(t+1)^{\eta}}$$

where we have used the inequality $\sum_{s=1}^{t}(s+1)^{-\eta} \le \int_0^t (u+1)^{-\eta}du$. Similarly, the second sum can be estimated as

$$\sum_{s=\lfloor t/2\rfloor+1}^{t}\frac{1}{(s+1)^{\eta}}\|R(t-s)\| \le \frac{\zeta_4}{(t+1)^{\eta}}\sum_{s=\lfloor t/2\rfloor+1}\|R(t-s)\| \le \frac{\zeta_5}{(t+1)^{\eta}},$$

for suitable $\zeta_4, \zeta_5 > 0$. In summary, we have $\sum_{s=0}^{t}\|R(t-s)\|\,\|H(s)\| \le \zeta_5(t+1)^{-\eta}$, hence $\sum_{s=0}^{t}\|R(t-s)\|\,\|H(s)\| = O(t^{-\eta})$ as $t\to\infty$. So by final value theorem

$$\lim_{z\to 1}(z-1)\tilde{r}(z)\tilde{h}(z) = 0,$$

and that's imply

$$\lim_{t\to\infty} y(t) = \lim_{z\to 1}(z-1)\tilde{y}(z) = \lim_{z\to 1}(z-1)((z-\varepsilon\tilde{r}(z))^{-1}z\tilde{r}(z)y(0) + \tilde{r}(z)\tilde{h}(z)) = 0.$$

Theorem 5.*Suppose that for any eigenvalues λ of J :*

$$d\left(\lambda, \mathbb{C}\backslash S^{\eta(0)}\right) > \rho. \tag{16}$$

Then System 4 is locally asymptotically stable. Where

$$\rho := \max\left\{\left(1 - \frac{\eta_{\min}}{\eta_{\max}} + \eta_{\min} - \eta(0)\right), \left(\frac{\eta_{\max}}{\eta_{\min}} - \eta_{\max} - 1 + \eta(0)\right)\right\},$$

*Proof.*With the same steps of prove Theorem 8: If

$$\left|z - \lambda - \sum_{k=0}^{\infty}(-1)^k\binom{\eta(0)}{k+1}z^{-k}\right| > \max\left\{\left(1 - \frac{\eta_{\min}}{\eta_{\max}} + \eta_{\min} - \eta(0)\right), \left(\frac{\eta_{\max}}{\eta_{\min}} - \eta_{\max} - 1 + \eta(0)\right)\right\},$$

then

$$\left|z - \lambda - \tilde{b}(z)\right| > 0,$$

therefore the condition 13 is fulfilled.

*Remark.*If either Matrix A in 8 or Matrix J in 5 is symmetric, their eigenvalues are real, making the calculations straightforward.

4 Numerical Simulations

What we will do in this section is to test the obtained result numerically through two numerical examples regarding the linear case and two other regarding the nonlinear case and to carry out numerical simulations that support the theoretical results.

4.1 Linear Systems

*Example 1.*Let us consider the symmetric system with a variable-order the following form:

$$\begin{cases} {}^{C}\Delta^{\eta(t)}u_1(t+1-\eta(t)) = -0.56u_1(t)+0.16462u_2(t) \\ {}^{C}\Delta^{\eta(t)}u_2(t+1-\eta(t)) = 0.16462u_1(t)-0.56u_2(t) \end{cases}, t\in\mathbb{N}_0, \tag{17}$$

with initial condition $(u_1,u_2)=(0.2,-0.3)$, where $\eta:\mathbb{N}_0\to[0,1], \eta(t)=0.4+0.1e^{-t}$. We have

$$\eta_{\min}=0.4, \eta_{\max}=0.5$$

$$\rho=\max\left\{\left(1-\frac{\eta_{\min}}{\eta_{\max}}+\eta_{\min}-\eta(0)\right),\left(\frac{\eta_{\max}}{\eta_{\min}}-\eta_{\max}-1+\eta(0)\right)\right\}=0.25.$$

The eigenvalues of the matrix are $\lambda_1=-0.39538, \lambda_2=-0.72462$
. We notice from the Figure 1 obtained by MATLAB that the condition 16 is fulfilled, and therefore the system 17 is asymptotically stable around the origin.

