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Surface family mate with Bertrand mate as mutual curvature lines in Galilean 3-space $\mathscr{G}_3$

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Abstract: The paper is an attempt to resolute the surface family mate ($\mathscr{SFM}$) with a Bertrand mate ($\mathscr{BM}$) as mutual curvature lines in Galilean 3-space $\mathscr{G}_3$. The $\mathscr{SFM}$ with the symmetry of $\mathscr{BM}$ can be specified as linear combinations of the components of the Serret–Frenet frames in $\mathscr{G}_3$. With these parametric representations, we resolved the indispensable and enough events for the specified $\mathscr{BM}$ to be the curvature lines on these surfaces. Afterword, the conclusion to ruled surface ($\mathscr{RS}$) is also gained.

Keywords: Isotropic normal, Bertrand mate, marching-scale functions.

1 Introduction

The curvature line ($\mathscr{CL}$) is curve which is permanently tangent to a principal direction. There will be two curvature lines throughout each non-umbilic point and the lines will meet at right angles. In fact, the $\mathscr{CL}$ is appointed as one of the considerable lines on a surface ($\mathscr{S}$) and it plays a paramount part in differential geometry and its achievements [1–4]. It is an beneficial device in $\mathscr{S}$ screening for displaying of the principal lines. The consistent curvature line and principal lines are fantastic and connected with the regular surfaces. Curvature lines can immediate the investigation of $\mathscr{S}$, placidly exercised in geometric design, and can appoint uniformity, the realization and the polygonization of $\mathscr{S}$. There exists a tremendous literature on the subject, including many monographs, for instance: Martin [5] defined methodical $\mathscr{S}$ patches limited by curvature lines, which are coined principal patches. He also offered that the turnout of such patches was contingently upon certain positions matching on the patch border lines. Alourdas et al. [6] initiated a style to emphasize a net curvature lines on a B-spline $\mathscr{S}$. Maekawa et al. [7] extended a manner to pick out the common distinctive of free-shape surfaces for derive examination. They examined the generic feature of the umbilic and attitude curvature lines that pass meanwhile an umbilic on a parametric free-shape $\mathscr{S}$. Che and Paul [8] developed a method to design and compute the curvature lines and their geometric monarchies pointed on an implicit $\mathscr{S}$. They also specified a novel gauge for non-umbilical and umbilical points on an implicit surface. Zhang et al. [9] proved a schema for enumerating and visualizing the curvature lines assigned on an implicit $\mathscr{S}$. Kalogerakis et al. [10] offered a robust construction for confirming curvature lines by point clouds. Their process is sensible for surfaces of random genus, with or without borderlines, and is statistically strong to utilize with outliers preserving $\mathscr{S}$ advantages. They proved the process to be active on an area of synthetic and real-world input data combinations via changing quantities of noise and outliers. In workable operations, however, pivotal work has concentrated on the backward reconnaissance or opposite issue: given a 3$\mathscr{D}$ curve, how can we pinpoint these ones surfaces that are countenanced with this curve as a characteristic curve, if feasible, rather than finding and providing curves on analytical curved surfaces?. Wang et al. [11] was the 1st to pick up the trouble of fabricating a $\mathscr{S}$ family with a nominated circumstantial geodesic line, by which each $\mathscr{S}$ can be a nominee for mode system. They expounded the indispensable and adequate situations for the coefficients to be fulfilled with both the isoparametric and the geodesic demands. This pattern has been utilized by numerous scholars [12–23].

Galilean 3-space $\mathscr{G}_3$ is the plainest fashion of a semi-Euclidean 3-space $\mathscr{E}_1^3$ for which the isotropic cone decreases to a plane. It is indicated as an edge from Euclidean space to special relativity. The major spine of $\mathscr{G}_3$ is its specific gravity, that is, it authorize the scholar to

