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Convergence of Cubic Renormalized FEM for Linear Elliptic Equations with L^1-Data

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Abstract: This paper contains a study based on the usual **cubic** FEM by which we extend all results obtained in [3,4] where the case of P_n $(n=1,2)$ renormalized FEM is considered to approximate the solution of linear elliptic equation with $L^\infty(\Omega)$-coefficients, L^1-data and which generalizes Laplace's equation.
By introducing a same techniques adopted in [3,4], where the dimension is $d=2,3$, the convergence for the unique discrete solution in $W_0^{1,q}(\Omega)$ for every $q\in[1,d/(d-1)[$ to the unique renormalized solution of the problem are proved, and the estimates of the error are derived. Thereby, similarly as the previous studies, in the case of a bounded Radon measure data, a weaker result is obtained. An error estimate in $W_0^{1,q}(\Omega)$ for smooth coefficients and $L^r(\Omega)$-data such that $T_k(f)\in H^1(\Omega)$ for all (k,r) in $\mathbb{R}^{+*}\times]1,\infty[$, is given.

Keywords: Cubic FEM, L^1-data, renormalized solution, diagonally dominant matrix, error estimate.

1 Introduction

Let Ω be an open bounded domain in $\mathbb{R}^d$, d = 2, 3, with the boundary $\partial\Omega$. A particular case is where Ω is an open bounded polyhedron. Let A be a coercive matrix with $L^\infty(\Omega)$-coefficients and f be a given $L^1(\Omega)$-data.

We consider the $\mathbb{P}_3$-finite element approximation (for short, a $\mathbb{P}_3$-FEA) of the Dirichlet problem:

$$\begin{cases} -\operatorname{div}(A\nabla u)=f \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

which, for a triangulation $\mathscr{T}_h$ of Ω, the usual $\mathbb{P}_3$ -FEA of (1) is the following:

$$\begin{cases} u_h\in V_h, \\ \forall v_h\in V_h,\ a(\nabla v_h,\nabla u_h)=(f,v_h), \end{cases} \tag{2}$$

with

$$V_h=\left\{v_h\in\mathscr{C}^0(\overline{\Omega}):\ \forall T\in\mathscr{T}_h,\ v_{h|T}\in\mathbb{P}_3,\ v_{h|\partial\Omega}=0\right\}, \tag{3}$$

has a unique solution, since the right-hand side of (2) $\int_\Omega f v_h dx$ is well defined for a $L^1(\Omega)$-data. Generally, in this case, one can't guarantee that the solution of (1) belongs to $H_0^1(\Omega)$. To overcome this problem, one has to consider the class of renormalized solution (for short r.s.) (cf. [1,2]), what gives the well-posedness of (1) in line with Hadamard.

The convergence of the unique solution u_h of (2) to the unique r.s. u of (1) is proved when $n=1,2$, respectively in [3,4] namely

$$\begin{cases} u_h \underset{h\to 0}{\longrightarrow} u \quad \text{strongly in } W_0^{1,q}(\Omega), \\ \Pi_h(T_k(u_h)) \underset{h\to 0}{\longrightarrow} T_k(u) \quad \text{strongly in } H_0^1(\Omega), \end{cases} \tag{4}$$

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for every $(k,q) \in \mathbb{R}^{+*} \times [1, d/(d-1)[$, where Π_h stands for the usual Lagrange interpolation operator in V_h and T_k stands for the usual truncation at height k.

For this purpose, the family of triangulations $\mathscr{T}_h$ is assumed to be regular in line with P.G. Ciarlet [5] and satisfies a similar hypothesis to the one verifying the discrete maximum principle.

It resulted an $O\left(h^{2\left(1-r^{-1}\right)}\right)$ error estimate in $W_0^{1,q}(\Omega)$ for a $L^r(\Omega)$-data, $r \in]1,2[$ and when the matrix A has a smooth coefficients.

In the same papers, the authors obtained a weaker convergence of the unique solution u_h (subsequence still denoted by h) of (2) with data a bounded Radon measure as follows:

$$\begin{cases} u_h \underset{h\to 0}{\rightharpoonup} u \quad \text{weakly in } W_0^{1,q}(\Omega), \\ \Pi_h(T_k(u_h)) \underset{h\to 0}{\rightharpoonup} T_k(u) \quad \text{weakly in } H_0^1(\Omega), \end{cases} \tag{5}$$

for every $(k,q) \in \mathbb{R}^{+*} \times [1, d/(d-1)[$, where u is a solution of

$$\begin{cases} u \in W_0^{1,q}(\Omega), T_k(u) \in H_0^1(\Omega), \\ -\text{div}(A\nabla u) = f \text{ in } \mathscr{D}'(\Omega). \end{cases} \tag{6}$$

This paper is arranged with the same technique adopted in [3,4] as follows: Section 2 presents Mathematical preliminaries of r.s., Problem formulation and Main result of $\mathbb{P}_3$-FEA for $d \in \{2,3\}$. In Section 3, a various results are proved to be used in the proof of our Main result seen in Section 4, and this paper is discussed and concluded in Section 5. Results of FEA case in [3,4] remain valid in our case, and a $4(1-r^{-1})^{th}$ order estimation error in $W_0^{1,q}(\Omega)$, when the coefficients of the matrix A are smooth and when f belongs to $L^r(\Omega)$ verifying $T_k(f) \in H^1(\Omega)$ for every (k,r) in $\mathbb{R}^{+*} \times]1,2[$, is deduced.

Notations

In the present paper, Axy means the scalar product of the vector Ax by the vector y.

The measure of a subset $S \subset \Omega$ is represented by $|S|$, its complement by S^c.

We keep the same standard notations of Sobolev spaces and their norms .

$\mathscr{M}_b(\Omega)$ represents the space of Radon measures on Ω with total bounded variation.

For every r in $]1,\infty[$, $L^{r,\infty}(\Omega)$ designates the Marcinkiewicz space equipped with the norm

$$\|v\|_{L^{r,\infty}(\Omega)} = \sup_{\mu>0} \mu \left|\{x \in \Omega : |v(x)| \geq \mu\}\right|.$$

We use the truncation T_k $(k>0)$, as defined in [3,4], namely

$$T_k(s) = \begin{cases} s & \text{for } |s| \leq k, \\ sgn(s)k & \text{else.} \end{cases}$$

For each $d-$simplex T in $\mathbb{R}^d$, we use the following notations:

- $a_{i,T}$ and $\lambda_{i,T}$ refer respectively to a vertex of T and its barycentric coordinates for every $i \in [\![0,d]\!]$;
- $a_{i,j,T} = \frac{1}{3}(2a_{i,T} + a_{j,T})$ for all (i,j) in $[\![0,d]\!]^2$ s.t $i<j$;
- $a_{i,j,T} = \frac{1}{3}(a_{i,T} + 2a_{j,T})$ for all (i,j) in $[\![0,d]\!]^2$ s.t $j<i$;
- $a_{i,j,k,T} = \frac{1}{3}\left(a_{i,T} + a_{j,T} + a_{k,T}\right)$ for all (i,j,k) in $[\![0,d]\!]^3$ s.t $i<j<k$;
- $\ddot{T} = \bigcup_{0\leq i<j<k\leq d} \left\{a_{i,T}, a_{i,j,T}, a_{j,i,T}, a_{i,j,k,T}\right\}$ represents the set of all vertices, points on edges and points on faces of T;
- the local basis is given by

$$\begin{cases} \varphi_{i,T} = \frac{1}{2}\lambda_{i,T}(3\lambda_{i,T}-1)(3\lambda_{i,T}-2),\ 0 \leq i \leq d, \\ \varphi_{i,j,T} = \frac{9}{2}\lambda_{i,T}(3\lambda_{i,T}-1)\lambda_{j,T},\ 0 \leq i < j \leq d, \\ \varphi_{i,j,T} = -\frac{9}{2}\lambda_{i,T}(3\lambda_{i,T}-2)\lambda_{j,T},\ 0 \leq j < i \leq d, \\ \varphi_{i,j,k,T} = 27\lambda_{i,T}\lambda_{j,T}\lambda_{k,T},\ 0 \leq i < j < k \leq d, \end{cases} \tag{7}$$

with $\varphi_{i,T}$, $\varphi_{i,j,T}$ and $\varphi_{i,j,k,T}$ are $\mathbb{P}_3$ shape functions related to T; and one can easily obtain for every $x \in T$:

$$\sum_{i=0}^{d} \varphi_{i,T}(x) + \sum_{\substack{i,j=0 \\ i\neq j}}^{d} \varphi_{i,j,T}(x) + \sum_{\substack{i,j,k=0 \\ i<j<k}}^{d} \varphi_{i,j,k,T}(x) = 1.$$