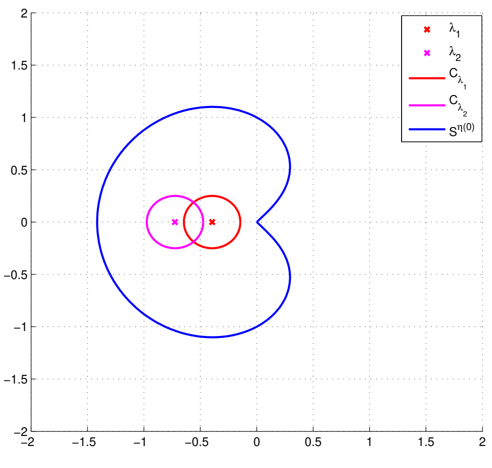

Fig. 1: The location of the eigenvalues and disks which are centred at these eigenvaluse with radius ρ

The following numerical simulations (Figure 2) show the stability which agree with theoretical result

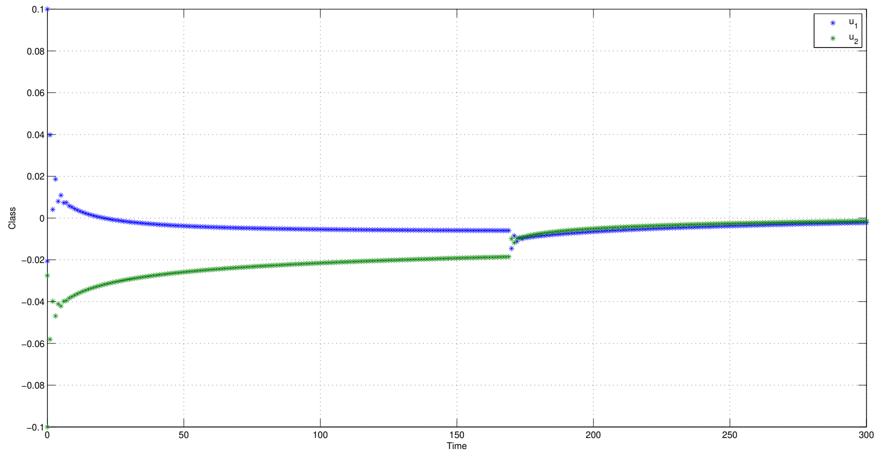

Fig. 2: The states of the FVoDS 17

*Example 2.*Let us consider a system with a variable-order in the following form:

$$\begin{cases} {}^{C}\Delta^{\eta(t)}u_1\left(t+1-\eta\left(t\right)\right) = -0.266905u_1\left(t\right)+0.56619u_2\left(t\right)-0.166905u_3\left(t\right) \\ {}^{C}\Delta^{\eta(t)}u_2\left(t+1-\eta\left(t\right)\right) = -0.166905u_1\left(t\right)-u_2\left(t\right)+0.166905u_3\left(t\right) \\ {}^{C}\Delta^{\eta(t)}u_3\left(t+1-\eta\left(t\right)\right) = 0.399285u_1\left(t\right)+0.56619u_2\left(t\right)-0.833095u_3\left(t\right) \end{cases}, t\in\mathbb{N}_0, \quad (18)$$

with initial condition $(u_1,u_2,u_3) = (2.2, 1.01, -1, 4)$, where $\eta : \mathbb{N}_0 \to [0,1], \eta\left(t\right) = 0.6+0.1\cos\frac{\pi}{4}t$. We have

$$\eta_{\min} = 0.5, \eta_{\max} = 0.7$$

$$\max\left\{\left(1-\frac{\eta_{\min}}{\eta_{\max}}+\eta_{\min}-\eta(0)\right), \left(\frac{(\eta_{\max})}{(\eta_{\min})}-(\eta_{\max})-1+\eta(0)\right)\right\} = 0.4.$$