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treat it in detail unaccompanied by employing massive amounts of energy and time . In other sides, the outlook of $\mathscr{G}_3$ shows its evolution with an humbles inquiry and a overall spread of a newfangled geometric society is pivotal for its effective resemblance with Euclidean 3-space $\mathscr{E}^3$. Further, complete growth is functional to provide the scholar with the psychological assurance of the constancy of the searched structure [24, 25]. In $\mathscr{G}_3$ the applicable action on surfaces with a curvature lines is rare. It is a practical and enchanting problem in workable employments, for example, Dede et al. [26, 27] analyzed tubular surfaces, the characterizations of the parallel surfaces. Yuzbasi et al. [28, 29] theorized a surface family ($\mathscr{SF}$) via a curve to be a mutual asymptotic and geodesic lines. Jiang et al. [30] meditated $\mathscr{SF}$ couple by $\mathscr{BM}$ couple as common asymptotic line. Almoneef and Abdel-Baky [31] prepared a surface family via $\mathscr{BM}$ to be geodesic line. AL-Jedani and Abdel-Baky [32] considered the $\mathscr{SF}$ and developable $\mathscr{SF}$ with a mutual geodesic curves, respectively. Alluhaibi and Abdel-Baky [33] investigated a similar idea but used the curvature lines instead of geodesic ones. In addition, a series of investigators, referred to as Li et al. and reported in [34–49], attitude theoretical works and promotions on soliton theory, submanifold theory, and other attached themes. Further mobilization can be found in these papers. Their attempts have considerable contribution to the advancement of research in these fields.

In this paper, we embrace a wonderful mode for resolving surfaces family with $\mathscr{BM}$ as mutual curvature lines. Given allowable curves, we foremost solve the question on the appropriate and indispensable events for the assigned curves to be curvature lines. In the execution of epilogue, the appropriate and indispensable events when the surfaces are ruled surfaces are also anatomized. Meantime, diverse curves are chosen to emphasize the pattern.

2 Basic concepts

The Galilean 3-space $\mathscr{G}_3$ is a Cayley–Klein geometry extended via the projective function of signature $(0,0,+,+)$ [24, 25]. The utter character of $\mathscr{G}_3$ concerning on the regulated set $\{\pi, \mathscr{L}, \mathscr{I}\}$, where π is the plane in the real 3-dimensional projective space $\mathscr{P}^3(\mathbb{R})$, $\mathscr{L}$ is the line (utter line) in π and $\mathscr{L}$ the steady elliptic detour of points of $\mathscr{L}$. Homogeneous coordinates in $\mathscr{G}_3$ are granted in such a pattern that π is specified by $w_0 = 0$, $\mathscr{L}$ by $w_0 = w_1 = 0$ and the points of $\mathscr{L}$ is assigned by $(0 : 0 : w_2 : w_3) \to (0 : 0 : w_3 : -w_2)$. A plane is coned Euclidean if it guaranty $\mathscr{L}$, in other respects it is coined isotropic, that is, planes z=const are Euclidean and so is the plane π. Other planes are isotropic. However, an isotropic plane does not include any isotropic direction.

For any $\mathbf{t} = (t_1, t_2, t_3)$, and $q = (q_1, q_2, q_3) \in \mathscr{G}_3$, their inner product is

$$< \mathbf{t}, \mathbf{e} > = \begin{cases} t_1 e_1, \text{ if } t_1 \neq 0 \vee e_1 \neq 0, \\ t_2 e_2 + t_3 e_3, \text{ if } t_1 = 0 \wedge e_1 = 0, \end{cases} \quad (1)$$

and their vectorial product is

$$\mathbf{t} \times \mathbf{e} = \begin{cases} \begin{vmatrix} \mathbf{0} & \mathbf{n}_2 & \mathbf{n}_3 \\ t_1 & t_2 & t_3 \\ e_1 & e_2 & e_3 \end{vmatrix}, \text{ if } t_1 \neq 0 \vee e_1 \neq 0, \\ \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ 0 & t_2 & t_3 \\ 0 & e_2 & e_3 \end{vmatrix}, \text{ if } t_1 = 0 \wedge e_1 = 0, \end{cases} \quad (2)$$

where $\mathbf{n}_1 = (1,0,0)$, $\mathbf{n}_2 = (0,1,0)$ and $\mathbf{n}_3 = (0,0,1)$ are the usual basis vectors in $\mathscr{G}_3$.