2 Statement of the main result

Consider the following hypothesis:

$$A(.) \in L^\infty(\Omega)^{d\times d}, \tag{8}$$

$$\alpha > 0, \forall y \in \mathbb{R}^d,\ \alpha|y|^2 \leq A(x)yy, \text{ a.e } x \in \Omega, \tag{9}$$

$$f \in L^1(\Omega), \tag{10}$$

and a function u is the r.s. of the problem (1), i.e. u satisfies the four following hypothesis, namely

$$u \in L^1(\Omega), \tag{11}$$

$$\forall k > 0, T_k(u) \in H_0^1(\Omega), \tag{12}$$

$$\lim_{k\to\infty} \frac{1}{k}\int_\Omega |\nabla T_k(u)|^2 \,dx = 0, \tag{13}$$

and

$$\begin{cases} \forall k > 0, \forall S \in \mathscr{C}_c^1(\mathbb{R}) \text{ with } \operatorname{supp}(S) \subset [-k,k], \\ \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ a\left(\nabla T_k(u), \nabla v S(u) + \nabla T_k(u) S'(u) v\right) = (S(u), v), \end{cases} \tag{14}$$

for each $h > 0$ a family of triangulation $\mathscr{T}_h$ is a finite collection of d-simplices T (triangles if $d = 2$, tetrahedra if $d = 3$) s.t.:

$$\begin{cases} \text{(i) } \Omega_h = \bigcup\{T : T \in \mathscr{T}_h\} \subset \overline{\Omega}, \\ \text{(ii) for every compact set } E \subset \Omega, \exists h_{0,E} > 0 \text{ s.t.} \\ E \subset \Omega_h \text{ for every } h < h_{0,E}, \\ \text{(iii) for } (T,T') \in \mathscr{T}_h^2 \text{ with } T \neq T': \left|T \cap T'\right| = 0, \\ \text{(iv) if F is a face of } T \in \mathscr{T}_h\text{: } F \subset \partial\Omega_h \text{ , or } F \subset T' \in \mathscr{T}_h. \end{cases} \tag{15}$$

The conformity of this triangulation is ensured by (iv). It could be taken as a particular case, when Ω is a polyhedron of $\mathbb{R}^d$, and where Ω_h coincides with Ω for every h,

$$h = \sup_{T\in\mathscr{T}_h} h_T \text{ decreases to zero}, \tag{16}$$

where h_T denotes the diameter of T,

we also assume the regularity of the family of triangulations $\mathscr{T}_h$ in the sense of ciarlet namely,

$$\forall h,\ \forall T \in \mathscr{T}_h,\ \frac{h_T}{\rho_T} \leq \sigma. \tag{17}$$

where σ is a positive constant and ρ_T denotes the diameter of the ball inscribed in T.

When $f \in H^{-1}(\Omega) \cap L^1(\Omega)$, as known in [1,2], one can confirm that the usual weak solution of (1), namely

$$\begin{cases} u \in H_0^1(\Omega), \\ \forall v \in H_0^1(\Omega),\ a(\nabla u, \nabla v) = (f, v), \end{cases} \tag{18}$$

is also a r.s. of (1) and conversely. The purpose of inserting this class of r.s. is to ensure the following existence, uniqueness and continuity theorem ([1,3,4]).

Theorem 1. *If A and f satisfy (8), (9) and (10), the problem (1) admits a unique r.s. $u \in W_0^{1,q}(\Omega)$ for every $q \in [1, d/(d-1)[$ with continued dependency on the data f in the following sense: let f^ε be a sequence s.t.*

$$f^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} f \text{ strongly in } L^1(\Omega),$$

then, the sequence u^ε of the r.s. of (1) for the data f^ε satisfies for every $(k,q) \in \mathbb{R}^{++} \times [1, d/(d-1)[$,

$$T_k(u^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} T_k(u) \text{ strongly in } H_0^1(\Omega),$$

$$u^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u \text{ strongly in } W_0^{1,q}(\Omega).$$

Finally, for $i = 1,2$, if $f_i \in L^1(\Omega)$, and if u_i are the r.s. of (1) for the data f_i, then, for every (k,q) in $\mathbb{R}^{++} \times [1, d/(d-1)[$, one has

$$T_k(u_1 - u_2) \in H_0^1(\Omega),$$

$$\alpha \left\|T_k(u_1 - u_2)\right\|^2_{H_0^1(\Omega)} \leq k \left\|f_1 - f_2\right\|_{L^1(\Omega)},$$

$$\alpha \left\|u_1 - u_2\right\|_{W_0^{1,q}(\Omega)} \leq C(d, |\Omega|, q) \left\|f_1 - f_2\right\|_{L^1(\Omega)}, \tag{19}$$

where the constant $C(d, |\Omega|, q) \neq C(h)$.

On every triangulation $\mathscr{T}_h$, we define the space V_h of $H_0^1(\Omega)$ by

$$V_h = \left\{ v_h \in \mathscr{C}^0(\overline{\Omega}) : v_h = 0 \text{ in } \overline{\Omega} \setminus \mathring{\Omega}_h, \forall T \in \mathscr{T}_h, v_h|_T \in \mathbb{P}_3 \right\}. \tag{20}$$

For every h, we denote by Γ_h the set of all vertices and midpoints of the d-simplices T of $\mathscr{T}_h$, and by $\mathring{\Gamma}_h$ the set of all interior vertices and midpoints of the d-simplices T of $\mathscr{T}_h$, namely

$$\Gamma_h = \bigcup\{\ddot{T} : T \in \mathscr{T}_h\} \text{ and } \mathring{\Gamma}_h = \Gamma_h \cap \mathring{\Omega}_h. \tag{21}$$

It is known that

$$dim(V_h) = card(\mathring{\Gamma}_h).$$

For reasons of simplification in terms of notation we put

$$\mathring{\Gamma}_h = \left\{a_1, a_2, ..., a_{N_h}\right\},$$

and

$$\Gamma_h = \left\{a_1, a_2, ..., a_{N_h}, a_{N_h+1}, ..., a_{N_h+b_h}\right\}.$$

The Lagrange basis $\left(\varphi_1, \varphi_2, ..., \varphi_{N_h}\right)$ of V_h related to $\mathring{\Gamma}_h$ is defined as follows

$$\begin{cases} \varphi_i \in \mathscr{C}^0(\Omega_h), \quad \varphi_i|_T \in \mathbb{P}_3(T) \text{ for every } T \in \mathscr{T}_h, \\ \varphi_i(a_i) = 1 \quad \text{and } \varphi_i(a_j) = 0 \text{ for every } a_j \in \Gamma_h,\ a_j \neq a_i, \\ \varphi_i|_{\overline{\Omega}\setminus\mathring{\Omega}_h} = 0, \end{cases} \tag{22}$$

for every $i \in [\![1, N_h]\!]$.

Using the notations of the previous section, we can explicit the φ_i's; for every $a_k \in \mathring{\Gamma}_h$ and for every $T \in \mathscr{T}_h$, if $a_k \notin \ddot{T}$, then, $\varphi_k \mid_T = 0$ else $\varphi_k \mid_T = \varphi_{i,T}$ in the case $a_k = a_{i,T}$ and $\varphi_k \mid_T = \varphi_{i,j,T}$ in the case $a_k = a_{i,j,T}$.

We define the interpolation operator Π_h by

$$\begin{cases} \forall v \in \mathscr{C}^0\left(\overline{\Omega}\right) \text{ with } v = 0 \text{ in } \overline{\Omega} \setminus \mathring{\Omega}_h, \\ \Pi_h(v) = \sum_{i=1}^{N_h} v(a_i)\,\varphi_i\,. \end{cases}$$

Finally, we define the $N_h \times N_h$ stiffness matrix $Q = (Q_{ij})$, namely

$$Q_{ij} = a(\nabla\varphi_i, \nabla\varphi_j) \text{ for } i, j \in [\![1, N_h]\!]. \tag{23}$$

As in [3], the main assumption of the present paper is that Q is a diagonally dominant matrix, namely

$$\forall i \in [\![1, N_h]\!] : Q_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{N_h} \left|Q_{ij}\right| \geq 0. \tag{24}$$

For the $\mathbb{P}_1$ finite elements approximation, this assumption is close to the usual assumption which ensures that the discrete maximum principle holds true (see [3] for more details).

For every triangulation $\mathscr{T}_h$, we consider the solution u_h of

$$\begin{cases} u_h \in V_h, \\ \forall v_h \in V_h : a(\nabla u_h, \nabla v_h) = (f, v_h). \end{cases} \tag{25}$$

Note that the right-hand side of (25) makes sense since $V_h \subset L^\infty(\Omega)$. The solution u_h of (25) exists and is unique.

As in [3], the main result of this paper is the following.

Theorem 2. *Assume that A, f and $\mathscr{T}_h$ satisfy (8), (9), (10), (15), (16), (17) and (24). Then, the unique solution u_h of (25) satisfies for every (k,q) in $\mathbb{R}^{+*} \times [1, d/(d-1)[$,*

$$u_h \underset{h\to 0}{\longrightarrow} u \quad \textit{strongly in } W_0^{1,q}(\Omega),$$

$$\Pi_h(T_k(u_h)) \underset{h\to 0}{\longrightarrow} T_k(u) \quad \textit{strongly in } H_0^1(\Omega),$$

where u is the unique r.s. of (1).