The eigenvalues of the matrix are $\lambda_1 = -0.43381, \lambda_2 = -0.66619, \lambda_3 = -1$. We notice from the Figure3 obtained by MATLAB that the condition 16 is fulfilled, and therefore the system 18 is asymptotically stable around the origin.

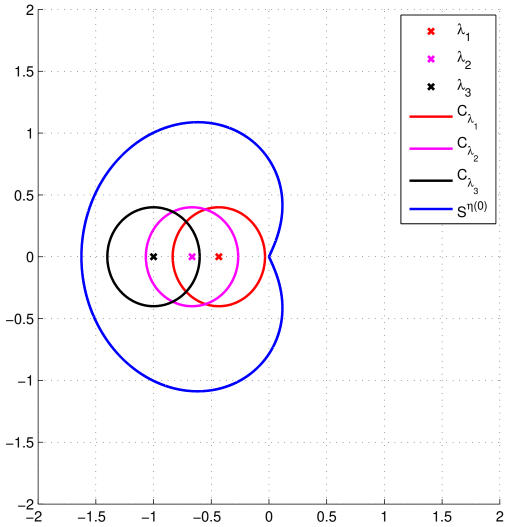

Fig. 3: The location of the eigenvalues and disks which are centred at these eigenvaluse with radius ρ

The following numerical simulations (Figure 4) show the stability which agree with theoretical result

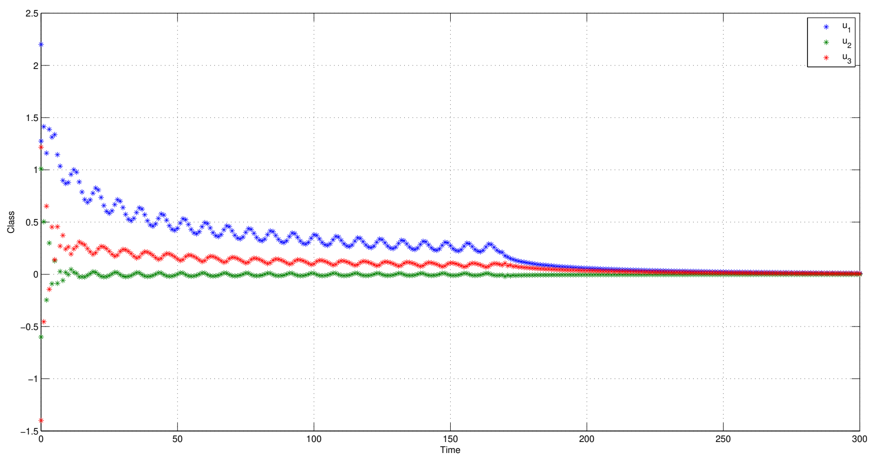

Fig. 4: The states of the FVoDS 18

4.2 Non-Linear Systems

*Example 3.*Let us consider the system with a variable-order the following form:

$$\begin{cases} {}^{C}\Delta^{\eta(t)}u_1\left(t+1-\eta\left(t\right)\right) = -0.41\sin\left(u_1\left(t\right)\right) + 1.4\sin\left(u_2\left(t\right)\right) \\ {}^{C}\Delta^{\eta(t)}u_2\left(t+1-\eta\left(t\right)\right) = 0.5u_1^2\left(t\right) - 0.6u_2\left(t\right) \end{cases}, t \in \mathbb{N}_0, \tag{19}$$

with initial condition $u(0) = (0.5, -0.5)$, where $\eta : \mathbb{N}_0 \to [0,1], \eta\left(t\right) = 0.3 + 0.05\cos(t)$. We have

$$\eta_{\min} = 0.25, \eta_{\max} = 0.35$$

$$\rho = \max\left\{\left(1-\frac{\eta_{\min}}{\eta_{\max}}+\eta_{\min}-\eta(0)\right),\left(\frac{\eta_{\max}}{\eta_{\min}}-\eta_{\max}-1+\eta(0)\right)\right\}=0.4.$$

