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A New Iterative Method for Non-linear Equations and Its Application

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Abstract: Newton's iterative method, which is quadratically convergent, is often used to calculate water surface profile of a river. But when the river is long, it will be spent much of time on calculating. In this paper, a new fourth-order iterative method for non-linear equations is proposed. Analysis of convergence shows that the method has at least third-order convergence, and if the parameter φ of the method is equal to 3, it will have fourth-order convergence. Several numerical examples are given to illustrate the efficiency and performance of the proposed method. In the end, the new method is used in water surface profile computing, which performs better than Newton's iterative method and Potra's method.

Keywords: Newton's method, fourth-order convergence, non-linear equations, water surface profile

1. Introduction

Non-linear Equations have been used to model a wide variety of phenomena that involve wave motion and the advective transport of substances, such as sediment transport, pollutant advection and heat transfer, and so on[1,2]. When it is applied to natural problems, it is hard to get the exact solution for the equations. Therefore, many numerical methods have been developed to get the numerical solution of the equations[3–5]. Newton's method is one of the most important tools in numerical analysis, especially in solving non-linear equations. In this paper, we consider the problem of numerical approximation of a real root α of the non-linear equation $f(x) = 0$. The α is called to be a simple root if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

Newton's method for a single non-linear equation is defined as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

The above method is quadratically convergent [6]. Many efficient modifications of Newton's method with cubic convergence have been developed [7–10]. One classical third-order modification of Newton's method is given by Potra [11]. It is defined as

$$x_{n+1} = x_n - \frac{f(x_n) + f(x^*_{n+1})}{f'(x_n)} \tag{2}$$

where $x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
Recently, Chun [12] gives a new approximation as

$$f'(x^*_{n+1}) = \frac{f(x_n) - f(x^*_{n+1})}{f(x_n) + f(x^*_{n+1})} f'(x_n) \tag{3}$$

Using the approximation in the following formula

$$x_{n+1} = x_n - \frac{3}{2}\frac{f(x_n)}{f'(x_n)} + \frac{1}{2}\frac{f(x_n)}{f'(x_n)}\frac{f'(y_n)}{f'(x_n)} \tag{4}$$

They obtain a new modifications of Newton's method as the following

$$x_{n+1} = x_n - \frac{f(x_n) + 2f(x^*_{n+1})}{f(x_n) + f(x^*_{n+1})}\frac{f(x_n)}{f'(x_n)} \tag{5}$$

The method also has third-order convergence. Eq. 2 and Eq. 5 do not require the second derivative, which are quite practical.

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In this paper, we propose a new method based on the combination of Eq. 2 and Eq. 5. The method has fourth-order convergence, and per iteration the new method requires two evaluations of the function and one evaluation of its first derivative. According to function evaluations, the new method is found to be efficient than the classical Potra's method. Furthermore, the efficiency and performance of the proposed method is illustrated through several numerical examples.

2. Development of method and convergence analysis

Eq. 2 and Eq. 5 both have cubical convergence. So we can consider the linear combination of Eq. 2 and Eq. 5 which can produce a method with high order convergence. Now, the equation can be written as

$$x_{n+1}=x_n-\Big(\varphi\frac{f(x_n)+f(x^*_{n+1})}{f'(x_n)} +(1-\varphi)\frac{f(x_n)+2f(x^*_{n+1})}{f(x_n)+f(x^*_{n+1})}\frac{f(x_n)}{f'(x_n)}\Big) \tag{6}$$

where x^*_{n+1} is Newton iteration.

Let α be a simple zero of f and using Taylor expansion, we get

$$f(x_n)=f'(\alpha)\Big(e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+o(e_n^5)\Big) \tag{7}$$

and

$$f\prime(x_n)=f'(\alpha)\Big(1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+o(e_n^4)\Big) \tag{8}$$

where $c_k=\frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$,$k=2,3,4\dots$ and $e_n=x_n-\alpha$. Dividing Eq. 7 by Eq. 8 gives us

$$\frac{f(x)}{f'(x)}=e_n-c_2e_n^2+2(c_2^2-c_3)e_n^3 +(7c_2c_3-4c_2^3-3c_4)e_n^4+o(e_n^5)) \tag{9}$$