A curve $\varkappa(u) = (\varkappa_1(u), \varkappa_2(u), \varkappa_3(u))$; $u \in I \subseteq \mathbb{R}$,is coined permissible if it has no inflection points, that is, $\dot{\varkappa} \times \ddot{\varkappa} \neq \mathbf{0}$ and no isotropic tangents $\dot{\varkappa}_1 \neq 0$. A permissible (allowable) curve is a comparable of a regular curve in Euclidean space. For a permissible curve $\varkappa$: $I \subseteq \mathbb{R} \to \mathscr{G}_3$ assigned by the Galilean invariant arc-length ς, we have:

$$\varkappa(\varsigma) = (\varsigma, \varkappa_2(\varsigma), \varkappa_3(\varsigma)). \quad (3)$$

The curvature $\kappa(\varsigma)$ and torsion $\tau(\varsigma)$ of the curve $\varkappa(\varsigma)$ are

$$\kappa(\varsigma) = \left\| \varkappa''(\varsigma) \right\| = \sqrt{(\varkappa_2''(\varsigma))^2 + (\varkappa_3''(\varsigma))^2},$$
$$\tau(\varsigma) = \frac{1}{\kappa^2(\varsigma)} \det \left(\varkappa', \varkappa'', \varkappa''' \right). \quad (4)$$

Note that a permissible curve has $\kappa(\varsigma) \neq 0$. The $\mathscr{SFF}$ is:

$$\mathbf{t}_1(\varsigma) = \varkappa'(s) = \left(1, \varkappa_2'(\varsigma), \varkappa_3'(\varsigma)\right),$$
$$\mathbf{t}_2(\varsigma) = \frac{1}{\kappa(\varsigma)} \varkappa''(\varsigma) = \frac{1}{\kappa(\varsigma)} \left(0, \varkappa_2''(\varsigma), \varkappa_3''(\varsigma)\right),$$
$$\mathbf{t}_3(\varsigma) = \frac{1}{\tau(\varsigma)} (0, (\frac{1}{\kappa(\varsigma)} \varkappa_2''(\varsigma))', (\frac{1}{\kappa(\varsigma)} \varkappa_3''(\varsigma))'), \quad (5)$$

where $\mathbf{t}(\varsigma)$, $\mathbf{t}_2(\varsigma)$, and $\mathbf{t}_3(\varsigma)$, respectively, are the tangent, principal normal, and binormal vectors. For every point of $\varkappa(\varsigma)$, the Serret-Frenet formulae read:

$$\begin{bmatrix} \mathbf{t}_1' \\ \mathbf{t}_2' \\ \mathbf{t}_3' \end{bmatrix} = \begin{bmatrix} 0 & \kappa(\varsigma) & 0 \\ 0 & 0 & \tau(\varsigma) \\ 0 & -\tau(\varsigma) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix}. \quad (6)$$

The planes $\mathfrak{Sp}\{\mathbf{t}_1, \mathbf{t}_2\}$, $\mathfrak{Sp}\{\mathbf{t}_2, \mathbf{t}_3\}$, and $\mathfrak{Sp}\{\mathbf{t}_3, \mathbf{t}_1\}$, respectively, are coined the osculating plane, normal plane, and rectifying plane.

Definition 2.1 [18–21]. Let $\varkappa(\varsigma)$ and $\widehat{\varkappa}(\varsigma)$ be two permissible curves in $\mathscr{G}^3$, $\mathbf{t}_2(\varsigma)$ and $\widehat{\mathbf{t}}_2(\varsigma)$ are their principal normal vectors respectively, the pair $\{\widehat{\varkappa}(\varsigma),$

$\varkappa(\varsigma)\}$ is coined $\mathscr{BM}$ if $\mathbf{t}_2(\varsigma)$ and $\widehat{\mathbf{t}}_2(\varsigma)$ are linearly dependent at the conformable points, $\varkappa(\varsigma)$ is named the $\mathscr{BM}$ of $\widehat{\varkappa}(\varsigma)$, and

$$\widehat{\varkappa}(\varsigma) = \varkappa(\varsigma) + r\mathbf{t}_2(\varsigma). \tag{7}$$

where r is a steady. Therefore, the consortium of the $\mathscr{SFF}$ of $\varkappa(\varsigma)$ and $\widehat{\varkappa}(\varsigma)$ is:

$$\begin{bmatrix} \widehat{\mathbf{t}}_1 \\ \widehat{\mathbf{t}}_2 \\ \widehat{\mathbf{t}}_3 \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix}, \tag{8}$$

where β is a constant angle [50].