The eigenvalues of the Jacobian matrix of $\left(-0.41\sin(u_1(t))+1.4\sin(u_2(t)),0.5u_1^2(t)-0.6u_2(t)\right)^t$,at $(0,0)$ are $\lambda_1 = -0.41, \lambda_2 = -0.6$. We notice from the following Figure 5 obtained by MATLAB that the condition 16 is fulfilled, and therefore the system 19 is asymptotically stable around the origin.

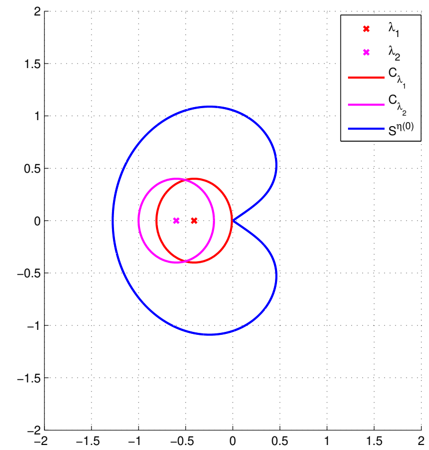

Fig. 5: The location of the eigenvalues and disks which are centered at these eigenvalues with radius ρ

The following numerical simulations (Figure 6) show the stability which agrees with theoretical result

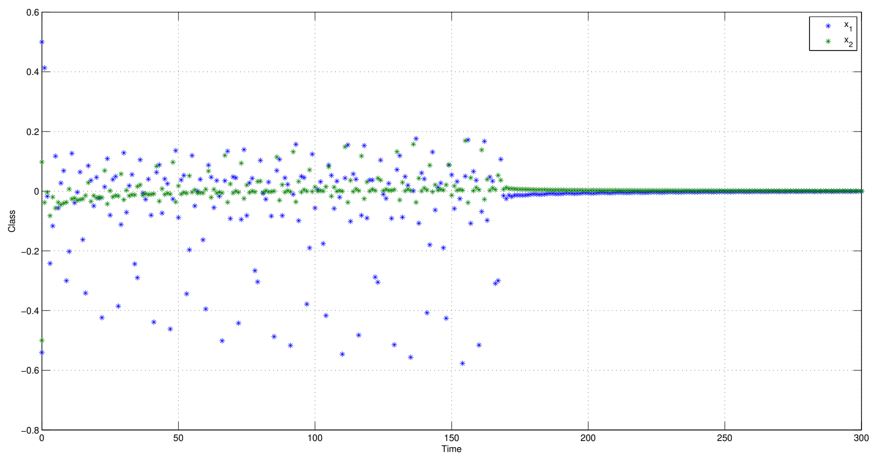

Fig. 6: The states of the FVoDS 19

*Example 4.*Let us consider the system with a variable-order the following form:

$$\begin{cases} {}^{C}\Delta^{\eta(t)}u_1(t+1-\eta(t)) = -0.3\sin(u_1(t))-\sin(u_2(t)) \\ {}^{C}\Delta^{\eta(t)}u_2(t+1-\eta(t)) = 0.09u_1(t)-0.8\sin(u_2(t)) \\ {}^{C}\Delta^{\eta(t)}u_3(t+1-\eta(t)) = \sin(u_1(t)-u_2(t)-u_3(t)) \end{cases} ,t\in\mathbb{N}_0, \tag{20}$$

with initial condition $u(0) = (2,3,-4)$, where $\eta : \mathbb{N}_0 \to [0,1], \eta(t) = 0.6 + 0.1\sin\frac{\pi}{2}t$. We have

$$\eta_{\min} = 0.5, \eta_{\max} = 0.7$$

$$\max\left\{\left(1 - \frac{\eta_{\min}}{\eta_{\max}} + \eta_{\min} - \eta(0)\right), \left(\frac{(\eta_{\max})}{(\eta_{\min})} - (\eta_{\max}) - 1 + \eta(0)\right)\right\} = 0.4.$$

