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A New Generalized Derivative and Related Properties

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Abstract: In this work we present a new definition of local derivative, with good properties, which is a natural generalization of the classical derivative. The main novelty is that this derivative allows us to expand the class of continuous functions and differentiable functions, one of the most requested issues in the definition of new differential operators. The above is an innovation with respect to other well-known local operators.

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1 Introduction

In [24] ([25] and [26]) the following fractional derivative is defined:

Definition 1. *If, for a function $f : [0,1] \longrightarrow R$, the limit*

$$D^q f(y) = \lim_{x \to y} \frac{d^q (f(x) - f(y))}{d(x-y)^q}, \tag{1}$$

exists and is finite, then we say that the local fractional derivative (LFD) of order q, at $x = y$, exists.

This derivative was used for the study of certain attractors of dynamic systems, which are examples of the occurrence of curves and continuous surfaces, but highly irregular and non-differentiable. Frequently these graphs are fractal sets and ordinary calculus is inadequate to characterize them. In this article the notion of "local fractional derivative" is developed, appropriately modifying the concepts of fractional calculus.

Additionally, they introduce the concept of local fractional integral based on the previous one, as we know today.

In the last three decades we have witnessed the development of new differential and integral operators, both fractional and generalized. The latter are generally defined as local derivatives and generate integral operators that may or may not be fractional. To date, the study of this field has attracted the attention of many researchers, not only in pure mathematics, but also in various areas of applied sciences. Due to its own theoretical development and variety of applications, the field has grown rapidly in recent years, resulting in a unified definition of "fractional derivatives" not existing, or at least not unanimously accepted, e.g., one One of the features that the authors present as typical of a fractional derivative is the failure to comply with the Leibniz Rule, however in [14] we build a local differential operator that also violates it, that is, we still need to clarify some points.

Although, as we have seen, local derivatives have been used since the end of the last century, see [3,6,7,8,40,42,43,46,47,48,49,50], it was not until 2014 when they were formalized with the work [23], where a local derivative, called conformable, is defined as follows

Definition 2. *Given a function $\gamma : [0,+\infty) \to \mathbb{R}$, then the conformable fractional derivative of γ of order α, with $0 < \alpha \leq 1$, is defined by*

$$T_\alpha \gamma(t) = \lim_{\varepsilon \to 0} \frac{\gamma(t + \varepsilon t^{1-\alpha}) - \gamma(t)}{\varepsilon}, \quad t > 0. \tag{2}$$

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*Remark.*If γ is α-differentiable in some $0 < \alpha \leq 1$, and $\lim_{t\to 0^+} T_\alpha\gamma(t)$ exists, then define $T_\alpha\gamma(0) = \lim_{t\to 0^+} T_\alpha\gamma(t)$. Additionally we have if γ is differentiable then $T_\alpha\gamma(t) = \gamma'(t)t^{(1-\alpha)}$, of the latter we see that if $\alpha \to 1$ we obtain the classical derivative.

In 2018 we introduced a new local derivative, named non conformable ([14]), with a very distinctive property: when $\alpha \to 1$ we do not get the ordinary derivative. We call this derivative non-conformable, to distinguish it from the previous known ones, since when $\alpha \to 1$ the slope of the tangent line to the curve at the point is not preserved (also see [30,34], and some applications can be consulted in [17, 15,29,30,44]).

Definition 3.*Given a function $\gamma : [t_0,+\infty) \to \mathbb{R}$, $t_0 > 0$. Then the $_1N$-derivative of γ of order α is defined by $_1N^\alpha\gamma(t) = \lim_{\varepsilon\to 0} \frac{\gamma(t+\varepsilon e^{t^{-\alpha}})-\gamma(t)}{\varepsilon}$ for all $t > 0$, $\alpha \in (0,1)$. If γ is $\alpha-$differentiable in some $(0,a)$, and $\lim_{t\to 0^+} {}_1N^{(\alpha)}\gamma(t)$ exists, then define $_1N^\alpha\gamma(0) = \lim_{t\to 0^+} {}_1N^{(\alpha)}\gamma(t)$.*

If the above derivative of the function $x(t)$ of order α exists and is finite in (t_0,∞), we will say that $x(t)$ is N_1-differentiable in $I = (t_0,\infty)$.

*Remark.*Other results that illustrate the aforementioned development can be consulted in [2,4,5,11,19,21,22,52, 45].

In [35] a generalized derivative was defined as follows (see also [13,51]).

Definition 4.*Given a function $\psi : [0,+\infty) \to \mathbb{R}$. Then the N-derivative of ψ of order α is defined by*

$$N_F^\alpha \psi(\tau) = \lim_{\varepsilon\to 0} \frac{\psi(\tau+\varepsilon F(\tau,\alpha)) - \psi(\tau)}{\varepsilon} \tag{3}$$

for all $\tau > 0$, $\alpha \in (0,1)$ being $F(\tau,\alpha)$ is some function. If ψ is N-differentiable in some $(0,\alpha)$, and $\lim_{\tau\to 0^+} N_F^\alpha\psi(\tau)$ exists, then define $N_F^\alpha\psi(0) = \lim_{\tau\to 0^+} N_F^\alpha\psi(\tau)$, note that if ψ is differentiable, then $N_F^\alpha\psi(\tau) = F(\tau,\alpha)\psi'(\tau)$ where $\psi'(\tau)$ is the ordinary derivative.

*Remark.*The generalized derivative defined above is not fractional (see [35]), but it does have a very desirable feature in applications, its dual dependency on both α and the kernel expression itself, with $0 < \alpha \leq 1$ in [23] the conformal derivative is defined by putting $F(t,\alpha) = t^{1-\alpha}$, while in [14] the nonconforming derivative is obtained with $F(t,\alpha) = e^{t^{-\alpha}}$ (see also [34]). This generalized derivative, in addition to the aforementioned cases, contains as particular cases practically all known local operators and has proved its utility in various applications, see, for example, [10,12,16,18,20,27,28, 29,30,31,32,33,36,37,38,39,44].

*Remark.*We must add that this generalized derivative does not comply with the Semigroup Law, that is, $N_F^{2\alpha} f(t) \neq N_F^\alpha(N_F^\alpha f(t))$. To indicate successive derivatives it is necessary to indicate the order in the second way. Obviously, if $F \equiv 1$, the ordinary derivative is obtained.