We tick a surface $\mathscr{M}$ in $\mathscr{G}_3$ by

$$\mathscr{M}: \mathbf{z}(\varsigma,t) = (z_1(\varsigma,t), z_2(\varsigma,t), z_3(\varsigma,t)), \ (\varsigma,t) \in \mathscr{D} \subseteq \mathbb{R}^2. \tag{9}$$

If $\mathbf{z}_j(\varsigma,t) = \frac{\partial \mathbf{z}}{\partial j}$, the isotropic surface normal is

$$\mathbf{u}(\varsigma,t) = \mathbf{z}_\varsigma \wedge \mathbf{z}_t, \text{ with } <\mathbf{z}_\varsigma, \mathbf{u}> = <\mathbf{z}_t, \mathbf{u}> = 0. \tag{10}$$

Theorem 2.1. A curve on a surface is a curvature line iff the surface normals on that curve originate a developable surface [1,2].

An iso-parametric curve is a curve $\varkappa(\varsigma)$ on a surface $\mathbf{z}(\varsigma,t)$ that has a constant ς or t-parameter value. In other words, there exists a parameter t_0 such that $\varkappa(\varsigma) = \mathbf{z}(\varsigma,t_0)$ or $\varkappa(t) = \mathbf{z}(\varsigma_0,t)$. Given a parametric curve $\varkappa(\varsigma)$, we call it an iso-principal line of the surface $\mathbf{z}(\varsigma,t)$ if it is both a principal line and a parameter curve on $\mathbf{z}(\varsigma,t)$.

3 Main results

This section prepares a modern side for making a $\mathscr{SFM}$ with a $\mathscr{BM}$ as mutual curvature lines in $\mathscr{G}_3$. For this objective, let $\widehat{\varkappa}(\varsigma)$ be a permissible curve, $\varkappa(\varsigma)$ be its $\mathscr{BM}$ and $\{\widehat{\kappa}(\varsigma), \widehat{\tau}(\varsigma), \widehat{\mathbf{t}}_1(\varsigma), \widehat{\mathbf{t}}_2(\varsigma), \widehat{\mathbf{t}}_3(\varsigma)\}$ is the $\mathscr{SFF}$ of $\widehat{\varkappa}(\varsigma)$ as in Eq. (6). The $\mathscr{SFM}$ with $\varkappa(\varsigma)$ can be assumed by

$$\mathscr{M}: \mathbf{z}(\varsigma,t) = \varkappa(\varsigma) + \mathfrak{a}(\varsigma,t)\mathbf{t}_1(\varsigma) + \mathfrak{b}(\varsigma,t)\mathbf{t}_2(\varsigma) + \mathfrak{c}(\varsigma,t)\mathbf{t}_3(\varsigma). \tag{11}$$

Likwise, the $\mathscr{SFM}$ with $\widehat{\varkappa}(\varsigma)$ is

$$\widehat{\mathscr{M}}: \widehat{\mathbf{z}}(\varsigma,t) = \widehat{\varkappa}(\varsigma) + \mathfrak{a}(\varsigma,t)\widehat{\mathbf{t}}_1(u) + \mathfrak{b}(\varsigma,t)\widehat{\mathbf{t}}_2(\varsigma) + \mathfrak{c}(\varsigma,t)\widehat{\mathbf{t}}_3(\varsigma), \tag{12}$$

where $\mathfrak{a}(\varsigma,t)$, $\mathfrak{b}(\varsigma,t)$, $\mathfrak{c}(\varsigma,t)$ are all $\mathscr{C}^1$ functions, and $0 \le t \le T$, $0 \le \varsigma \le L$. If the variable t is marked as the time, the functions $\mathfrak{a}(\varsigma,t)$, $\mathfrak{b}(\varsigma,t)$ and $\mathfrak{c}(\varsigma,t)$ can then be cleared as oriented marching spaces of a point at the t in the orientations $\widehat{\mathbf{t}}_1$; $\widehat{\mathbf{t}}_2$; and $\widehat{\mathbf{t}}_3$, respectively, and the vector $\widehat{\varkappa}(\varsigma)$ is explicate as the initialization of this point.