As $x^*_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$ we get

$$x^*_{n+1}=\alpha+c_2e_n^2-2(c_2^2-c_3)e_n^3+o(e_n^4) \tag{10}$$

Expanding $f(x^*_{n+1})$ about α and using (10), we obtain

$$f(x^*_{n+1})=f'(\alpha)\Big(c_2e_n^2-2(c_2^2-c_3)e_n^3 -(7c_2c_3-5c_2^3-3c_4)e_n^4+o(e_n^5)\Big) \tag{11}$$

From Eq. 7 and Eq. 11, we have

$$f(x_n)+f'(x_{n+1})=f'(\alpha)\Big(e_n+2c_2e_n^2-(2c_2^2 -3c_3)e_n^3-(7c_2c_3-5c_2^3-4c_4)e_n^4+o(e_n^5)\Big) \tag{12}$$

and

$$f(x_n)+2f(x^*_{n+1})=f'(\alpha)\Big(e_n+3c_2e_n^2-(4c_2^2- 5c_3)e_n^3-(14c_2c_3-10c_2^3-7c_4)e_n^4+o(e_n^5)\Big) \tag{13}$$

Dividing Eq. 12 by Eq. 8 gives us

$$\frac{f(x_n)+f(x^*_{n+1})}{f\prime(x_n)}=e_n-2c_2^2e_n^3-(7c_2c_3-9c_2^3)e_n^4+o(e_n^5) \tag{14}$$

Again, dividing Eq. 13 by Eq. 8 gives us

$$\frac{f(x_n)+2f(x^*_{n+1})}{f(x_n)+f(x^*_{n+1})}=1+c_2e_n-(4c_2^2 -2c_3)e_n^2-(14c_2c_3-15c_2^3-3c_4)e_n^3+o(e_n^4) \tag{15}$$

From Eq. 9 and Eq. 15 we have

$$\frac{f(x_n)+2f(x^*_{n+1})}{f(x_n)+f(x^*_{n+1})}\frac{f(x_n)}{f'(x_n)}=e_n-3c_2^3e_n^3 +(17c_2^3-11c_2c_3)e_n^4+o(e_n^5) \tag{16}$$

From Eq. 6, we get

$$e_{n+1}=e_n-\Big(\varphi\frac{f(x_n)+f(x^*_{n+1})}{f'(x_n)} +(1-\varphi)\frac{f(x_n)+2f(x^*_{n+1})}{f(x_n)+f(x^*_{n+1})}\frac{f(x_n)}{f'(x_n)}\Big) \tag{17}$$

Substituting Eq. 14 and Eq. 16 into Eq. 17, we obtain

$$e_{n+1}=(3-\varphi)c_2^2e_n^3-\Big((17-8\varphi)c_2^3+(4\varphi-11)c_2c_3\Big)e_n^4+o(e_n^5) \tag{18}$$

This means that the method defined by Eq. 9 has at least cubical convergence. If we take $\varphi=3$, the convergence is fourth-order. And from Eq. 18 we have the error equation as

$$e_{n+1}=(17c_2^3-c_2c_3)e_n^4+o(e_n^5) \tag{19}$$

The new method has fourth-order convergence, which can be expressed as

$$x_{n+1}=x_n-\Big(3\frac{f(x_n)+f(x^*_{n+1})}{f'(x_n)} -2\frac{f(x_n)+2f(x^*_{n+1})}{f(x_n)+f(x^*_{n+1})}\frac{f(x_n)}{f'(x_n)}\Big) \tag{20}$$

where $x^*_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$ It is easy to know that the Eq. 20 requires two evaluations of the function and one evaluation of its first derivative. If we consider the definition of efficiency index [13] as $p^{\frac{1}{m}}$, where p is the order of the method and m is the number of function evaluations per iteration required by the method, the method defined by Eq. 20 has the efficiency index equal to $4^{\frac{1}{3}}\approx 1.587$, which is better than Newton's method $2^{\frac{1}{2}}\approx 1.414$.