*Remark.*From the above definition, it is not difficult to extend the order of the derivative for $0 \leq n-1 < \alpha \leq n$ by putting

$$N_F^{n,\alpha} h(\tau) = \lim_{\varepsilon\to 0} \frac{h^{(n-1)}(\tau+\varepsilon F(\tau,\alpha)) - h^{(n-1)}(\tau)}{\varepsilon}, \tag{4}$$

denoting $N_F^{1,\alpha} = N_F^\alpha$. If $h^{(n)}$ exists on some interval $I \subseteq \mathbb{R}$, then we have $N_F^{n,\alpha}h(\tau) = F(\tau,\alpha)h^{(n)}(\tau)$, with $0 \leq n-1 < \alpha \leq n$.

In this paper, we define a new local differential operator, which is a natural generalization of the classical derivative and has very interesting properties. The main novelty is that this derivative allows us to expand the class of continuous functions and differentiable functions, one of the most requested issues in the definition of new differential operators. The above represents an innovation compared to other recognized local operators.

2 First Results

Fractional, or Non-integer order calculus, was introduced over 300 years ago. At that time Leibniz wrote a letter to L'H'opital in which he raised the possibility of generalizing the meaning of derivatives of whole orders to derivatives not of whole orders. L'Hópital wanted to know the result for the derivative of order $n = 1/2$. Leibniz replied that "one day, useful consequences will be drawn" and, in fact, his vision became a reality ([41]).

Therefore, from its very origins, the notion of derivative is a "local" notion, opposed to the globality of the integral, hence they are not inverse operators in the strict sense. It has always been referred to instants, points, specific magnitudes and not at intervals. The classical notions of fractional derivatives "forgot" this fact and built an operator that is not local, therefore, from its conception, the classical fractional derivatives are "not derivatives", it is an operator of another nature. As we have said, it is impossible to compare them, so Tarasov's statements should be reformulated as follows: "No nonlocality. No derivative operator".

We know that the derivative of a function can be expressed as follows:

$$\gamma'(a) = \lim_{x\to a} \frac{\gamma(x) - \gamma(a)}{x-a} \tag{5}$$

if the limit exists. In this case it is said that the function γ is derivable at the point $x = a$. From a geometric perspective, the derivative of a function at the point $x = a$ is the slope of the tangent line to the curve $y = \gamma(x)$ at the point $x = a$.

Let's take a closer look at (5). We can consider that $\gamma(x)$ is the composition of the function γ with the identity I, therefore, our idea is to consider a more general composition $(\gamma \circ N)(x)$ thus $(\gamma \circ N)(x) = \gamma(N(x,\alpha))$, where N is a real function absolutely continuous over all $\mathbb{R}$. Thus we have the following definition:

Definition 5. *Given a function $\gamma : [0,+\infty) \to \mathbb{R}$. Then the CN-derivative of γ of order α (the composite N derivative) is defined by*

$$\gamma^{\alpha}_{CN}(a) = \lim_{x\to a} \frac{\gamma(N(x,\alpha)) - \gamma(N(a,\alpha))}{x-a}, \qquad \alpha \in (0,1]. \tag{6}$$

Remark. If $N(x,1) = 1$ then the CN-derivative is a conformable derivative, in the case that this equality is not fulfilled, we will say that the CN-derivative is non conformable.

Remark. Obviously, if $N(x,\alpha) = x$ then from (6) we obtain the classic derivative of (5). In the case

$$N(x,\alpha) = \begin{cases} x^{\alpha}, x \geq 0 \\ -(-x)^{\alpha}, x < 0 \end{cases} \quad \alpha \in (0,1]$$

then we obtain the α-derivative of [1].

For simplicity, we will denote $\gamma_N(x) = \gamma(N(x,\alpha))$. So, we have the following definitions.

Definition 6. *A function $\gamma : [0,+\infty) \to \mathbb{R}$ is called CN-continuous if $|\gamma_N(x) - \gamma_N(a)| < \varepsilon$, when $|x-a| < \delta$. So, we have*

$$\lim_{x\to a} \gamma_N(x) = \gamma_N(a). \tag{7}$$

This implies that the function $\gamma_N(x)$ is defined at the point $x = a$, that its limit exists at that point and that both values are equal. The function is continuous on the set D if it is continuous at every point on the set.

Remark. Let's see an important detail: a function can be CN-continuous and not be continuous in the classical sense. The function $f(x) = \frac{1}{x-a}$ is not continuous at $x = a$, however we have the following:

$$f_N(x) = \frac{1}{x^{\alpha} - a},$$
$$f_N(a) = \frac{1}{a^{\alpha} - a}$$
$$lim_{x\to a} f_N(x) = f_N(a),$$

i.e. is CN-continuous at any point $x = a$.

Let us now consider the function $f(x) = \frac{x}{x-2}, \quad x \neq 2$, as before, this function is not continuous at $x = 2$. But it is easy to check that

$$f_N(x) = \frac{x^{\alpha}}{x^{\alpha} - 2},$$
$$f_N(2) = \frac{2^{\alpha}}{2^{\alpha} - 2},$$

that is, it is CN-continuous at $x = 2$. Let us calculate its CN-derivative in $x = 2$ since it is clear that it will be CN-derivative in the other real numbers. Taking into account that $f_N(x) = \frac{x^{\alpha}}{x^{\alpha}-2}$ and $f_N(2) = \frac{2^{\alpha}}{2^{\alpha}-2}$ we have

$$f^{\alpha}_{CN}(2) = lim_{x\to 2} \frac{f(N(x,\alpha)) - f(N(2,\alpha))}{x-2}$$
$$= \frac{-\alpha}{2^{\alpha} - 2} \frac{2^{\alpha}}{2^{\alpha-1} - 2}.$$

Therefore, there are CN-derivable functions that are not derivable in the classical sense.