Our excitations is to deliberate adequate and needful events for $\varkappa(\varsigma)$ is an iso-parametric curvature line on $\mathscr{M}$.

Firstly, let's determine a unit vector $\mathbf{t}(\varsigma)$ such that $<\mathbf{t}, \mathbf{t}_1> = \mathbf{0}$, that is,

$$\mathbf{t}(\varsigma) = \cos\phi\mathbf{t}_2(\varsigma) + \sin\phi\mathbf{t}_3(\varsigma), \text{ with } \phi = \phi(\varsigma). \tag{13}$$

Suppose that

$$\mathbf{z}(\varsigma,t) = \varkappa(\varsigma) + t\mathbf{t}(\varsigma); \ \ t \in \mathbb{R}, \tag{14}$$

is a developable surface, that is,

$$\det(\varkappa', \mathbf{t}(\varsigma), \mathbf{t}'(\varsigma)) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \cos\phi \\ 0 & -\phi'\sin\phi - \tau\sin\phi & \phi'\cos\phi + \tau\cos\phi \end{vmatrix} = 0,$$

which leads to,

$$\phi'(\varsigma) + \tau(\varsigma) = 0 \Rightarrow \phi(\varsigma) = \phi_0(\varsigma)d\varsigma, \tag{15}$$

where $\phi_0 = \phi(\varsigma_0)$ and ς_0 is the premier amount of arc length.

Secondly, since $\varkappa(\varsigma)$ is an iso-parametric curve on $\mathscr{M}$ there exists a value $t = t_0$ such that $\varkappa(\varsigma) = \mathbf{z}(\varsigma,t_0)$. Then,

$$\mathfrak{a}(\varsigma,t_0) = \mathfrak{b}(\varsigma,t_0) = \mathfrak{c}(\varsigma,t_0) = 0,$$
$$\frac{\partial \mathfrak{a}(\varsigma,t_0)}{\partial \varsigma} = \frac{\partial \mathfrak{b}(\varsigma,t_0)}{\partial \varsigma} = \frac{\partial \mathfrak{c}(\varsigma,t_0)}{\partial \varsigma} = 0.$$

So,

$$\mathbf{u}(\varsigma,t_0) := \frac{\partial \mathbf{z}(\varsigma,t_0)}{\partial \varsigma} \times \frac{\partial \mathbf{z}(\varsigma,t_0)}{\partial t} = -\frac{\partial \mathfrak{c}(\varsigma,t_0)}{\partial t}\mathbf{t}_2(u) + \frac{\partial \mathfrak{b}(\varsigma,t_0)}{\partial t}\mathbf{t}_3(u), \tag{16}$$

is the isotropic surface normal. Further, via Theorem 2.1, $\varkappa(\varsigma)$ is a curvature line on $\mathscr{M}$ iff $\mathbf{u}(\varsigma) \parallel \mathbf{t}(\varsigma,t_0)$. Therefore, from Eqs. (13), and (16) there exists a function $\beta(\varsigma) \neq 0$ such that

$$-\frac{\partial \mathfrak{c}(\varsigma,t_0)}{\partial t} = \beta(\varsigma)\cos\phi, \ \ \frac{\partial \mathfrak{b}(\varsigma,t_0)}{\partial t} = \beta(\varsigma)\sin\phi, \tag{17}$$

where $\phi(\varsigma)$ is designated by Eq. (15). The functions $\beta(\varsigma)$ and $\phi(\varsigma)$ are controlling functions.

Hence, we give the following theorem.