We will follow the same approach as that adopted by the authors in [3]. Whenever deemed necessary, and for the reader's convenience, we reproduce the adaptation of the proofs given in [3] even when only small changes are needed.

This Theorem will be proved in Section 4, using the tools that we will prepare in Section 3. In Section 5, we will explain why the results of [3] about error estimate and the case where f is a bounded Radon measure, remain valid in our case. We also show that we obtain an $O\left(h^{4(1-r^{-1})}\right)$ error estimate in $W_0^{1,q}(\Omega)$, if we assume in addition that $T_k(f) \in H^1(\Omega)$ for every $k > 0$.

3 Tools

To prove the theorem 2, we begin by proving the following result which is a piecewise $\mathbb{P}_3$ variant of a result of ([2,3,4,8]).

Theorem 3. *Assume that $v_h \in V_h$ satisfies*

$$\forall k > 0, \quad \int_\Omega |\nabla \Pi_h(T_k(v_h))|^2 \,\mathrm{d}x \leq kM \tag{26}$$

for some $M > 0$. Then, for every $q \in [1, d/(d-1)[$

$$\|v_h\|_{W_0^{1,q}(\Omega)} \leq c(d, |\Omega|, q)\,M \tag{27}$$

where the constant $c(d, |\Omega|, q) \neq c(h)$.

The proof of this theorem, as in [3,4], needs the following lemmas.

Lemma 1. *Let $v_h \in V_h$ and let $k > 0$. If for some $T \in \mathscr{T}_h$ there exists $y \in T$ with $|v_h(y)| \geq k$, then, there exists a d-simplex $S \subset T$ with $|S| = c(d)\,|T|$ such that*

$$\forall x \in S : |\Pi_h(T_k(v_h(x)))| \geq \frac{k}{2},$$

where the strictly positive constant $c(d) \neq c(h)$.

Proof. By applying Mean Value Theorem to the polynomial function v in $\mathbb{P}_3\left(\mathbb{R}^d\right)$, one have

$$v(x) - v(y) = \nabla v(\theta)(x - y), \tag{28}$$

for all x, y in $\mathbb{R}^d$, where $\theta = x + t(x - y)$ with $t \in]0,1[$.

Consider now,

$$w_{h,k} = \Pi_h(T_k(v_h)) \in \mathbb{P}_3(T),$$

where $T \in \mathscr{T}_h, v_h \in V_h$ and $k > 0$, s.t.

$$\sup_T |v_h| \geq k.$$

At first, one can confirm that $|w_{h,k}| \leq k$ on $\ddot{T}$, but contrary to its property of being bounded on T by k in the affine case [3], the present cubic case, imposes the existence of some $y \in T$ s.t. $\left|w_{h,k}(y)\right| \geq k$. Indeed, and as [4], this can be justified by the fact that

$$\begin{cases} w_{h,k} = v_h, \text{ if } |v_h| < k \text{ on } \ddot{T}, \textit{else} \\ |w_{h,k}(y)| = k, \text{ for some } y \in \ddot{T}. \end{cases}$$

Now on, without any confusion, one can assume that the inequality

$$\lambda_{0,T}(y) \geq \frac{1}{128},$$

holds for $d \in \{2;3\}$.

Since

$$x - y = \sum_{i=1}^{d} (\lambda_{i,T}(x) - \lambda_{i,T}(y))\,(a_{i,T} - a_{0,T}),$$

and from (28), we obtain for all $x \in T$:

$$w_{h,k}(x) - w_{h,k}(y) = = \sum_{i=1}^{d} (\lambda_{i,T}(x) - \lambda_{i,T}(y)) \nabla w_{h,k}(\theta)(a_{i,T} - a_{0,T}) \tag{29}$$

Moreover, for all $x \in T$, (see (7)):

$$w_{h,k}(x) = \sum_{i=0}^{d} w_{h,k}(a_{i,T}) \varphi_{i,T}(x) + \sum_{\substack{i,j=0 \\ i \neq j}}^{d} w_{h,k}(a_{i,j,T}) \varphi_{i,j,T}(x) + \sum_{\substack{i,j,k=0 \\ i<j<k}}^{d} w_{h,k}(a_{i,j,k,T}) \varphi_{i,j,k,T}(x).$$

Since $\lambda_{i,T} \in \mathbb{P}_1(\mathbb{R}^d)$, $\nabla \lambda_{i,T}$ is a constant vector in $\mathbb{R}^d$ and

$$\lambda_{i,T}(a) - \lambda_{i,T}(b) = \nabla \lambda_{i,T}(a-b)$$

for all a, b in $\mathbb{R}^d$.

So for all $i \neq j \neq k$ in $[\![0,d]\!]$

$$\begin{cases} \nabla \lambda_{i,T}(a_{i,T} - a_{j,T}) = 1, \\ \nabla \lambda_{i,T}(a_{j,T} - a_{k,T}) = 0, \end{cases} \tag{30}$$

From (7), one can compute the gradient vector of basis related on each T as follows:

$$\begin{cases} \nabla \varphi_{i,T} = \frac{1}{2}(27\lambda_{i,T}^2 - 18\lambda_{i,T} + 2)\nabla \lambda_{i,T}, \\ \nabla \varphi_{i,j,T} = \frac{9}{2}\Big[\Big(6\lambda_{i,T} - 1\Big)\lambda_{j,T}\nabla \lambda_{i,T} + \\ \qquad + (3\lambda_{i,T} - 1)\lambda_{i,T}\nabla \lambda_{j,T}\Big], \\ \nabla \varphi_{i,j,k,T} = 27\lambda_{i,T}\lambda_{j,T}\nabla \lambda_{k,T}. \end{cases} \tag{31}$$

Therefore,

$$\begin{cases} \nabla w_{h,k} = \frac{1}{2}\sum_{i=0}^{d} w_{h,k}(a_{i,T})(27\lambda_{i,T}^2 - 18\lambda_{i,T} + 2)\nabla \lambda_{i,T} + \\ \quad + \frac{9}{2}\sum_{\substack{i,j=0 \\ i \neq j}}^{d} w_{h,k}(a_{i,j,T})\Big[\Big(6\lambda_{i,T} - 1\Big)\lambda_{j,T}\nabla \lambda_{i,T} + \\ \quad + (3\lambda_{i,T} - 1)\lambda_{i,T}\nabla \lambda_{j,T}\Big] + \\ \quad + 27\sum_{\substack{i,j,k=0 \\ i \neq j \neq k}}^{d} w_{h,k}(b_{i,j,k,T})\lambda_{i,T}\lambda_{j,T}\nabla \lambda_{k,T}, \end{cases} \tag{32}$$

where, for $i_1 < i_2 < i_3$ and $\{i_1, i_2, i_3\} = \{i, j, k\}$:

$$b_{i,j,k} = a_{i_1,i_2,i_3,T}.$$

Taking into account (30), one can have, for all $x \in T$ and $m \in [\![0,d]\!]$,

$$\begin{aligned} &\nabla w_{h,k}(x)(a_{m,T} - a_{0,T}) = \\ &= \frac{1}{2} w_{h,k}(a_{m,T})\left[\lambda_{m,T}(x)(27\lambda_{m,T}(x) - 18) + 2\right] + \\ &- \frac{1}{2} w_{h,k}(a_{0,T})\left[\lambda_{0,T}(x)(27\lambda_{0,T}(x) - 18) + 2\right] + \\ &+ \frac{9}{2}\Bigg[\sum_{\substack{j=0 \\ j \neq m}}^{d} w_{h,k}(a_{m,j,T})\lambda_{j,T}(x)(6\lambda_{m,T} - 1) + \\ &+ \sum_{\substack{i=0 \\ i \neq m}}^{d} w_{h,k}(a_{i,m,T})\lambda_{i,T}(x)(3\lambda_{i,T} - 1)\Bigg] + \\ &- \frac{9}{2}\Bigg[\sum_{j=1}^{d} w_{h,k}(a_{0,j,T})\lambda_{j,T}(x)(6\lambda_{0,T} - 1) + \\ &+ \sum_{i=1}^{d} w_{h,k}(a_{i,0,T})\lambda_{i,T}(x)(3\lambda_{i,T} - 1)\Bigg] + \\ &+ 27\Bigg[\sum_{\substack{i,j,m=0 \\ i \neq j \neq m}}^{d} w_{h,k}(b_{i,j,m,T})\lambda_{i,T}(x)\lambda_{j,T}(x) + \\ &- \sum_{\substack{i,j,=1 \\ i \neq j}}^{d} w_{h,k}(b_{i,j,0,T})\lambda_{i,T}(x)\lambda_{j,T}(x)\Bigg]. \end{aligned}$$