Consider another kernel N. Let's consider

$$N(x,\alpha) = \begin{cases} x(1+(1-\alpha)x), \, x \geq 0 \\ (-x)(1+(1-\alpha)(-x)), \, x < 0 \end{cases} \alpha \in (0,1]$$

and be the function $f(x) = \frac{1}{x-5}, \quad x \neq 5$. This function is not continuous, nor is it obviously differentiable, at $x = 5$ (it is a particular case of the function $f(x) = \frac{1}{x-a}$ studied above). In this case we have $f_N(x) = \frac{1}{x(1+(1-\alpha)x)-5}$ and $f_N(5) = \frac{1}{5(1+(1-\alpha)5)-5} = \frac{1}{(1-\alpha)5^2}$, so it is CN-continuous at $x = 5$. If we now consider the function $f(x) = \frac{x}{x-2}$, which is neither continuous nor differentiable at $x = 2$. However we have to

$$f_N(x) = \frac{x(1+(1-\alpha)x)}{x(1+(1-\alpha)x) - 2},$$
$$f_N(2) = \frac{2(1+(1-\alpha)2)}{2(1+(1-\alpha)2) - 2} = \frac{1+2(1-\alpha)}{2(1-\alpha)}$$

where do you have to $lim_{x\to 2} f_N(x) = f_N(2)$, that is, it is CN-continuous at $x = 2$. Let us calculate its CN-derivative at $x = 2$.

$$f^{\alpha}_{CN}(2) = lim_{x\to 2} \frac{f(N(x,\alpha)) - f(N(2,\alpha))}{x-2}$$
$$= lim_{x\to 2} \frac{\frac{x(1+(1-\alpha)x)}{x(1+(1-\alpha)x)-2} - \frac{1+2(1-\alpha)}{2(1-\alpha)}}{x-2}$$
$$= \frac{-\left(1+2^2(1-\alpha)\right)}{16(1-\alpha)}.$$

These examples demonstrate that there are significant qualitative differences between classical theory and our definitions.

Let's look at some results on α-continuous functions, which will be useful in the future.

Definition 7. *A function $\gamma : I \to \mathbb{R}$ is said to be CN-bounded, $\alpha \in (0,1]$, on I, $N(I,\alpha) \subset N(I,\alpha)$, if $\gamma_N : I \to \mathbb{R}$, $\alpha \in (0,1]$, is bounded on I, i.e., there exists some $K \in \mathbb{R}$ such that $\|\gamma_N(x)\| \leq K$, for all $x \in I$.*

Remark. From the previous examples it is clear that if a function is bounded on I it is CN-bounded, but the converse is not true.

Theorem 1. *If $\gamma : I \to \mathbb{R}$ is CN-continuous, $\alpha \in (0,1]$, in $x_0 \in I$ then is CN-bounded in some neighborhood of x_0, with $\alpha \in (0,1]$.*

Proof. It is a direct consequence of the definition of CN-continuity.

Theorem 2. *If $\gamma : [c,d] \to \mathbb{R}$ is CN-continuous, $\alpha \in (0,1]$, and $[a,b]$ is such that $N(x,\alpha) \subset [c,d]$, with $x \in [a,b]$, then γ is CN-bounded, with $\alpha \in (0,1]$, on $[a,b]$.*

Proof. Suppose that γ is not bounded on $[a,b]$. Let c_1 be the midpoint of $[a,b]$, in one of the intervals $[a,c_1]$ or $[c_1,b]$ the function is not bounded (otherwise it would be bounded in the total). For example, let this interval be $[a,c_1]$. Let c_2 be the midpoint of $[a,c_1]$, in one of the intervals $[a,c_2]$ or $[c_2,c_1]$ the function is not bounded. We call that interval $[a,c_2]$. We take its midpoint, c_3, and reason again in the same way. Continuing with this process we obtain a succession of closed, nested intervals, $[a,b] \supset [a,c_1] \supset [a,c_2] \supset [a,c_3]$...... such that, in each one, γ is not bounded. Since each interval of the sequence has a length half of the previous one, the sequence of its lengths $\frac{b-a}{2^n}$ tends to zero. Now, by Cantor's Axiom, we can ensure the existence of a point c common to all of them. Using the Theorem 1 we have some neighborhood of c where the function is bounded. As within this neighborhood there are infinite intervals of the previous sequence (because their lengths tended to zero and c is in all of them) in which the function was not bounded, we arrive at a contradiction, which arises from assuming that γ is not bounded in $[a,b]$. Therefore, γ is bounded in $[a,b]$, as we wanted to prove.

Theorem 3. *If $\gamma : [c,d] \to \mathbb{R}$ is CN-continuous, $\alpha \in (0,1]$, and $[a,b]$ is such that $N(x,\alpha) \subset [c,d]$, with $x \in [a,b]$, then γ has an absolute CN-maximum value and an absolute CN-minimum value, with $\alpha \in (0,1]$, on $[a,b]$.*

Proof. Let $M = sup\gamma_N(x)$, on $x \in [a,b]$, $\alpha \in (0,1]$. Assume that $\gamma_N(x) < M$ for $x \in [a,b]$. Then $\gamma_N(x) = \frac{1}{M-\gamma_N(x)}$ is CN-continuous and by Theorem 2, γ is CN-bounded, $\alpha \in (0,1]$, and for some $K > 0$, $\gamma \leq K$,$x \in [a,b]$. Thus, $\gamma_N \leq M - \frac{1}{K}$, $x \in [a,b]$. Since M is the supremum of $\gamma_N(x)$, $\alpha \in (0,1]$, then $M \leq M - \frac{1}{K} < M$ which is a contradiction. Thus, there is an $x \in [a,b]$ such that $M = \gamma_N(x)$, $\alpha \in (0,1]$. Similarly, we can prove that there is an $x \in [a,b]$ such that $\gamma_N(x) = \gamma_N(x)$, $\alpha \in (0,1]$.

We will now demonstrate various properties of the CN-differentiability of functions.

Theorem 4. *Let γ, $\lambda : K \to \mathbb{R}$ be α-differentiable functions, $\alpha \in (0,1]$, at $a \in I$, where I and K are intervals such that $N(I) \subseteq K$. Then*

1. *If γ is differentiable at N, and N is differentiable at a, then $\gamma^{\alpha}_{CN}(a) = \gamma'_N(a)N'(a,\alpha)$.*
2. *If $\gamma(x) = c$, where c is a constant, for all $x \in I$, then $\gamma^{\alpha}_{CN}(c) = 0$.*
3. *$(c\gamma)^{\alpha}_{CN}(a) = c\gamma^{\alpha}_{CN}(a)$, where c is a constant.*
4. *$(c\gamma+d\lambda)^{\alpha}_{CN}(a) = c\gamma^{\alpha}_{CN}(a) + d\lambda^{\alpha}_{CN}(a)$, for all c, $d \in \mathbb{R}$.*
5. *$(\gamma\lambda)^{\alpha}_{CN}(a) = \lambda_N(a)\gamma^{\alpha}_{CN}(a) + \gamma_N(a)\lambda^{\alpha}_{CN}(a)$.*
6. *$\left(\frac{\gamma}{\lambda}\right)^{\alpha}_{CN}(a) = \frac{\lambda_N(a)\gamma^{\alpha}_{CN}(a) - \gamma_N(a)\lambda^{\alpha}_{CN}(a)}{\lambda^2_N(a)}$.*