Theorem 3.1. $\varkappa(\varsigma)$ is a curvature line on $\mathscr{M}$ iff

$$\left.\begin{array}{l} \mathfrak{a}(\varsigma,t_0) = \mathfrak{b}\varsigma,t_0) = \mathfrak{c}(\varsigma,t_0) = 0, \ \ 0 \le t_0 \le T, \ \ 0 \le \varsigma \le L, \\ -\frac{\partial \mathfrak{c}(\varsigma,t_0)}{\partial t} = \beta(\beta)\cos\phi, \ \ \frac{\partial \mathfrak{b}(\varsigma,t_0)}{\partial t} = \beta(\varsigma)\sin\phi, \ \beta(\varsigma) \neq 0, \\ \phi(\varsigma) = \phi_0(\varsigma)du, \ \phi_0 = \phi(\varsigma_0), \text{ with } \phi_0 = \phi(\varsigma_0). \end{array}\right\} \tag{18}$$

Any surface $\mathscr{M}: \mathbf{z}(\varsigma,t)$ recognized by Eq. (11) and matching Theorem 3.1 is a member of the $\mathscr{SFM}$. As cited in [8], for the objective of determination and experimentation, we also check the issue when $\mathfrak{a}(\varsigma,t)$, $\mathfrak{b}(\varsigma,t)$ and $\mathfrak{c}(\varsigma,t)$ can be recognized by

$$\mathfrak{a}(\varsigma,t) = \mathfrak{l}(\varsigma)\mathfrak{a}(t), \ \mathfrak{b}(\varsigma,t) = \mathfrak{m}(\varsigma)\mathfrak{b}(t), \ \mathfrak{c}(\varsigma,t) = \mathfrak{n}(\varsigma)\mathfrak{c}(t). \tag{19}$$

Here $\mathfrak{l}(\varsigma)$, $\mathfrak{m}(\varsigma)$, $\mathfrak{n}(\varsigma)$, $\mathfrak{a}(t)$, $\mathfrak{b}(t)$, and $\mathfrak{c}(t)$ are $\mathfrak{c}^1$ functions are not identically vanish. Then, from Theorem 3.1, we gain:

Corollary 3.1. $\varkappa(\varsigma)$ is a curvature line on $\mathscr{M}$ iff

$$\left.\begin{array}{l} \mathfrak{a}(t_0)=\mathfrak{b}(t_0)=\mathfrak{c}(t_0)=0,\ \ 0\le t_0\le T,\ \ 0\le u\le L, \\ -\mathfrak{n}(\varsigma)\frac{d\mathfrak{c}(t_0)}{dt}=\beta(\varsigma)\cos\phi,\ \ \mathfrak{m}(\varsigma)\frac{d\mathfrak{b}(t_0)}{dt}=\beta(\varsigma)\sin\phi. \\ \phi(\varsigma)=\phi_0(\varsigma)d\varsigma,\ \phi_0=\phi(\varsigma_0),\ \text{with}\ \phi_0=\phi(\varsigma_0). \end{array}\right\} \tag{20}$$

Nevertheless, we can write that $\mathfrak{a}(\varsigma,t)$, $\mathfrak{b}(\varsigma,t)$ and $\mathfrak{c}(\varsigma,t)$ depend only on the variable t; that is, $\mathfrak{l}(\varsigma)=m(\varsigma)=n(\varsigma)=1$. Then, we address the Eqs. (21) via the various expressions of $\phi(\varsigma)$ as follows:
(i) If $\tau(\varsigma)\neq 0$, then $\phi(\varsigma)$ is a non-steady function of ς and the Eqs. (21) can be recognized by

$$\left.\begin{array}{c} \mathfrak{a}(t_0)=\mathfrak{b}(t_0)=\mathfrak{c}(t_0)=0, \\ -\frac{d\mathfrak{c}(t_0)}{dt}=\beta(\varsigma)\cos\phi,\ \frac{d\mathfrak{b}(t_0)}{dt}=\beta(\varsigma)\sin\phi. \end{array}\right\} \tag{21}$$

(ii) If $\tau(\varsigma)=0$, that is, $\varkappa(\varsigma)$ is a planar curve, then $\phi(\varsigma)=\phi_0$ is a constant and we have:
(a) If $\phi_0\neq 0$, the Eqs. (21) can be recognized by

$$\left.\begin{array}{c} \mathfrak{a}(t_0)=\mathfrak{b}(t_0)=\mathfrak{c}(t_0)=0, \\ -\frac{d\mathfrak{c}(t_0)}{dt}=\beta(\varsigma)\cos\phi_0,\ \frac{d\mathfrak{b}(t_0)}{dt}=\beta(\varsigma)\sin\phi_0. \end{array}\right\} \tag{22}$$