Thus, since $|w_{h,k}| \leq k$ in $\ddot{T}$, one can obtain:

$$|\nabla w_{h,k}(x)(a_{m,T} - a_{0,T})| \leq (11 + 45 + 18 + 54)k \leq 128\,k, \tag{33}$$

Back to (29) and with (33), one can deduce for all $x \in T$:

$$|w_{h,k}(y) - w_{h,k}(x)| \leq 128\,k \sum_{i=1}^{d} |\lambda_{i,T}(y) - \lambda_{i,T}(x)|.$$

Finally, in order to have $|w_{h,k}(x)| \geq \frac{k}{2}$ it suffices to have

$$\sum_{i=1}^{d} |\lambda_{i,T}(x) - \lambda_{i,T}(y)| \leq \frac{1}{256}.$$

Let S be the $d-$simplex contained in T and similar to T defined as follows:

$$S = (y - a_{0,T}) + S_0,$$

where

$$S_0 = \left\{ x \in T : \lambda_{0,T}(x) \geq 1 - \frac{1}{256} = \frac{255}{256} \right\},$$

so, one can claim that

$$|S| = |S_0|.$$

Furthermore, if we consider the invertible affine application Φ_T s.t.

$$\Phi_T(\hat{T}) = T,$$

where $\hat{T}$ is the reference unit $d-$simplex, then

$$\Phi_T^{-1}(S_0) = \left\{ x \in \hat{T} : \Phi_T^{-1} \circ \lambda_{0,T}(x) \geq \frac{255}{256} \right\},$$

which allows one to claim that

$$C(d) = \frac{\left|\Phi_T^{-1}(S_0)\right|}{\left|\Phi_T^{-1}(T)\right|} \neq C(h).$$

This proves the result. □

Lemma 2. *Assume that $v_h \in V_h$ satisfies (26), then,*

$$|B_k(v_h)| \leq C(d, |\Omega|, 2^*) \left(\frac{M}{k}\right)^{\frac{2^*}{2}}, \tag{34}$$

for every $k > 0$, where $2^* = \dfrac{2d}{d-2} = 6$ if $d = 3$ and 2^* is any real number with $2^* \geq 1$ if $d = 2$; $B_k(v_h)$ is defined by

$$B_k(v_h) = \bigcup \left\{ T \in \mathscr{T}_h : \max_T |v_h| \geq k \right\}, \tag{35}$$

$c(d, |\Omega|, 2^*) \neq c(h)$.

Proof. Sobolev's theorem asserts that,

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{L^{2^*}(\Omega)} \leq \mu_S \|\nabla v\|_{L^2(\Omega)^d}, \tag{36}$$

here $d = 2$ or $d = 3$ so $2^* = \dfrac{2d}{d-2} = 6$ if $d = 3$ and 2^* can be any real number with $1 \leq 2^*$.

Fix $k > 0$. If $T \subset B_k(v_h)$, from lemma 1, we know that there exists $S \subset T$, with $|S| = c(d)|T|$ and

$$\forall x \in S, \quad |\Pi_h(T_k(v_h(x)))| \geq \frac{k}{2}.$$

Therefore

$$\int_T |\Pi_h(T_k(v_h(x)))|^{2^*} \mathrm{d}x \geq \int_S |\Pi_h(T_k(v_h(x)))|^{2^*} \mathrm{d}x \geq c(d)|T| \left(\frac{k}{2}\right)^{2^*}.$$

Hence,

$$|B_k(v_h)| = \sum_{T \subset B_k(v_h)} |T| \leq \frac{1}{c(d)} \left(\frac{2}{k}\right)^{2^*} \int_\Omega |\Pi_h(T_k(v_h(x)))|^{2^*} \mathrm{d}x.$$

This combined with (36) yields

$$|B_k(v_h)| \leq \frac{1}{c(d)} \left(\frac{2\mu_S}{k}\right)^{2^*} \left(\int_\Omega |\nabla \Pi_h(T_k(v_h(x)))|^2 \mathrm{d}x\right)^{\frac{2^*}{2}}.$$

So, from (26), one can observe that

$$|B_k(v_h)| \leq \frac{(2\mu_S)^{2^*}}{c(d)} \left(\frac{M}{k}\right)^{\frac{2^*}{2}},$$

which is (34) with $c(d, |\Omega|, 2^*) = \dfrac{(2\mu_S)^{2^*}}{c(d)} \neq c(h)$. □

Proof. [Proof of Theorem 3 see [3]]

Fix $q \in [1, \dfrac{d}{d-1}[$. Taking $r = \dfrac{2 \times 2^*}{2 + 2^*} = \dfrac{3}{2}$ if $d = 3$, and verifying $\dfrac{2 \times 2^*}{2 + 2^*} > q$, in the case $d = 2$.

To prove Theorem 3, it is sufficient to estimate

$$\||\nabla v_h|\|_{L^{r,\infty}(\Omega)} = \sup_{\mu > 0} \mu \left|\{x \in \Omega : |\nabla v_h(x)| \geq \mu\}\right|^{r^{-1}},$$

and use the following embedding inequality

$$\||\nabla v_h|\|_{L^q(\Omega)} \leq C(q, r, |\Omega|) \||\nabla v_h|\|_{L^{r,\infty}(\Omega)}.$$

So, let $\mu > 0$. For every $k > 0$, we can write

$$|\{x \in \Omega : |\nabla v_h(x)| \geq \mu\}| \leq |B_k(v_h)| + |\widehat{B_k(v_h)}|,$$

where,

$$\widehat{B_k(v_h)} := \{x \in \Omega : |\nabla v_h(x)| \geq \mu\} \cap [B_k(v_h)]^c.$$

But $\widehat{B_k(v_h)}$ coincides, up to a set of measure zero, with

$$\{x \in \Omega : |\nabla v_h(x)| \geq \mu\} \cap \bigcup \left\{ T \in \mathscr{T}_h : \max_T |v_h| < k \right\}.$$

Furthermore, under the hypothesis $\max_T |v_h| < k$, one have

$$\Pi_h(T_k(v_h))\,|_T = v_h\,|_T,$$

so

$$|\widehat{B_k(v_h)}| \leq |\{x \in \Omega : |\nabla \Pi_h(T_k(v_h(x)))| \geq \mu\}| \leq \frac{1}{\mu^2} \int_\Omega |\nabla \Pi_h T_k(v_h(x))|^2 \mathrm{d}x,$$

and by (26),

$$|\widehat{B_k(v_h)}| \leq \frac{kM}{\mu^2}.$$

We now fix $k = \mu^{2-r} M^{r-1}$, s.t.

$$\frac{kM}{\mu^2} = \left(\frac{M}{\mu}\right)^r,$$

using (34), it follows that

$$|\{x\in\Omega:|\nabla v_h(x)|\geq\mu\}|\leq(C(d,|\Omega|,2^*)+1)\left(\frac{M}{\mu}\right)^r,$$

for $d\in\{2,3\}$,
which implies, for some fixed r and for every $\mu>0$;

$$\mu\,|\{x\in\Omega:|\nabla v_h(x)|\geq\mu\}|^{r^{-1}}\leq C(d,|\Omega|,q)M.$$

This finishes the proof. □

Lemma 3. *Let T be a $d-$simplex in $\mathbb{R}^d$ and $v\in\mathbb{P}_3(T)$. For every s,k with $0\leq s<k$, if there exist a,b in T s.t.*

$$|v(a)|\geq k \text{ and } |v(b)|\leq s,$$

then,

$$|T|\leq\frac{8\times 9^2}{5}\frac{|a-b|^2}{(k-s)^2}\int_T|\nabla v|^2\mathrm{d}x, \text{ for } d=2 \quad (37)$$

$$|T|\leq\frac{8\times 21^2}{9}\frac{|a-b|^2}{(k-s)^2}\int_T|\nabla v|^2\mathrm{d}x, \text{ for } d=3 \quad (38)$$

Proof. In this proof, without loss of generality, $\{\varphi_{i,T},\varphi_{i,j,T}\}_{0\leq i<j\leq d}$ represents the local basis of $\mathbb{P}_2(T)$.