Proof. For (1), we have

$$\begin{aligned}
\gamma^{\alpha}_{CN}(a) &= \lim_{x\to a} \frac{\gamma_N(x) - \gamma_N(a)}{x-a} \\
&= \lim_{x\to a} \frac{\gamma(N(x,\alpha)) - \gamma(N(a,\alpha))}{x-a} \\
&= \lim_{N(x,\alpha)\to N(a,\alpha)} \frac{\gamma(N(x,\alpha)) - \gamma(N(a,\alpha))}{N(x,\alpha) - N(a,\alpha)} \\
&\cdot \lim_{x\to a} \frac{N(x,\alpha) - N(a,\alpha)}{x-a} \\
&= \gamma'[N(a,\alpha)]N'(a,\alpha) \\
&= \gamma'_N(a)N'(a,\alpha).
\end{aligned}$$

The proofs of (2), (3) and (4) follow from the definition 5. For (5), we have

$$\begin{aligned}
&(\gamma\lambda)^{\alpha}_{CN}(a) = \lim_{x\to a} \frac{(\gamma\lambda)_{CN}(x) - (\gamma\lambda)_{CN}(a)}{x-a} \\
&= \lim_{x\to a} \frac{\gamma_N(x)\lambda_N(x) - \gamma_N(a)\lambda_N(a)}{x-a} \\
&= \lim_{x\to a} \frac{\gamma_N(x)\lambda_N(x) - \gamma_N(a)\lambda_N(x) + \gamma_N(a)\lambda_N(x) - \gamma_N(a)\lambda_N(a)}{x-a} \\
&= \lim_{x\to a} \frac{\lambda_N(x)\left(\gamma_N(x) - \gamma_N(a)\right) + \gamma_N(a)\left(\lambda_N(x) - \lambda_N(a)\right)}{x-a} \\
&= \lim_{x\to a} \lambda_N(x) \lim_{x\to a} \frac{\gamma_N(x) - \gamma_N(a)}{x-a} + \gamma_N(a) \lim_{x\to a} \frac{\lambda_N(x) - \lambda_N(a)}{x-a} \\
&= \lambda_N(a)\gamma^{\alpha}_{CN}(a) + \gamma_N(a)\lambda^{\alpha}_{CN}(a).
\end{aligned}$$

To prove (6), we have

$$\begin{aligned}
&\left(\frac{\gamma}{\lambda}\right)^{\alpha}_{CN}(a) = \lim_{x\to a} \frac{\frac{\gamma_N(x)}{\lambda_N(x)} - \frac{\gamma_N(a)}{\lambda_N(a)}}{x-a} \\
&= \lim_{x\to a} \frac{\gamma_N(x)\lambda_N(a) - \gamma_N(a)\lambda_N(x)}{\lambda_N(x)\lambda_N(a)(x-a)} \\
&= \lim_{x\to a} \frac{1}{\lambda_N(x)} \lim_{x\to a} \frac{\gamma_N(x) - \gamma_N(a)}{x-a} \\
&- \lim_{x\to a} \frac{\lambda_N(x) - \lambda_N(a)}{x-a} \lim_{x\to a} \frac{\gamma_N(a)}{\lambda_N(x)\lambda_N(a)} \\
&= \frac{\lambda_N(a)\gamma^{\alpha}_N(a) - \gamma_N(a)\lambda^{\alpha}_N(a)}{\lambda^2_N(a)}.
\end{aligned}$$

Theorem 5.*Let $\gamma : K \to \mathbb{R}$ is α-differentiable, $\alpha \in (0,1]$, at $a \in I$, where I and K are intervals such that $N(I) \subseteq K$, then γ is α-continuous, $\alpha \in (0,1]$, at a.*

*Proof.*Since $\gamma^{\alpha}_{CN}(a)$, $\alpha \in (0,1]$, exists and for all $x \in I$, $x \neq a$, that

$$\gamma_N(x) - \gamma_N(a) = \frac{\gamma_N(x) - \gamma_N(a)}{x-a}(x-a).$$

Then

$$\lim_{x\to a}[\gamma_N(x) - \gamma_N(a)] = \lim_{x\to a}\left[\frac{\gamma_N(x) - \gamma_N(a)}{x-a}\right]\lim_{x\to a}(x-a),$$

thus

$$\lim_{x\to a}\gamma_N(x) = \gamma_N(a).$$

Theorem 6.*(Chain Rule for Fractional Derivative) Let $\gamma : K_1 \to \mathbb{R}$ and $\lambda : K_2 \to \mathbb{R}$ be such that $\gamma_N(K_1) \subseteq K_2$, where K_1 and K_2 are intervals. If γ is α-differentiable, $\alpha \in (0,1]$, at $a \in I$, $N(I) \subseteq K_1$, and λ is differentiable at $\gamma_N(a)$, then $(\lambda \circ \gamma)^{\alpha}_{CN}(a) = \lambda'[\gamma_N(a)]\,\gamma^{\alpha}_{CN}(a)$.*

*Proof.*Since λ is differentiable at $r_0 = \gamma_N(a)$, $\alpha \in (0,1]$, there exists $\lambda'(r_0)$ such that

$$\lim_{\Delta r\to 0}\frac{\lambda(r_0+\Delta r) - \lambda(r_0)}{\Delta r} = \lambda'(r_0).$$

Define u by

$$u(\Delta r) = \frac{\lambda(r_0+\Delta r) - \lambda(r_0)}{\Delta r} - \lambda'(r_0).$$

Then

$$\lambda(r_0+\Delta r) - \lambda(r_0) = \left(\lambda'(r_0) + u(\Delta r)\right)\Delta r.$$

Taking $\Delta r = \gamma_N(a+\Delta x) - \gamma_N(a)$ with $\Delta x \neq 0$, we get

$$\begin{aligned}&\lambda(\gamma_N(a+\Delta x)) - \lambda(\gamma_N(a))\\ &= \left(\lambda'(\gamma_N(a)) + u(\Delta r)\right)(\gamma_N(a+\Delta r) - \gamma_N(a)).\end{aligned}$$

Dividing both sides by Δx yields

$$\begin{aligned}&\frac{\lambda(\gamma_N(a+\Delta x)) - \lambda(\gamma_N(a))}{\Delta x}\\ &= \left(\lambda'(\gamma_N(a)) + u(\Delta r)\right)\left(\frac{\gamma_N(a+\Delta x) - \gamma_N(a)}{\Delta x}\right).\end{aligned}$$

Since γ is α-differentiable, $\alpha \in (0,1]$, at a, it is α-continuous, $\alpha \in (0,1]$, at a. Letting $\Delta x \to 0$, then $\Delta r \to 0$, and consequently that $u(\Delta r) \to 0$. Then,

$$\begin{aligned}&\lim_{\Delta x\to 0}\frac{\lambda(\gamma_N(a+\Delta x)) - \lambda(\gamma_N(a))}{\Delta x}\\ &= \lambda'(\gamma_N(a))\lim_{\Delta x\to 0}\left(\frac{\gamma_N(a+\Delta x) - \gamma_N(a)}{\Delta x}\right),\end{aligned}$$

implies that
$(\lambda \circ \gamma)^{\alpha}_{CN}(a) = \lambda'[\gamma_N(a)]\,\gamma^{\alpha}_{CN}(a)$.