(b) If $\phi_0=0$, the Eqs. (20) can be recognized by

$$\left.\begin{array}{c} \mathfrak{a}(t_0)=\mathfrak{b}(t_0)=\mathfrak{c}(t_0)=0, \\ -\frac{d\mathfrak{c}(t_0)}{dt}=\beta(\varsigma),\ \frac{d\mathfrak{b}(t_0)}{dt}=0, \end{array}\right\} \tag{23}$$

and from Eqs. (16), and (17) we have $\mathbf{u}(\varsigma,t_0)\parallel t_2$. In this position, $\varkappa$=$\varkappa(\varsigma)$ is not only a curvature line but also a geodesic. We also let $\{\widehat{\mathscr{M}},\mathscr{M}\}$ to reference the $\mathscr{SFM}$ with $\{\widehat{\varkappa}(\varsigma),\ \varkappa(\varsigma)\}$ as mutual curvature lines. In the distinctive case in Eq. (8) if $\beta=0$, and $\beta=\pi/2$ then the $\mathscr{SFM}$ are coined parallel mate and right mate, respectively.

Example 3.1. Let $\varkappa(\varsigma)$ be

$$\varkappa(\varsigma)=(\varsigma,\sin\varsigma,\cos\varsigma),\ 0\le\varsigma\le 2\pi.$$

Then,

$$\begin{aligned} \varkappa'(\varsigma)&=(1,\cos\varsigma,-\sin\varsigma),\ \varkappa''(\varsigma)\\ &=(0,-\sin\varsigma,-\cos\varsigma),\ \varkappa'''(\varsigma)\\ &=(0,-\cos\varsigma,\sin\varsigma). \end{aligned}$$

In view of Eqs. (2), (3), (4), (5) we gain $\kappa(\varsigma)=-\tau(\varsigma)=1$ and

$$\begin{aligned} \mathbf{t}_1(\varsigma)&=(1,\cos\varsigma,-\sin\varsigma),\\ \mathbf{t}_2(\varsigma)&=(0,-\sin\varsigma,-\cos\varsigma),\\ \mathbf{t}_3(\varsigma)&=(0,\cos\varsigma,-\sin\varsigma). \end{aligned}$$

Then $\phi(\varsigma)=-\varsigma+\phi_0$. If $\phi_0=0$, we acquire $\phi(\varsigma)=-\varsigma$. For

$$\begin{aligned} &\mathfrak{l}(\varsigma)=\mathfrak{m}(\varsigma)=\mathfrak{n}(\varsigma)=1,\\ &\mathfrak{a}(t)=t,\ \mathfrak{b}(t)=-t\beta(\varsigma)\sin\varsigma,\ \ \mathfrak{c}(t)=-t\beta(\varsigma)\cos\varsigma,\ \beta(\varsigma)\neq 0. \end{aligned}$$

The $\mathscr{SFM}$ with $\varkappa(\varsigma)$ is

$$\mathscr{M}:\mathbf{z}(\varsigma,t)=\ (\varsigma,\sin\varsigma,\cos\varsigma)+t(1,-\beta\sin\varsigma,-\beta\cos\varsigma) \times\begin{pmatrix} 1 & \cos\varsigma & -\sin\varsigma \\ 0 & -\sin\varsigma & -\cos\varsigma \\ 0 & \cos\varsigma & -\sin\varsigma \end{pmatrix}.$$

The $\mathscr{SFM}$ with $\widehat{\varkappa}(\varsigma)$ as mutual curvature line is as follows: Let $r=2$ in Eq. (7), we derive $\widehat{\varkappa}(\varsigma)=(\varsigma,-\sin\varsigma,-\cos\varsigma)$. The Serret-Frenet vectors of $\widehat{\varkappa}(\varsigma)$ are

$$\begin{aligned} \widehat{\mathbf{t}}_1(\varsigma)&=(\cos\beta,(\cos\beta+\sin\beta)\cos\varsigma,-(\cos\beta+\sin\beta)\sin\varsigma),\\ \widehat{\mathbf{t}}_2(\varsigma)&=(0,-\sin\varsigma,-\cos\varsigma),\\ \widehat{\mathbf{t}}_3(\varsigma)&=(-\sin\beta,(\cos\beta-\sin\beta)\cos\varsigma,(\sin\beta-\cos\beta)\sin\varsigma. \end{aligned}$$