3. Numerical examples

In this section, the methods including Classical Newton's method (NM) defined by Eq. 1, Potra's method (PM) defined by Eq. 2, Chun's method (CM) defined by Eq. 5 and the new method proposed in this paper (CNK) defined by Eq. 20 are applied to solve some non-linear equations, and the efficiency and performance of them are compared. Depending on the precision of the computer, an approximate solution rather than the exact root is accepted. A stopping criterion is used for the computing programs. When the stopping criterion is satisfying,x_{n+1} is taken as the solution. The stopping criterion is set as 10^{-14} here. The following test functions, referred in reference [13], are used, and the rootα for each function that we have computed is displayed as:

$$f_1(x) = x^3 + 4x^2 - 10 \quad \alpha = 1.365230013414097 \quad (21)$$

$$f_2(x) = x^2 - e^x - 3x + 2 \quad \alpha = 0.2575302854398607 \quad (22)$$

$$f_3(x) = xe^{x^2} - sin^2(x) + 3cos(x) + 5 \quad (23)$$
$$\alpha = -1.207647827130919$$

$$f_4(x) = (x-1)^2 - 1 \quad \alpha = 2.0000000000000000 \quad (24)$$

$$f_5(x) = (x-1)^3 - 2 \quad \alpha = 2.259921049894873 \quad (25)$$

$$f_6(x) = (x-1)^6 - 1 \quad \alpha = 2.000000000000000 \quad (26)$$

$$f_7(x) = cos(x) - x \quad \alpha = 0.7390851332151606 \quad (27)$$

$$f_8(x) = sin^2(x) - x^2 + 1 \quad \alpha = 1.404491648215341 \quad (28)$$

$$f_9(x) = e^{x^2+7x-30} + 1 \quad \alpha = 3.0000000000000000 \quad (29)$$

The number of iterations (n) and function evaluations (NFE) are shown in Table 1 and Table 2. The computational results indicate that the total number of functional evaluations and iterations required for the CNK method is less than NM method, PM method and CM method. The proposed method converges more rapidly than Newton's method and the other two methods.

Table 1 The number of iterations of various iterative methods

f(x)	x_0	NM	PM	CM	CNK
f_1	1	5	4	4	3
	2	5	4	4	3
f_2	2	5	4	4	3
	3	6	4	4	4
f_3	-3	14	10	10	9
	-2	8	6	6	5
f_4	2.5	5	3	4	3
	3.5	6	4	4	3
f_5	3	6	4	4	3
	3.2	6	4	4	4
f_6	2.1	5	3	4	3
	3	9	6	6	5
f_7	0.5	4	3	3	3
	1	4	3	3	2
f_8	1.5	4	3	3	2
	2	5	4	4	3
f_9	3.25	8	6	6	5
	3.5	12	8	9	7

Table 2 Function evaluations (NFE) of various iterative methods

f(x)	x_0	NM	PM	CM	CNK
f_1	1	10	12	12	9
	2	10	12	12	9
f_2	2	10	12	12	9
	3	12	12	12	12
f_3	-3	28	28	30	27
	-2	16	18	18	15
f_4	2.5	10	9	12	9
	3.5	12	12	12	9
f_5	3	12	12	12	9
	3.2	12	12	12	12
f_6	2.1	10	9	12	9
	3	18	18	18	15
f_7	0.5	8	9	9	9
	1	8	9	9	6
f_8	1.5	8	9	9	6
	2	10	12	12	9
f_9	3.25	16	18	18	15
	3.5	24	24	27	21

4. Application of the new method

Channel regulation is one of the most important methods to keep a good navigation condition for rivers. On some rivers, channel regulations are carried out every year. To verify the feasibility of the regulation schemes, hydraulic computation on the regulation projects is necessary. Water surface profile is the key hydraulic parameter on channel regulation, and some studies have been done on calculating the water surface profile [14]. The classical method to calculate the water surface profile is by the following equation

$$Z_{(2)}+\frac{\alpha_{(2)}V_{(2)}^2}{2g}=Z_{(1)}+\frac{\alpha_{(1)}V_{(1)}^2}{2g}+\frac{1}{2}\Big(\frac{n^2V_{(1)}^2}{R_{(1)}^{\frac{4}{3}}}+\frac{n^2V_{(2)}^2}{R_{(2)}^{\frac{4}{3}}}\Big)\Delta L \quad (30)$$