The eigenvalues of the Jacobian matrix of

$$(-0.3\sin(u_1(t)) - \sin(u_2(t)), 0.09u_1(t) - 0.8\sin(u_2(t)), \sin(u_1(t) - u_2(t) - u_3(t)))^t,$$

at $(0,0,0)$ are $\lambda_1 = -0.55 + 0.16583i, \lambda_2 = -0.55 - 0.16583i, \lambda_3 = -1$. We notice from the following Figure 7 obtained by MATLAB that the condition 16 is fulfilled, and therefore the system 20 is asymptotically stable around the origin.

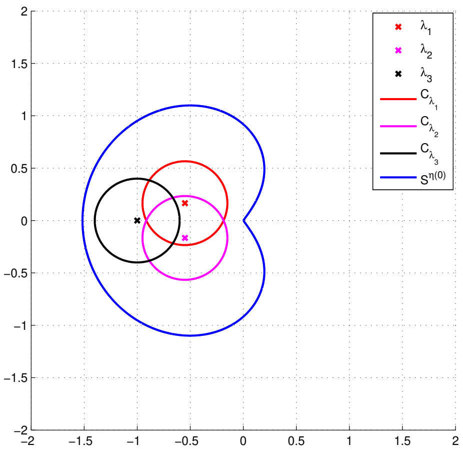

Fig. 7: The location of the eigenvalues and disks which are centered at thees eigenvalues with radius ρ

The following numerical simulations (Figure 8) show the stability which agrees with theoretical result

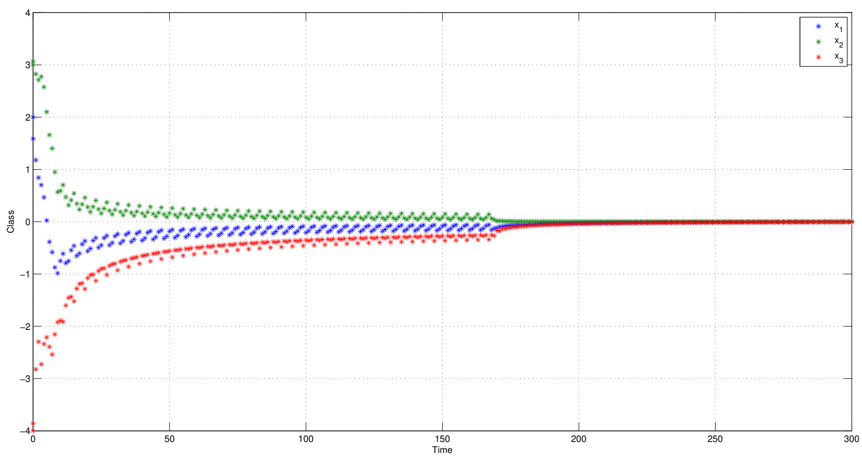

Fig. 8: The states of the FVoDS 20

5 Conclusion

In this study, we have presented innovative stability criteria rigorously validated for fractional discrete-time systems with variable order. This work considers a generalization of the results obtained in the past years regarding the stability of discrete-time systems with constant order. These criteria were confirmed through numerical demonstrations with multiple illustrative examples, offering valuable insights for a wide range of discrete-time systems, thus contributing significantly to the field. This work motivates us to study many models in the case of variable orders and to apply this type of model in many applied aspects such as biology and engineering.

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