For every $w\in\mathbb{P}_3(T)$ and every x in $\mathbb{R}^d$, $\nabla w\in\mathbb{P}_2(T)^d$, so

$$\nabla w(x)=\sum_{i=0}^d\nabla w(a_{i,T})\varphi_{i,T}(x)+\sum_{\substack{i,j=0\\i<j}}^d\nabla w(a_{i,j,T})\varphi_{i,j,T}(x).$$

Therefore,

$$|\nabla w|^2=\sum_{i=0}^d|\nabla w(a_{i,T})|^2\varphi_{i,T}^2+\sum_{\substack{i,j=0\\i<j}}^d|\nabla w(a_{i,j,T})|^2\varphi_{i,j,T}^2+2\Big[\sum_{\substack{i,j=0\\i\neq j}}^d\nabla w(a_{i,T})\nabla w(a_{j,T})\varphi_{i,T}\varphi_{j,T}+\sum_{\substack{i,j,k,l=0\\i<j,k<l,(i,j)\neq(k,l)}}^d\nabla w(a_{i,j,T})\nabla w(a_{k,l,T})\varphi_{i,j,T}\varphi_{k,l,T}+\sum_{\substack{i,j,k=0\\i<j}}^d\nabla w(a_{i,j,T})\nabla w(a_{k,T})\varphi_{i,j,T}\varphi_{k,T}\Big]$$

$$\leq 2\Big[\sum_{i=0}^d|\nabla w(a_{i,T})|^2\varphi_{i,T}^2+\sum_{\substack{i,j=0\\i<j}}^d|\nabla w(a_{i,j,T})|^2\varphi_{i,j,T}^2\Big].$$

In the otherwise, for $d=2$:

$$\int_T|\nabla w(z)|^2dz=|T|\Big[\frac{1}{30}\sum_{i=0}^d|\nabla w(a_{i,T})|^2+\frac{8}{45}\sum_{\substack{i,j=0\\i<j}}^d|\nabla w(a_{i,j,T})|^2\Big]+2|T|\Big[-\frac{1}{180}\sum_{\substack{i,j=0\\i\neq j}}^d\nabla w(a_{i,T})\nabla w(a_{j,T})+\frac{4}{45}\sum_{\substack{i,j,k,l=0\\i<j,k<l,(i,j)\neq(k,l)}}^d\nabla w(a_{i,j,T})\nabla w(a_{k,l,T})-\frac{1}{45}\sum_{\substack{i,j,k=0\\i<j,\ k\notin\{i,j\}}}^d\nabla w(a_{i,j,T})\nabla w(a_{k,T})\Big]$$

$$\geq|T|\Big[\frac{1}{30}\sum_{i=0}^d|\nabla w(a_{i,T})|^2+\frac{8}{45}\sum_{\substack{i,j=0\\i<j}}^d|\nabla w(a_{i,j,T})|^2\Big]-|T|\Big[\frac{1}{180}\sum_{i=0}^d|\nabla w(a_{i,T})|^2+\frac{4}{45}\sum_{\substack{i,j=0\\i<j}}^d|\nabla w(a_{i,j,T})|^2-\frac{1}{25}\sum_{\substack{i,j=0\\i<j}}^d|\nabla w(a_{i,j,T})|^2-\frac{1}{81}\sum_{i=0}^d|\nabla w(a_{i,T})|^2\Big]$$

$$\geq|T|\Big[\Big(\frac{1}{30}-\frac{1}{180}-\frac{1}{81}\Big)\sum_{i=0}^d|\nabla w(a_{i,T})|^2+\Big(\frac{8}{45}-\frac{4}{45}-\frac{1}{25}\Big)\sum_{\substack{i,j=0\\i<j}}^d|\nabla w(a_{i,j,T})|^2\Big]$$

$$\geq|T|\Big[\frac{5}{4\times 81}\sum_{i=0}^d|\nabla w(a_{i,T})|^2+\frac{11}{9\times 25}\sum_{\substack{i,j=0\\i<j}}^d|\nabla w(a_{i,j,T})|^2\Big].$$

Therefore,

$$\int_T|\nabla w(z)|^2dz\geq\frac{5|T|}{8\times 81}|\nabla w(x)|^2.$$

In the case $d=3$, we obtain

$$\int_T|\nabla w(z)|^2dz=|T|\Big[\frac{1}{70}\sum_{i=0}^d|\nabla w(a_{i,T})|^2+$$

$$+\frac{8}{105}\sum_{\substack{i,j=0\\i<j}}^{d}|\nabla w(a_{i,j,T})|^2\Big]+$$

$$+2|T|\Big[\frac{1}{420}\sum_{\substack{i,j=0\\i\neq j}}^{d}\nabla w(a_{i,T})\nabla w(a_{j,T})+$$

$$+\frac{2}{105}\sum_{\substack{i,j,k,l=0\\i<j,k<l,k,l\notin\{i,j\}}}^{d}\nabla w(a_{i,j,T})\nabla w(a_{k,l,T})+$$

$$+\frac{4}{105}\sum_{\substack{i,j,k,l=0\\i<j,k<l,k,l\in\{i,j\}\\(k,l)\neq(i,j)}}^{d}\nabla w(a_{i,j,T})\nabla w(a_{k,l,T})+$$

$$-\frac{1}{105}\sum_{\substack{i,j,k=0\\i<j,\ k\in\{i,j\}}}^{d}\nabla w(a_{i,j,T})\nabla w(a_{k,T})+$$

$$-\frac{1}{70}\sum_{\substack{i,j,k=0\\i<j,\ k\notin\{i,j\}}}^{d}\nabla w(a_{i,j,T})\nabla w(a_{k,T})\Big]$$

$$=|T|\Big[\frac{1}{70}\sum_{i=0}^{d}|\nabla w(a_{i,T})|^2+$$

$$+\frac{8}{105}\sum_{\substack{i,j=0\\i<j}}^{d}|\nabla w(a_{i,j,T})|^2\Big]+$$

$$+2|T|\Big[\frac{1}{420}\sum_{\substack{i,j=0\\i\neq j}}^{d}\nabla w(a_{i,T})\nabla w(a_{j,T})+$$

$$+\frac{2}{105}\sum_{\substack{i,j,k,l=0\\i<j,k<l,(k,l)\neq(i,j)}}^{d}\nabla w(a_{i,j,T})\nabla w(a_{k,l,T})+$$

$$+\frac{2}{105}\sum_{\substack{i,j,k,l=0\\i<j,k<l,k,l\in\{i,j\}\\(k,l)\neq(i,j)}}^{d}\nabla w(a_{i,j,T})\nabla w(a_{k,l,T})+$$

$$-\frac{2}{210}\sum_{\substack{i,j,k=0\\i<j}}^{d}\nabla w(a_{i,j,T})\nabla w(a_{k,T})+$$

$$-\frac{1}{210}\sum_{\substack{i,j,k=0\\i<j,\ k\notin\{i,j\}}}^{d}\nabla w(a_{i,j,T})\nabla w(a_{k,T})\Big]$$

$$\geq|T|\Big[\frac{1}{70}\sum_{i=0}^{d}|\nabla w(a_{i,T})|^2+$$

$$+\frac{8}{105}\sum_{\substack{i,j=0\\i<j}}^{d}|\nabla w(a_{i,j,T})|^2\Big]+$$

$$-|T|\Big[\frac{1}{420}\sum_{i=0}^{d}|\nabla w(a_{i,T})|^2+$$

$$+\frac{4}{105}\sum_{\substack{i,j=0\\i<j}}^{d}|\nabla w(a_{i,j,T})|^2+$$

$$+\frac{3}{100}\sum_{\substack{i,j=0\\i<j}}^{d}|\nabla w(a_{i,j,T})|^2+$$

$$+\frac{3}{(21)^2}\sum_{i=0}^{d}|\nabla w(a_{i,T})|^2\Big]$$

$$\geq|T|\Big[\Big(\frac{1}{70}-\frac{1}{420}-\frac{3}{(21)^2}\Big)\sum_{i=0}^{d}|\nabla w(a_{i,T})|^2+$$

$$+\Big(\frac{8}{105}-\frac{4}{105}-\frac{3}{100}\Big)\sum_{\substack{i,j=0\\i<j}}^{d}|\nabla w(a_{i,j,T})|^2\Big]$$

$$\geq|T|\Big[\frac{9}{4\times(21)^2}\sum_{i=0}^{d}|\nabla w(a_{i,T})|^2+$$

$$+\frac{17}{100\times 21}\sum_{\substack{i,j=0\\i<j}}^{d}|\nabla w(a_{i,j,T})|^2\Big].$$

Therefore,

$$\int_T|\nabla w(z)|^2dz\geq\frac{9|T|}{8\times(21)^2}|\nabla w(x)|^2.$$

Using (28), one can observe that

$$k-s\leq|w(a)|-|w(b)|\leq|\nabla w(x)|\,|a-b|. \tag{39}$$

Combining with (39), we get

$$(k-s)^2\leq|\nabla w(x)|^2\,|a-b|^2. \tag{40}$$

Therefore,

$$\int_T|\nabla w(z)|^2dz\geq\frac{(k-s)^2}{|a-b|^2}\frac{|T|}{C(d)}$$

where $C(2)=\dfrac{8\times 81}{5}$ and $C(3)=\dfrac{8\times(21)^2}{9}$, and this implies (38). □

The following lemma is then, obvious.

Lemma 4. *Let $v\in V_h$. For every s, k with $0\leq s<k$, the set $B_{k,s}(v)$ defined by*

$$B_{k,s}(v)=\bigcup\Big\{T\in\mathscr{T}_h:\min_T|v|\leq s,\max_T|v|\geq k\Big\}, \tag{41}$$

satisfies

$$|B_{k,s}(v)|\leq C(d)\frac{h^2}{(k-s)^2}\int_\Omega|\nabla(v)|^2\,\mathrm{d}x. \tag{42}$$

We can also derive the following lemma.