where $Z_{(2)}$ is the upstream section water level, $Z_{(1)}$ is the downstream section water level, $V_{(2)}$ is the average velocity of the upstream section, $V_{(1)}$ is the average velocity of the downstream section, $\alpha_{(2)}$ is the upstream section momentum coefficient, $\alpha_{(1)}$ is the downstream section momentum coefficient, and g is the gravitational acceleration. ΔL is the distance between the upstream section and the downstream section. From Eq. 30, a new function can be constructed as

$$f(Z)=Z+\frac{\alpha V^2}{2g}-\frac{1}{2}\frac{n^2V^2}{R^{\frac{4}{3}}}\Delta L-\Big(Z_{(1)}+\frac{\alpha_{(1)}V_{(1)}^2}{2g}+\frac{n^2V_{(1)}^2}{R_{(1)}^{\frac{4}{3}}}\Delta L\Big) \quad (31)$$

And $Z_{(2)}$ is the solution of formula Eq. 31. Again, from Eq. 31 it can get

$$f'(Z)=1-\frac{\alpha Q^2B}{gA^3}+\frac{n^2V^2}{R^{\frac{4}{3}}}\Delta L\Big(\frac{B}{A}+\frac{2}{3}\frac{dR}{RdZ}\Big) \quad (32)$$

where R is hydraulic radius, Z is water level and H is water depth. Substituting Eq. 32 into the following Eq. 33, we can get the CNK method's solution as

$$Z_{n+1}=Z_n-\Big(3\frac{f(Z_n)+f(Z_{n+1}^*)}{f'(Z_n)}-2\frac{f(Z_n)+2f(Z_{n+1}^*)}{f(Z_n)+f(Z_{n+1}^*)}\frac{f(Z_n)}{f'(Z_n)}\Big) \quad (33)$$

Select a river is 2550 meter long, and is divided into four sections. The characteristic parameters about the river are list in table 3 [15].

The discharge is 700 m^3/s , and roughness coefficient is 0.025. Downstream water level is 43.9 m, $\alpha=1$ and g=9.81 m/s^2. The water surface profile of the river is required to calculate. According to the data, Eq. 34 is adopted to calculate the water surface profile of the river. The computing precision is controlled by the following formula [16].

$$-0.00001\leq\frac{Z_{n+1}-Z_n}{Z_{n+1}}\leq 0.00001 \quad (34)$$

Table 3 Characteristics of the river

		Elevation (m)	Surface width (m)	Area (m^2)	Interval (m)
	1	41	16	30	
Section 1	2	43.5	360	500	0
	3	44	465	706.3	
	4	44.5	510	950	
	1	42	407	507	
Section 2	2	43.5	583	1249.5	575
	3	44	690	1567.8	
	4	45	815	2320.3	
	1	43	488	482	
Section 3	2	43.8	554	898.8	1140
	3	44.3	612	1190.3	
	4	45	674	1640.4	
	1	42.6	303	355	
Section 4	2	43.8	427	793	1840
	3	44.4	491	1068.4	
	4	45	532	1375.3	
	1	42.2	553	547	
Section 5	2	43.7	637	1477	2550
	3	44.2	735	1832.5	
	4	44.8	781	2287.3	

Table 4 and Table 5 is the computing results by various iterative methods. The results indicate that the water surface profiles calculated by Newton's method and CNK method are very close. They all well meet to the precision required. But the iterative times by CNK method are two, which is less than that by Newton's method. The CNK method has a more rapid convergence speed than Newton's method in calculating water surface profile, especially in a long river with large numbers of computing sections.

Table 4 Computing results of Newton's method

Section number	Water level (m)	Iterative times
1	43.90000	0
2	44.07402	3
3	44.10602	3
4	44.18548	3
5	44.25760	3

5. Conclusions

(1) CNK method proposed in the paper has fourth-order convergence and requires two evaluations of the function and one evaluation of its first derivative. Through the

Table 5 Computing results of CNK method

Section number	Water level (m)	Iterative times
1	43.90000	0
2	44.07402	2
3	44.10603	2
4	44.18548	2
5	44.25761	2

efficiency index analysis and numerical examples comparison, the method is indicated to be better than Potra's method and the other representative methods in the paper.

(2) CNK method can be used to calculate water surface profile of a river. The result proves it highly efficient. Compared with Newton's method, it can decrease the iterative times and increase computation speed.

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