Definition 8.*Let $\gamma : I \to \mathbb{R}$ be a function, where $I \subseteq \mathbb{R}$ is an interval, and let $\omega \in (n, n+1]$, $n \in \mathbb{N}$. The ω-derivative of at $a \in I$, $N(K) \subseteq I$, is defined by*

$$\gamma^{\alpha}_{CN_\omega}(a) = \lim_{x\to a}\frac{(\gamma^{\lceil\alpha\rceil-1}_{CN})(x) - (\gamma^{\lceil\alpha\rceil-1}_{CN})(a)}{x-a},$$

$\alpha \in (0,1]$, provided that the limit exists, where $\gamma^{\lceil\alpha\rceil-1}$ is the $(n-1)^{th}$ derivative of γ.

Remark.

1. The ω-derivative in Definition 8 is the fractional derivative of γ of order ω, $\omega \in (n, n+1]$, $n \in \mathbb{N}$.
2. If γ^{α} is differentiable at $N(x,\alpha)$ and $N(x,\alpha)$ is differentiable at a, then

$$\begin{aligned}\gamma^{\alpha}(N(a,\omega)) &= \lim_{x\to a}\frac{(\gamma^{\lceil\alpha\rceil-1}_{CN})(x) - (\gamma^{\lceil\alpha\rceil-1}_{CN})(a)}{x-a}\\ &= \lim_{x\to a}\frac{(\gamma^{\lceil\alpha\rceil-1})N(x,\alpha) - \gamma^{\lceil\alpha\rceil-1})N(a,\alpha)}{x-a}\\ &= \lim_{N(x,\alpha)\to N(a,\alpha)}\frac{(\gamma^{\lceil\alpha\rceil-1})(N(x)) - (\gamma^{\lceil\alpha\rceil-1})(N(a))}{N(x,\alpha) - N(a,\alpha)}\\ &\cdot \lim_{x\to a}\frac{N(x,\alpha) - N(a,\alpha)}{x-a}\\ &= \gamma^{\alpha}_{CN}(a)N'(a,\alpha).\end{aligned}$$

Theorem 7.*(Rolle's Theorem for Fractional Derivative). If $\gamma : [j,k] \to \mathbb{R}$, $j < k$, is α-differentiable, $\alpha \in (0,1]$, on (h,i) and α-continuous, $\alpha \in (0,1]$, on $[h,i]$, $N([h,i]) \subseteq [j,k]$, with $\gamma_N(h) = \gamma_N(i)$, then $\gamma^{\alpha}_{CN}(l) = 0$ for some $l \in (h,i)$.*

*Proof.*By Theorem 3 $\gamma_N : [h,i] \to \mathbb{R}$, $\alpha \in (0,1]$, has a finite minimum and maximum value, m and M respectively on $[h,i]$. If $m = M$, then γ_N is a constant on (h,i) and $\gamma^{\alpha}_{CN}(x) = 0$, $\forall x \in (h,i)$.If $m \neq M$, their either $\gamma_N(l) = m$ or $\gamma_N(l) = M$ for some $l \in (h,i)$ because $\gamma_N(h) = \gamma_N(i)$. Suposse $\gamma_N(l) = m$. Then $\gamma_N(h+\varepsilon) - \gamma_N(h) \geq 0$, $\forall\varepsilon$ such that $h+\varepsilon \in (h,i)$. Therefore,

$$\lim_{\varepsilon\to 0^+}\frac{\gamma_N(h+\varepsilon) - \gamma_N(h)}{\varepsilon} \geq 0,$$

if $\varepsilon > 0$, and

$$\lim_{\varepsilon\to 0^-}\frac{\gamma_N(h+\varepsilon) - \gamma_{CN}(h)}{\varepsilon} \leq 0,$$

if $\varepsilon < 0$. Thus,

$$\gamma^{\alpha}_{CN}(a) = 0.$$

Theorem 8.*(Mean Value Theorem for Fractional Derivative) If $\gamma : [j,k] \to \mathbb{R}$, is α-differentiable, $\alpha \in (0,1]$, on (h,i), $N([h,i]) \subseteq [j,k]$, and α-continuous, $\alpha \in (0,1]$, on $[h,i]$, then*

$$[N(i,\alpha) - N(h,\alpha)]\,\gamma^{\alpha}_{CN}(l) = [\gamma_N(i) - \gamma_N(h)]\,N'(l,\alpha),$$

for $l \in (h,i)$.

*Proof.*Let $r_N : [h,i] \to \mathbb{R}$ be defined by

$$r_N(x) = \gamma_N(x) - N(x,\alpha)\frac{\gamma_N(i)-\gamma_N(h)}{N(i,\alpha)-N(h,\alpha)}, \quad (8)$$

$\alpha \in (0,1]$. Then

$$r^{\alpha}_{CN}(x) = \gamma^{\alpha}_{CN}(x) - N'(x,\alpha)\frac{\gamma_N(i)-\gamma_N(h)}{N(i,\alpha)-N(h,\alpha)}. \quad (9)$$

Note that r is α-continuous, $\alpha \in (0,1]$, on $[h,i]$ and α-differentiable, $\alpha \in (0,1]$, on (h,i) with $r_{CN}(h) = r_{CN}(i)$. By Theorem 7 there is $l \in (h,i)$ such that $r^{\alpha}_{CN}(l) = 0$ and

$$\gamma^{\alpha}_{CN}(l) = N^{\alpha}(l,\alpha)\frac{\gamma_{CN}(i)-\gamma_{CN}(h)}{N(i,\alpha)-N(h,\alpha)}.$$

Definition 9.*Let $\gamma : [j,k] \to \mathbb{R}$ be an α-bounded function, $\alpha \in (0,1]$, on $[h,i]$, $N([h,i]) \subseteq [j,k]$, and let $P = \{x_d\}_{d=1}^{n}$ be a partition of $[h,i]$ such that $h = x_0 < x_1 < \cdots < x_n = i$. The CN-Riemann sum, $\alpha \in (0,1]$, of γ over P is defined by*

$$S_{CN}(\gamma,P) = \sum_{i=1}^{n} \gamma_{CN}(x_i^*)N'(x_i^*,\alpha)(x_i - x_{i-1}),$$

for any $x_i^* \in [x_{i-1},x_i]$.