Lemma 5. *Let $v_h \in V_h$ and $0 \leq s < k$, the set $\tilde{B}_{k,s}(v_h)$ defined by*

$$\tilde{B}_{k,s}(v_h) = \bigcup \left\{ T \in \mathscr{T}_h : \max_T |v_h| \geq k, \min_T |v_h| \leq s \right\}, \tag{43}$$

satisfies

$$\left|\tilde{B}_{k,s}(v_h)\right| \leq C(d) \frac{h^2}{(k-s)^2} \int_\Omega |\nabla \Pi_h(T_k(v_h))|^2 \, dx. \tag{44}$$

Proof. In deed, if $T \subset \tilde{B}_{k,s}(v_h)$, then, there are two possibilities: $\max_T |v_h| \leq k$ and so $\Pi_h(T_k(v_h)) = v_h$, or $\max_T |v_h| > k$ and so $\max_T |\Pi_h(T_k(v_h))| \geq k$ and obviously $\min_T |\Pi_h(T_k(v_h))| \leq s$. in the two cases

$$T \subset B_{k,s}(\Pi_h(T_k(v_h))).$$

The estimate (44) follows. □

*Remark.*It is clear that, under hypothesis (26),

$$\left|\tilde{B}_{k,s}(v_h)\right| \underset{h\to 0}{\longrightarrow} 0 \quad \text{and} \quad \left|B_{k,s}(\Pi_h(T_k(v_h)))\right| \underset{h\to 0}{\longrightarrow} 0\,. \tag{45}$$

In addition, one has:

Proposition 1. *Let $v_h \in V_h$ and $0 \leq s < k$. If v_h satisfies (26), then,*

$$\left|B_{k,s}(v_h)\right| \underset{h\to 0}{\longrightarrow} 0. \tag{46}$$

Proof. Fix $k > 0$ and $s > 0$, $s < k$. For $h > 0$ such that $\frac{1}{h} \geq k$, we can write

$$B_{k,s}(v_h) = \left(B_{k,s}(v_h) \cap B_{\frac{1}{h}}(v_h)\right) \cup \left(B_{k,s}(v_h) \cap \left[B_{\frac{1}{h}}(v_h)\right]^c\right).$$

On the one hand, with $2^* = 6$ in (34), one has

$$\left|B_{k,s}(v_h) \cap B_{\frac{1}{h}}(v_h)\right| \leq \left|B_{\frac{1}{h}}(v_h)\right| \leq C(|\Omega|)\, h^3. \tag{47}$$

On the other hand,

$$B_{k,s}(v_h) \cap \left[B_{\frac{1}{h}}(v_h)\right]^c \subset B_{k,s}\left(\Pi_h(T_{\frac{1}{h}}(v_h))\right).$$

Indeed; if $x \in B_{k,s}(v_h) \cap \left[B_{\frac{1}{h}}(v_h)\right]^c$ and $T \in \mathscr{T}_h$ such that $x \in T$, then, $\max_T |v_h| \geq k$, $\min_T |v_h| \leq s$ and, for every y in T, $|v_h(y)| \leq \frac{1}{h}$, what means $\Pi_h T_{\frac{1}{h}}(v_h)\,|_T = v_h\,|_T$, so

$$T \subset B_{k,s}\left(\Pi_h(T_{\frac{1}{h}}(v_h))\right).$$

Therefore, with lemma 4, and (26), one has

$$\begin{aligned} \left|B_{k,s}(v_h) \cap B^c_{\frac{1}{h}}(v_h)\right| &\leq \left|B_{k,s}\left(\Pi_h(T_{\frac{1}{h}}(v_h))\right)\right| \\ &\leq C(d) \frac{h^2}{(k-s)^2} \int_\Omega \left|\nabla \Pi_h(T_{\frac{1}{h}}(v_h))\right|^2 dx \\ &\leq C(d) \frac{h}{(k-s)^2} M. \end{aligned} \tag{48}$$

The convergence (46) is then a consequence of (47) and (48). □

Lemma 6. *Let $v_h \in V_h$. For every s, k s.t $0 < s < k$, one has*

$$\{x \in \Omega : T_s(\Pi_h(T_k(v_h))) \neq T_s(v_h)\} \subset B_{k,s}(v_h, \Pi_h(T_k(v_h))), \tag{49}$$

where,
$B_{k,s}(v_h, \Pi_h(T_k(v_h))) := B_{k,s}(v_h) \cup B_{k,s}(\Pi_h(T_k(v_h)))$.

Proof. Let$T \in \mathscr{T}_h$ with $x \in T$. s.t.

$$T_s(\Pi_h(T_k(v_h))) \neq T_s(v_h)\,.$$

One can easily see that

$$\Pi_h(T_k(v_h))\,|_T \neq v_h\,|_T,$$

$$\max_T |v_h| \geq k,$$

and

$$\max_T |\Pi_h(T_k(v_h))| \geq k.$$

This gives the three following possibilities to be discussed (i) $|\Pi_h(T_k(v_h(x)))| \geq s$ and $|v_h(x)| < s$, then,

$$T \subset B_{k,s}(v_h)\,.$$

(ii) $|\Pi_h(T_k(v_h(x)))| < s$ and $|v_h(x)| \geq s$, then,

$$T \subset B_{k,s}(\Pi_h(T_k(v_h))).$$

(iii) $|\Pi_h(T_k(v_h(x)))| < s$ and $|v_h(x)| < s$, then,

$$T \subset B_{k,s}(\Pi_h(T_k(v_h))) \cap B_{k,s}(v_h)\,.$$

In all cases $x \in T \subset B_{k,s}(v_h, \Pi_h(T_k(v_h)))$, and (49) follows. □

Using (45) and (46), we are now in measure to have the result of Proposition 2.6 in [3], but we are going to state a result more precise, useful to prove Theorem 2 in our case ($\mathbb{P}_3$ -FEA; the assumption $|\Pi_h(T_k(v_h))| \leq k$ does not hold in this case).

Proposition 2. *Assume that $v_h \in V_h$ satisfies (26). Then, for every $k > 0$, one has*

$$\Pi_h(T_k(v_h)) - T_k(v_h) \underset{h\to 0}{\longrightarrow} 0 \qquad \textit{in measure.} \tag{50}$$

Proof. Let $k > 0$, η s.t. $0 < \eta < k$ and consider

$$\mathscr{E}_\eta = \{x \in \Omega : |\Pi_h(T_k(v_h(x))) - T_k(v_h(x))| \geq \eta\}\,.$$

Let $x \in \mathscr{E}_\eta$ and $T \in \mathscr{T}_h$ with $x \in T$. It is easily checked that

$$\Pi_h(T_k(v_h))_{|T} \neq T_k(v_h)_{|T},$$

what implies that

$$\max_T |v_h| > k.$$

So there are four possibilities.
(i) v_h changes sign in T , then, by continuity,

$$T \subset B_{k,s}(v_h) \text{ for every } s \in]0,k[.$$

(ii) $\Pi_h(T_k(v_h))$ changes sign in T, then,

$$T \subset B_{k,s}(\Pi_h(T_k(v_h))) \text{ for every } s \in]0,k[.$$

(iii) $|v_h|_{|\ddot{T}} \le k$ and $v_{h|T} \ge 0$ (or $v_{h|T} \le 0$), so

$$\Pi_h(T_k(v_h))_{|T} = v_{h|T}.$$

- If $v_{h|T} \ge 0$, then, $v_h(x) \ge k+\eta$ and

$$T \subset B_{k+\eta,k}(\Pi_h(T_k(v_h))).$$

- If $v_{h|T} \le 0$, then, $v_h(x) \le -k-\eta$ and

$$T \subset B_{k+\eta,k}(\Pi_h(T_k(v_h))).$$

(iv) $\max_T |v_h| > k$ and $v_{h|T} \ge 0$ (or $v_{h|T} \le 0$). So

$$|\Pi_h T_k(v_h(x)) - T_k(v_h(x))| = ||\Pi_h T_k(v_h(x))| - |T_k(v_h(x))||.$$

- If $|\Pi_h(T_k(v_h(x)))| - |T_k(v_h(x))| \ge \eta$. There are three possibilities :
case 1 : $|v_h(x)| \ge k$, so

$$|\Pi_h(T_k(v_h(x)))| \ge k+\eta,$$

and

$$T \subset B_{k+\eta,k}(\Pi_h(T_k(v_h))),$$

case 2 : $|v_h(x)| < k - \dfrac{\eta}{2}$, so

$$T \subset B_{k,k-\frac{\eta}{2}}(v_h),$$

case 3 : $k - \dfrac{\eta}{2} \le |v_h(x)| < k$, so

$$|\Pi_h(T_k(v_h(x)))| \ge k + \frac{\eta}{2},$$

and

$$T \subset B_{k+\frac{\eta}{2},k}(\Pi_h(T_k(v_h))).$$

- If $|T_k(v_h(x))| - |\Pi_h(T_k(v_h(x)))| \ge \eta$, then,

$$|\Pi_h(T_k(v_h(x)))| \le |T_k(v_h(x))| - \eta,$$

so

$$T \subset B_{k,k-\eta}(\Pi_h(T_k(v_h))).$$

We can then, conclude that

$$\mathscr{E}_\eta \subset B_{k+\frac{\eta}{2},k}(\Pi_h(T_k(v_h))) \cup B_{k,k-\eta}(\Pi_h(T_k(v_h))) \cup B_{k,k-\frac{\eta}{2}}(v_h).$$

Convergence (50) is then, consequence of (45) and (46). □

The result and the proof of the proposition (2.7) in [3], can be conserved without changes.