Definition 10.*We sat that $\gamma : [j,k] \to \mathbb{R}$ is an CN-Riemann integrable function, $\alpha \in (0,1]$, on $[h,i]$, $N([h,i]) \subseteq [j,k]$, and a real number R_α is the 'a-definite integral, $\alpha \in (0,1]$, of γ over $[h,i]$ if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that for every partition $P = \{x_d\}_{d=1}^{n}$ of $[h,i]$ with $\|P\| = \max_{1\le i\le n}\{x_i - x_{i-1}\} < \delta$ and for any $x_i^* \in [x_{i-1},x_i]$, then*

$$\|S_{CN}(\gamma,P) - R_\alpha\| < \varepsilon$$

and we write it as

$$\alpha - \int_h^i \gamma(x)\,dx = R_\alpha = \int_h^i \gamma_{CN}(x)N'(x,\alpha)\,dx. \quad (10)$$

And we will denote to the set of all $\alpha-$Riemann integrable functions on $[h,i]$, $N([h,i]) \subseteq [j,k]$, by $\mathbf{R}_{RC}$.

*Remark.*The α-integral $\alpha - \int_h^i \gamma(x)\,dx$, $\alpha \in (0,1]$, in Definition 10 is the fractional Riemann integral of γ of order α.

Proposition 1.*Every Riemann integrable function is $\alpha-$Riemann integrable of order $\alpha = 1$ but no every $\alpha-$Riemann integrable function is Riemann integrable.*

Proposition 2.*Suppose that $\gamma : [j,k] \to \mathbb{R}$ is α_p-continuous and α_p-Riemann integrable, $\alpha_p \in (0,1]$, $p \in \mathbb{N}$, on $[h,i]$, $N([h,i]) \subseteq [j,k]$. If α_p converges to α, then γ is α-Riemann integrable on $[h,i]$.*

Theorem 9.*Let $\gamma_p : A \to \mathbb{R}$, $p \in \mathbb{N}$, be a sequence of α-continuous, α-Riemann integrable functions, $\alpha \in (0,1]$, on $[h,i]$, $N([h,i]) \subseteq [j,k]$. If $\gamma_p : A \to \mathbb{R}$, converges uniformly to $\gamma : [j,k] \to \mathbb{R}$ then γ is $\alpha-$integrable on $[h,i]$.*

Theorem 10.*If $\gamma \in \mathbb{R}_{RC}[h,i]$, then the α-Riemann integral, $\alpha \in (0,1]$, of γ is unique.*

*Proof.*Assume that J_1^{α} and J_2^{α} are α-Riemann integrals, $\alpha \in (0,1]$, of γ and let $\varepsilon > 0$ be given. Then for $j = 1,2$, there exists $\delta_j = \delta_j\left(\frac{\varepsilon}{2}\right) > 0$ such that

$$\|P\| < \delta_j \Longrightarrow \|S_\alpha(\gamma,p) - J_1^{\alpha}\| < \frac{\varepsilon}{2},$$

where P is any partition of $[h,i]$. Letting $\delta = min\{\delta_1,\delta_2\}$, we get

$$\begin{aligned} 0 \le \|J_1^{\alpha} - J_2^{\alpha}\| &= \|J_1^{\alpha} - S_\alpha(\gamma,p) + S_\alpha(\gamma,p) - J_2^{\alpha}\| \\ &= \|J_1^{\alpha} - S_\alpha(\gamma,p)\| + \|S_\alpha(\gamma,p) - J_2^{\alpha}\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since ε was arbitrary, then

$$0 \le \|J_1^{\alpha} - J_2^{\alpha}\| < \varepsilon$$

holds for all $\varepsilon > 0$. Thus,

$$\|J_1^{\alpha} - J_2^{\alpha}\| = 0,$$

and $J_1^{\alpha} = J_2^{\alpha}$.

Proposition 3.*The $\alpha-$definite integral, $\alpha \in (0,1]$, in (10) can be written as follows*

$$\alpha - \int_h^i \gamma(x)\,dx = \int_{N(h,\alpha)}^{N(i,\alpha)} \gamma(\mu)\,d\mu. \quad (11)$$

*Proof.*The proof is directly from the Theorem (Change of Variables for Continuous Integrands).

Now we use the formula (11) of the fractional integral to introduce the following theorems.

Theorem 11.*Let $\gamma,\lambda \in \mathbf{R}_{RC}[h,i]$. Then, for $\alpha \in (0,1]$,*

1. $\int_{N(h,\alpha)}^{N(i,\alpha)} \gamma(\mu)\,d\mu = -\int_{N(i,\alpha)}^{N(h,\alpha)} \gamma(\mu)\,d\mu.$
2. $\int_{N(h,\alpha)}^{N(h,\alpha)} \gamma(\mu)\,d\mu = 0.$
3. $\int_{N(h,\alpha)}^{N(i,\alpha)} r\gamma(\mu)\,d\mu = r\int_{N(h,\alpha)}^{N(i,\alpha)} \gamma(\mu)\,d\mu.$
4. $\int_{N(h,\alpha)}^{N(i,\alpha)} (\gamma(\mu) \pm \lambda(\mu))\,d\mu = \int_{N(h,\alpha)}^{N(i,\alpha)} \gamma(\mu)\,d\mu \pm \int_{N(h,\alpha)}^{N(i,\alpha)} \lambda(\mu)\,d\mu.$
5. $\int_{N(h,\alpha)}^{N(i,\alpha)} \gamma(\mu)\,d\mu + \int_{N(i,\alpha)}^{N(j,\alpha)} \gamma(\mu)\,d\mu = \int_{N(h,\alpha)}^{N(j,\alpha)} \gamma(\mu)\,d\mu.$
6. *If γ is $\alpha-$continuous on $[h,i]$, then*

$$\min_x \gamma_{CN}(x) \le \frac{1}{N(i,\alpha)-N(h,\alpha)}\int_{N(h,\alpha)}^{N(i,\alpha)} \gamma(\mu)\,d\mu \le \max_x \gamma_{CN}(x).$$

7. *If $\gamma_{CN}(x) \le \lambda_{CN}(x)$ on $[h,i]$, then*

$$\int_{N(h,\alpha)}^{N(i,\alpha)} \gamma(\mu)\,d\mu \le \int_{N(h,\alpha)}^{N(i,\alpha)} \lambda(\mu)\,d\mu.$$

Theorem 12.*(Mean Value Theorem for Fractional Definite Integrals)*

1. If $\gamma:[j,k]\to\mathbb{R}$ is $\alpha-$continuous, $\alpha\in(0,1]$, on $[h,i]$, $N([h,i])\subseteq[j,k]$, and N is continuously differentiable on $[h,i]$, then

$$\frac{d}{dN(x,\alpha)}\left[\int_{N(h,\alpha)}^{N(x,\alpha)}\gamma(\mu)\,d\mu\right]=\gamma_{RC}(x),$$

for each $x\in[h,i]$.