Proposition 3. *Under assumption (24), one has for every* $v_h \in V_h$ *and every* $k > 0$

$$a(\nabla(v_h - \Pi_h(T_k(v_h))), \nabla \Pi_h(T_k(v_h))) \ge 0. \tag{51}$$

Proof. [Proof of Proposition 2.7 in [3]] Since

$$v_h = \sum_{i=1}^{N_h} v_h(a_i)\varphi_i \text{ and } \Pi_h(T_k(v_h)) = \sum_{i=1}^{N_h} T_k(v_h(a_i))\varphi_i,$$

using the definition of $Q_{i,j}$, we have

$$\begin{aligned}
&a(\nabla(v_h - \Pi_h(T_k(v_h))), \nabla \Pi_h(T_k(v_h))) = \\
&= a\Big(\sum_{i=1}^{N_h}(v_h(a_i) - T_k(v_h(a_i)))\nabla\varphi_i, \sum_{j=1}^{N_h} T_k(v_h(a_j))\nabla\varphi_j\Big) \\
&= a\Big(\sum_{i=1}^{N_h}(v_h(a_i) - T_k(v_h(a_i)))\nabla\varphi_i, \sum_{j=1}^{N_h} T_k(v_h(a_j))\nabla\varphi_j\Big) \\
&= \sum_{i=1}^{N_h}(v_h(a_i) - T_k(v_h(a_i))) \sum_{j=1}^{N_h} T_k(v_h(a_j))\, a(\nabla\varphi_i, \nabla\varphi_j) \\
&= \sum_{i=1}^{N_h}(v_h(a_i) - T_k(v_h(a_i))) \sum_{j=1}^{N_h} T_k(v_h(a_j))\, Q_{i,j} \\
&= \sum_{i=1}^{N_h} S_i,
\end{aligned}$$

where

$$\begin{aligned}
S_i &= (v_h(a_i) - T_k(v_h(a_i)))T_k(v_h(a_i))Q_{i,i} + \\
&+ (v_h(a_i) - T_k(v_h(a_i))) \sum_{\substack{j=1\\ j\neq i}}^{N_h} T_k(v_h(a_j))Q_{i,j}.
\end{aligned}$$

Fix $i \in [\![1, N_h]\!]$. If $|v_h(a_i)| \le k$, then,

$$v_h(a_i) - T_k(v_h(a_i)) = 0,$$

and

$$S_i = 0.$$

If $|v_h(a_i)| > k$, then,

$$(v_h(a_i) - T_k(v_h(a_i)))T_k(v_h(a_i)) = k|v_h(a_i) - T_k(v_h(a_i))|.$$

Since $\left|T_k(v_h(a_j))\right| \le k$ for every j, one has

$$\begin{aligned}
S_i &\ge k|v_h(a_i) - T_k(v_h(a_i))|Q_{i,i} + \\
&- k|v_h(a_i) - T_k(v_h(a_i))| \sum_{\substack{j=1\\ j\neq i}}^{N_h} |Q_{i,j}| \\
&= k|v_h(a_i) - T_k(v_h(a_i))| \Big(Q_{i,i} - \sum_{\substack{j=1\\ j\neq i}}^{N_h} |Q_{i,j}|\Big) \\
&\ge 0,
\end{aligned}$$

in the light of the assumption (24).

This proves that $S_i \geq 0$ for every $i \in [\![1, N_h]\!]$; and so (51). □

4 Proof of the main theorem

We first show an a priori estimate (compare with (26) on the solution u_h of (25).

Proposition 4. *Under the assumption of Theorem 2, the solution u_h of (25) satisfies for every $k>0$ and every $h>0$*

$$a\left(\nabla\Pi_h(T_k(u_h)), \nabla\Pi_h(T_k(u_h))\right) \leq \left(f, \Pi_h(T_k(u_h))\right), \quad (52)$$

in particular, u_h satisfies

$$\int_\Omega |\nabla\Pi_h(T_k(u_h))|^2 \mathrm{d}x \leq k\frac{5d^2+6(d+1)}{5\alpha}\|f\|_{L^1(\Omega)}. \quad (53)$$

Proof. Using $\Pi_h(T_k(u_h))$ as a test function in (25) and (51), we can have (52).

To prove (53), we need the following lemma. □

Lemma 7. *Let $v_h \in V_h$, $T \in \mathscr{T}_h$ and $k>0$ s.t.*

$$\max_{\ddot{T}} |v_h| \leq k.$$

Then,

$$\max_T |v_h| \leq k\frac{5d^2+6(d+1)}{5}. \quad (54)$$

Proof. Let $v_h \in V_h$, $T \in \mathscr{T}_h$ and $k>0$ s.t.

$$\max_{\ddot{T}} |v_h| \leq k.$$

For every $x \in T$,

$$\sum_{i=0}^{d} v_h(a_{i,T})\varphi_{i,T}(x) = \sum_{\substack{i=0 \\ \lambda_{i,T}(x)\in]\frac{1}{3},\frac{2}{3}[}}^{d} v_h(a_{i,T})\varphi_{i,T}(x) + \sum_{\substack{i=0 \\ \lambda_{i,T}(x)\notin]\frac{1}{3},\frac{2}{3}[}}^{d} v_h(a_{i,T})\varphi_{i,T}(x),$$

and,

$$\sum_{\substack{i,j=0 \\ i\neq j}}^{d} v_h(a_{i,j,T})\varphi_{i,j,T}(x) = \sum_{\substack{i,j=0 \\ \lambda_{i,T}(x)<\frac{1}{3}}}^{d} v_h(a_{i,j,T})\varphi_{i,j,T}(x) + \sum_{\substack{i,j=0 \\ \lambda_{i,T}(x)\geq\frac{1}{3}}}^{d} v_h(a_{i,j,T})\varphi_{i,j,T}(x).$$

Therefore,

$$\begin{aligned}
|v_h(x)| &\leq \sum_{i=0}^{d} |v_h(a_{i,T})||\varphi_{i,T}(x)| + \\
&+ \sum_{\substack{i,j=0 \\ i\neq j}}^{d} |v_h(a_{i,j,T})||\varphi_{i,j,T}(x)| + \\
&+ \sum_{\substack{i,j,k=0 \\ i<j<k}}^{d} |v_h(a_{i,j,k,T})|\varphi_{i,j,k,T}(x) \\
&\leq -k\Bigg(\sum_{\substack{i=0 \\ \lambda_{i,T}(x)\in]\frac{1}{3},\frac{2}{3}[}}^{d} \varphi_{i,T}(x) + \sum_{\substack{i,j=0,\ i\neq j \\ \lambda_{i,T}(x)<\frac{1}{3}}}^{d} \varphi_{i,j,T}(x)\Bigg) + \\
&+ k\Bigg(\sum_{\substack{i=0 \\ \lambda_{i,T}(x)\notin]\frac{1}{3},\frac{2}{3}[}}^{d} \varphi_{i,T}(x) + \sum_{\substack{i,j=0,\ i\neq j \\ \lambda_{i,T}(x)\geq\frac{1}{3}}}^{d} \varphi_{i,j,T}(x) + \\
&+ \sum_{\substack{i,j,k=0 \\ i<j<k}}^{d} \varphi_{i,j,k,T}(x)\Bigg) \\
&\leq k\Bigg(1 - 2\Bigg(\sum_{\substack{i=0 \\ \lambda_{i,T}(x)\in]\frac{1}{3},\frac{2}{3}[}}^{d} \varphi_{i,T}(x) + \sum_{\substack{i,j=0,\ i\neq j \\ \lambda_{i,T}(x)<\frac{1}{3}}}^{d} \varphi_{i,j,T}(x)\Bigg)\Bigg) \\
&\leq k\left(1 + 2\left(\frac{\sqrt{3}}{27}(d+1) + \frac{7^{\frac{3}{2}}-10}{27}d(d+1)\right)\right) \\
&\leq k\left(1 + 2(d+1)\left(\frac{1}{15} + \frac{d}{3}\right)\right) \\
&\leq k\left(1 + \frac{2(d+1)(5d+1)}{15}\right).
\end{aligned}$$

The inequality (54) is then, proved. □

As in [3], the main estimate (53) together with Theorem 3 implies the boundless of u_h in $W_0^{1,q}(\Omega)$ for every $q \in [1, d/(d-1)[$, and the following result remains valid.