2. If $\gamma:[j,k]\to\mathbb{R}$ is $\alpha-$differentiable, $\alpha\in(0,1]$, on $[h,i]$, $N([h,i])\subseteq[j,k]$, γ' is $\alpha-$continuous on $[h,i]$ and N is continuously differentiable on $[h,i]$, then

$$\int_{N(h,\alpha)}^{N(i,\alpha)}\gamma'(\mu)\,d\mu=\gamma_{RC}(i)-\gamma_{RC}(h).$$

Proof. To prove part (1), for any x, $x+\Delta x\in[h,i]$, assume that $N(x,\Delta x,\alpha)=N(x,\alpha)+\Delta N(x,\alpha)$ and let

$$\Gamma_{CN}(x)=\int_{N(h,\alpha)}^{N(x,\alpha)}\gamma(\mu)\,d\mu$$

and

$$\Gamma_{CN}(x+\Delta x)=\int_{N(h,\alpha)}^{N(x,\alpha)+\Delta N(x,\alpha)}\gamma(\mu)\,d\mu,$$

for $\gamma_{CN}:[h,i]\to\mathbb{R}$, $\alpha\in(0,1]$. Substracting the last two equalities gives

$$\begin{aligned}\Gamma_{CN}(x+\Delta x)-\Gamma_{CN}(x)&=\int_{N(h,\alpha)}^{N(x,\alpha)+\Delta N(x,\alpha)}\gamma(\mu)\,d\mu\\&-\int_{N(h,\alpha)}^{N(x,\alpha)}\gamma(\mu)\,d\mu\\&=\int_{N(x,\alpha)}^{N(x,\alpha)+\Delta N(x,\alpha)}\gamma(\mu)\,d\mu.\end{aligned}$$

By Theorem 12, we have

$$\Gamma_{CN}(x+\Delta x)-\Gamma_{CN}(x)=\gamma_{CN}(a)\Delta N(x,\alpha),$$

for some point $a\in[h,i]$. Dividing both sides by $\Delta N(x,\alpha)$ gives

$$\frac{\Gamma_{CN}(x+\Delta x)-\Gamma_{CN}(x)}{\Delta N(x,\alpha)}=\gamma_{CN}(a).$$

Letting $\Delta N(x,\alpha)\to0$ $(\Delta x\to0)$ on both sides of the equation, we get

$$\begin{aligned}\frac{d}{dN(x,\alpha)}\Gamma(N(x,\alpha))&=\lim_{\Delta N(x,\alpha)\to0}\frac{\Gamma_{CN}(x+\Delta x)-\Gamma_{CN}(x)}{\Delta N(x,\alpha)}\\&=\lim_{\Delta N(x,\alpha)\to0}\gamma_{CN}(a)=\gamma_{CN}(x),\end{aligned}$$

where $\lim_{\Delta x\to0}a=x$ for the Squeeze Theorem. For part (2), let

$$\lambda_{CN}(x)=\int_{N(h,\alpha)}^{N(i,\alpha)}\gamma'(\mu)\,d\mu,$$

for $\gamma_{CN}:[h,i]\to\mathbb{R}$. By part (1), we have that $\lambda'_{CN}(x)=\gamma'_{CN}(x)$. Thus, there is a constant z such that $\lambda_{CN}(x)-\gamma_{CN}(x)=z$ for all $x\in[h,i]$. Since

$$\lambda_{CN}(h)=\int_{N(h,\alpha)}^{N(h,\alpha)}\gamma'(\mu)\,d\mu=0,$$

and $\lambda_{CN}(h)-\gamma_{CN}(h)=z$, then $z=-\gamma_{CN}(h)$ and $\lambda_{CN}(x)-\gamma_{CN}(x)=-\gamma_{CN}(h)$ for all $x\in[h,i]$. Moreover, since $\lambda_{CN}(i)-\gamma_{CN}(i)=-\gamma_{CN}(h)$, then $\lambda_{CN}(i)=\gamma_{CN}(i)-\gamma_{CN}(h)$.

3 Complements

Theorem 13. *Fractional Derivative of Certain Functions of Order α, $\alpha\in(0,1]$.*

a) $[x^n]_{CN}^{\alpha}=n\,[N(x,\alpha)]^{n-1}N'(x,\alpha)$.
b) $[e^{cx}]_{CN}^{\alpha}=ce^{cN(x,\alpha)}N'(x,\alpha)$.
c) $[ln(x)]_{CN}^{\alpha}=\frac{N'(x,\alpha)}{N(x,\alpha)}$.
d) $[a^x]_{CN}^{\alpha}=a^{N(x,\alpha)}ln(a)N'(x,\alpha)$.
e) $[sin(x)]_{CN}^{\alpha}=cos\,[N(x,\alpha)]\,N'(x,\alpha)$.
f) $[cos(x)]_{CN}^{\alpha}=-sin\,[N(x,\alpha)]\,N'(x,\alpha)$.

Proof. They are obtained directly from the Definition 5 and Theorem 4.

Below we illustrate the Chain Rule, presented in Theorem 6.

Example 1. Let $f(x)=sin^2x$.
Let's calculate the derivative by two paths:

i) By the Chain Rule

$$\begin{aligned}\left[sin^2x\right]_{CN}^{\alpha}&=\lambda'\left[\gamma_N(x)\right]\gamma_{CN}^{\alpha}(x)\\&=2sin\,[N(x,\alpha)]\,cos\,[N(x,\alpha)]\,N'(x,\alpha).\end{aligned}$$

ii) From Theorem 13

$$\begin{aligned}\left[sin^2x\right]_{CN}^{\alpha}&=\left[\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2\right]_{CN}^{\alpha}\\&=\left[\frac{e^{2ix}+e^{-2ix}-2}{-4}\right]_{CN}^{\alpha}\\&=-\frac{1}{4}\left[\left(e^{2ix}\right)_{CN}^{\alpha}+\left(e^{-2ix}\right)_{CN}^{\alpha}+(-2)_{CN}^{\alpha}\right]\\&=-\frac{1}{4}\left[2ie^{2iN(x,\alpha)}N'(x,\alpha)-2ie^{-2iN(x,\alpha)}N'(x,\alpha)\right]\\&=-\frac{i}{2}N'(x,\alpha)\left[e^{2iN(x,\alpha)}-e^{-2iN(x,\alpha)}\right]\\&=N'(x,\alpha)\left[\frac{e^{2iN(x,\alpha)}-e^{-2iN(x,\alpha)}}{2i}\right]\\&=N'(x,\alpha)\,sin\,(2N(x,\alpha))\\&=2sin\,[N(x,\alpha)]\,cos\,[N(x,\alpha)]\,N'(x,\alpha).\end{aligned}$$

Analogously, we have

$$\left[cos^2 x\right]_{CN}^{\alpha} = -2cos\left[N\left(x,\alpha\right)\right]sin\left[N\left(x,\alpha\right)\right]N'\left(x,\alpha\right).$$

*Example 2.*Let $f(x) = sin^n x$.

Let's calculate the derivative by two paths:

i)By the Chain Rule

$$\begin{aligned}[sin^n x]_{CN}^{\alpha} &= \lambda'\left[\gamma_N(x)\right]\gamma_{CN}^{\alpha}(x)\\ &= n\cdot sin^{n-1}\left[N\left(x,\alpha\right)\right]cos\left[N\left(x,\alpha\right)\right]N'\left(x,\alpha\right).\end{aligned}$$

ii)From Theorem 4

$$\begin{aligned}[sin^n x]_{CN}^{\alpha} &= \left[sin(x)sin^{n-1}x\right]_{CN}^{\alpha}\\ &= sin^{n-1}\left[N\left(x,\alpha\right)\right]\\ &\quad\cdot\ cos\left[N\left(x,\alpha\right)\right]N'\left(x,\alpha\right)\\ &\quad+ sin\left[N\left(x,\alpha\right)\right]\left[sin^{n-1}x\right]_{CN}^{\alpha}\\ sin\left[N\left(x,\alpha\right)\right]\left[sin^{n-1}x\right]_{CN}^{\alpha} &= sin^{n-1}\left[N\left(x,\alpha\right)\right]\\ &\quad\cdot\ cos\left[N\left(x,\alpha\right)\right]N'\left(x,\alpha\right)\\ &\quad+ sin^2\left[N\left(x,\alpha\right)\right]\left[sin^{n-2}x\right]_{CN}^{\alpha}\\ sin^2\left[N\left(x,\alpha\right)\right]\left[sin^{n-2}x\right]_{CN}^{\alpha} &= sin^{n-1}\left[N\left(x,\alpha\right)\right]\\ &\quad\cdot\ cos\left[N\left(x,\alpha\right)\right]N'\left(x,\alpha\right)\\ &\quad+ sin^3\left[N\left(x,\alpha\right)\right]\left[sin^{n-3}x\right]_{CN}^{\alpha}\\ &\ \ \vdots\\ sin^{n-1}\left[N\left(x,\alpha\right)\right]\left[sin(x)\right]_{CN}^{\alpha} &= sin^{n-1}\left[N\left(x,\alpha\right)\right]\\ &\quad\cdot\ cos\left[N\left(x,\alpha\right)\right]N'\left(x,\alpha\right)\\ &\quad+ sin^n\left[N\left(x,\alpha\right)\right]\left[sin^{n-n}x\right]_{CN}^{\alpha}.\end{aligned}$$

Adding member by member, we obtain

$$[sin^n x]_{CN}^{\alpha} = n\cdot sin^{n-1}\left[N\left(x,\alpha\right)\right]cos\left[N\left(x,\alpha\right)\right]N'\left(x,\alpha\right).$$

*Example 3.*Let $f(x) = sin(x^2)$, this is a differentiable function. So, from Theorem 4, we have

$$\left[sin(x^2)\right]_{CN}^{\alpha} = 2N\left(x,\alpha\right)\cdot N'\left(x,\alpha\right)cos\left[N^2\left(x,\alpha\right)\right].$$

By the Chain Rule we obtain,

$$\begin{aligned}\left[sin(x^2)\right]_{CN}^{\alpha} &= \lambda'\left[\gamma_N(x)\right]\gamma_{CN}^{\alpha}(x)\\ &= cos\left[N^2\left(x,\alpha\right)\right]2N\left(x,\alpha\right)\cdot N'\left(x,\alpha\right).\end{aligned}$$

4 Final Remarks

The essential question that can be derived from this work is the following: is the development of new local differential operators really important? Before providing our answer, we believe it is appropriate to point out a detail: in the Analysis of Several Variables, different notions of derivability and differentiability coexist, although the strongest notion is the latter. Why not accept that other notions of derivability can coexist?

The development of new local differential operators is important for several reasons:

- **Solving complex problems**: Differential operators are fundamental in the formulation and resolution of partial differential equations (PDEs), which model a wide variety of physical, chemical, biological and engineering phenomena. New operators may offer more precise or efficient ways to address these problems.
- **Improvement in numerical techniques**: In many cases, the problems described by EDPs do not have analytical solutions and must be solved numerically. New differential operators can improve the precision and stability of numerical methods, such as finite element or finite difference methods.
- **Advances in mathematical theory**: The development of new operators can lead to theoretical advances in mathematics, providing new tools and perspectives for research in functional analysis, distribution theory, and other related fields.
- **Interdisciplinary applications**: New differential operators can open opportunities in emerging fields and interdisciplinary applications. For example, in computational biology, advanced materials, image processing, and data sciences.
- **Optimization and control**: In engineering and applied sciences, differential operators are key in optimal control and optimization problems. New operators can improve optimization techniques, making them more robust and efficient.
- **Nonlinear and complex phenomena**: Many natural phenomena are inherently nonlinear and complex. New differential operators can be specifically designed to better capture these characteristics and provide more realistic models.
- **Technological innovation**: The development of new operators can lead to technological advances and new practical applications, from improvements in engineering designs to new techniques in medical diagnosis and treatment.

In summary, the new local differential operators are essential to advance both mathematical theory and its numerous practical applications, allowing increasingly complex and diverse problems to be addressed in a more effective and precise way.

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