Theorem 4. *Under the assumptions of Theorem 2, the solution u_h of (25) satisfies for every $q \in [1, d/(d-1)[$*

$$u_h \underset{h\to 0}{\longrightarrow} u \quad \textit{strongly in } W_0^{1,q}(\Omega), \quad (55)$$

where u is the unique r.s. of (1).

Proof. [Proof of Theorem 3.2 in [3]] Consider a sequence f^ε in $L^2(\Omega)$, converging strongly in $L^1(\Omega)$ to f (for example $f^\varepsilon = T_{\varepsilon^{-1}}(f)$). Let u_h^ε to be the unique solution of (25) for the right-hand side f^ε. Then, $u_h - u_h^\varepsilon$ satisfies

$$\begin{cases} u_h - u_h^\varepsilon \in V_h, \\ \forall v_h \in V_h : a(\nabla v_h, \nabla(u_h - u_h^\varepsilon)) = (f - f^\varepsilon, v_h). \end{cases}$$

Applying estimate (53) to this problem, we obtain for every $k>0$, every $h>0$ and every $\varepsilon>0$

$$\int_\Omega |\nabla \Pi_h T_k(u_h-u_h^\varepsilon)|^2\,\mathrm{d}x \le k\frac{5d^2+6(d+1)}{5\alpha}\|f-f^\varepsilon\|_{L^1(\Omega)},$$

which implies by Theorem 3 that

$$\|u_h-u_h^\varepsilon\|_{W_0^{1,q}(\Omega)} \le C(d,|\Omega|,q)\frac{1}{\alpha}\|f-f^\varepsilon\|_{L^1(\Omega)}, \quad (56)$$

for every $q\in[1,d/(d-1)[$, every $h>0$ and every $\varepsilon>0$.

On the other hand, since $f^\varepsilon\in L^2(\Omega)$ and $\mathscr{T}_h$ satisfies (15), (16) and (17), it is known (see [5]) that for every fixed ε

$$u_h^\varepsilon \underset{h\to 0}{\longrightarrow} u^\varepsilon \quad \text{strongly in } H_0^1(\Omega), \quad (57)$$

where u^ε is the unique solution of

$$\begin{cases} u^\varepsilon\in H_0^1(\Omega), \\ -\mathrm{div}(A\nabla u^\varepsilon)=f^\varepsilon \text{ in } \mathscr{D}'(\Omega). \end{cases} \quad (58)$$

Finally, the function u^ε is also the unique renormalized solution of the problem

$$\begin{cases} -\mathrm{div}(A\nabla u^\varepsilon)=f^\varepsilon \text{ in } \Omega, \\ u^\varepsilon=0 \quad \text{on } \partial\Omega. \end{cases} \quad (59)$$

By the estimate (19) we have

$$\|u^\varepsilon-u\|_{W_0^{1,q}(\Omega)} \le C(d,|\Omega|,q)\frac{1}{\alpha}\|f-f^\varepsilon\|_{L^1(\Omega)}, \quad (60)$$

for every $q\in[1,d/(d-1)[$, where u is the unique r.s. of (1).

Writing now

$$\|u_h-u\|_{W_0^{1,q}(\Omega)} \le \|u_h-u_h^\varepsilon\|_{W_0^{1,q}(\Omega)}+\|u_h^\varepsilon-u^\varepsilon\|_{W_0^{1,q}(\Omega)}+ + \|u^\varepsilon-u\|_{W_0^{1,q}(\Omega)}. \quad (61)$$

Using (56), (57) and (60), we get for every $\varepsilon>0$ and every $q\in[1,d/(d-1)[$

$$\limsup_{h\longrightarrow 0}\|u_h-u\|_{W_0^{1,q}(\Omega)} \le C(d,|\Omega|,q)\frac{1}{\alpha}\|f-f^\varepsilon\|_{L^1(\Omega)},$$

passing to the limit when ε tends to zero proves (55). □

To complete the proof of Theorem 2, it remains to prove that $\Pi_h(T_k(u_h))$ converges strongly to $T_k(u)$ in $H_0^1(\Omega)$.

Proposition 5. *Under the assumptions of Theorem 2, the solution u_h of (25) satisfies*

$$\Pi_h(T_k(u_h)) \underset{h\to 0}{\longrightarrow} T_k(u) \quad \textit{strongly in } H_0^1(\Omega), \quad (62)$$

for every $k>0$.

Proof. [Proof of Proposition 3.3 in [3]] First by result of Proposition 2 and the estimate (53) one can have

$$\Pi_h(T_k(u_h)) \underset{h\to 0}{\rightharpoonup} T_k(u) \quad \text{weakly in } H_0^1(\Omega), \quad (63)$$

for every $k>0$.

On the other hand, using (54) one has

$$|f\Pi_h(T_k(u_h))| \le k\frac{5d^2+6(d+1)}{5\alpha}|f|\in L^1(\Omega),$$

so, by Lebesgue's dominated convergence theorem combined with Rellich-Kondrashov's compactness theorem one has

$$(f,\Pi_h T_k(u_h)) \underset{h\to 0}{\longrightarrow} (f,T_k(u)).$$

Therefore passing to the limit with respect to h in (52) yields

$$\limsup_{h\longrightarrow 0} a(\nabla\Pi_h T_k(u_h),\nabla\Pi_h(T_k(u_h))) \le (f,T_k(u)). \quad (64)$$

And since u is the r.s. of (1), it is known that (see [3])

$$a(\nabla T_k(u),\nabla T_k(u))=(f,T_k(u)). \quad (65)$$

Finally, from (64) and (65), we deduce that

$$\limsup_{h\longrightarrow 0} a(\nabla\Pi_h T_k(u_h),\nabla\Pi_h(T_k(u_h))) \le a(\nabla T_k(u_h),\nabla T_k(u_h)),$$

which combined with the weak convergence (63) implies the strong convergence (62). □

5 Some remarks

5.1 The case where $f\in\mathscr{M}_b(\Omega)$

The proof of Proposition 2 needs only properties of v_h. So, (63) remain valid when $f\in\mathscr{M}_b(\Omega)$ which is stated in the following convergence result.

Theorem 5. *[Theorem 4.1 in [3]] Under the assumptions (8), (9),(15), (16), (17) and (24), if $f\in\mathscr{M}_b(\Omega)$, there exist a subsequence u_h and a function u s.t. for every $(k,q)\in\mathbb{R}^{+*}\times[1,d/(d-1)[$*

$$\Pi(T_k(u_h)) \underset{h\to 0}{\rightharpoonup} T_k(u) \quad \textit{weakly in } H_0^1(\Omega),$$

$$u_h \underset{h\to 0}{\rightharpoonup} u \quad \textit{weakly in } W_0^{1,q}(\Omega),$$

along this subsequence, where u satisfies

$$T_k(u)\in H_0^1(\Omega),$$

$$u\in W_0^{1,q}(\Omega),$$

and

$$\forall v\in\mathscr{C}_c^\infty(\Omega), \quad a(\nabla u,\nabla v)=\int_\Omega v\,\mathrm{d}f.$$

5.2 Error estimate

For $r \in]1,2[$, if $f \in L^{r,\infty}(\Omega)$, under suitable hypotheses on Ω, $\mathscr{T}_h$ and A together with (61), the unique solution u_h^ε of (25) with $f^\varepsilon = T_{\varepsilon^{-1}}(f) \in L^2(\Omega)$ satisfies

$$\|u_h^\varepsilon - u^\varepsilon\|_{H_0^1(\Omega)} \leq Ch\|f^\varepsilon\|_{L^2(\Omega)}. \tag{66}$$

So, in our case, for $q \in [1, d/(d-1)[$, one can obtain an $O\left(h^{2\left(1-r^{-1}\right)}\right)$ error estimate in $W_0^{1,q}(\Omega)$ (see [3]).

And if in addition,

$$\forall k > 0 : T_k(f) \in H^1(\Omega), \tag{67}$$

then, it is known ([5]) that

$$\|u_h^\varepsilon - u^\varepsilon\|_{H_0^1(\Omega)} \leq Ch^2\|f^\varepsilon\|_{L^2(\Omega)}$$

for some constant $C \neq C(h)$.

Therefore, one can under a suitable modification of the proof given in [3] provides an $O\left(h^{4\left(1-r^{-1}\right)}\right)$ error estimate in $W_0^{1,q}(\Omega)$.

5.3 Conclusion and future aims

This work is based on the main hypothesis (24) imposed on the stiffness matrix Q. To guarantee this, it will be necessary to seek an appropriate triangulation. Which turns out to be difficult in our case of cubic approximation (see [4]). But faced with this difficulty, we hope to improve the convergence approximation rate. Our future work will focus on the search for an alternative to hypothesis (24) accompanied by an illustration of our studies by intervening an example of numerical approximation treating an elliptic partial differential equation with Dirichlet boundary conditions involving measure data. This type of problem is part of the applications to turbulence and heat transfer modeling.

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