Then,

$$\widehat{\mathscr{M}}:\widehat{\mathbf{z}}(\varsigma,t)=(\varsigma,-\sin\varsigma,-\cos\varsigma)+t(1,-\beta\sin\varsigma,-\beta\cos\varsigma) \times\begin{bmatrix} \cos\beta & (\cos\beta+\sin\beta)\cos\varsigma & -(\cos\beta-\sin\beta)\sin\varsigma \\ 0 & -\sin\varsigma & -\cos\varsigma \\ -\sin\beta & (\cos\beta-\sin\beta)\cos\varsigma & (\sin\beta-\cos\beta)\sin\varsigma \end{bmatrix}.$$

By $\beta(\varsigma)=1$, $-1.5\le t\le 1.5$, $0\le\varsigma\le 2\pi$, then Eq. (19) is displaced. The parallel $\mathscr{SFM}$ is exhibited in Figure 1. Figure 2 exhibit the right $\mathscr{SFM}$. The blue curve draws $\varkappa(\varsigma)$ on $\mathscr{M}$ and the green curve is $\widehat{\varkappa}(\varsigma)$ on $\widehat{\mathscr{M}}$.

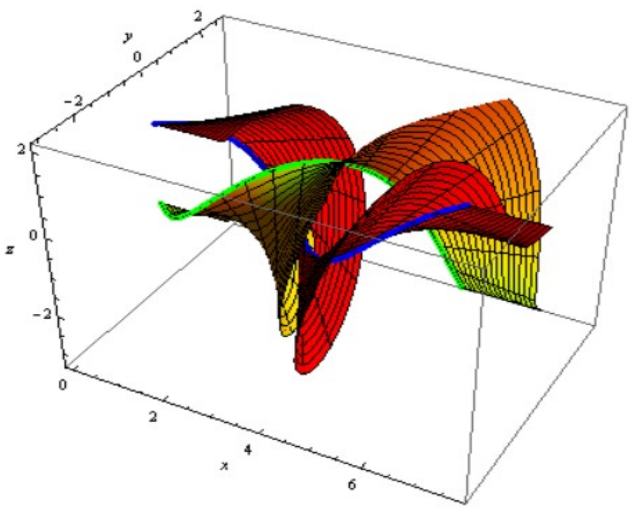

Fig. 1: Parallel $\mathscr{SFM}$.

Example 3.2. Let

$$\varkappa(\varsigma)=(\varsigma,1+\sin\varsigma,\sin\varsigma),\ \ 0\le\varsigma\le 2\pi.$$

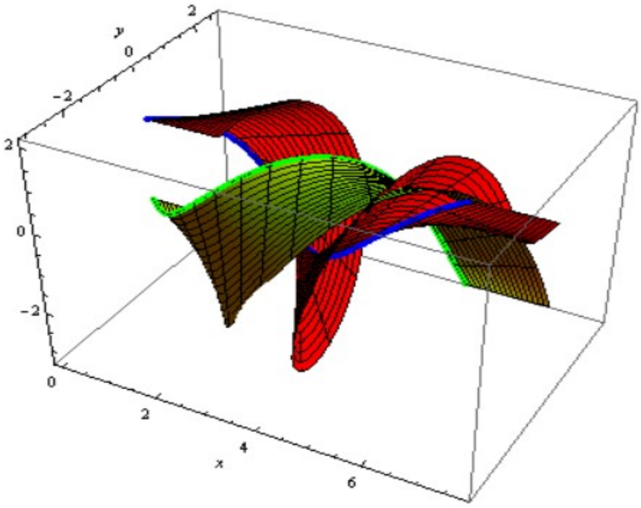

Fig. 2: Right $\mathscr{SFM}$.

Then,

$$\mathbf{t}_1(\varsigma) = (1, \cos\varsigma, \cos\varsigma),\ \mathbf{t}_2(\varsigma) = (0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}),$$
$$\mathbf{t}_3(\varsigma) = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}),$$

with $\kappa(\varsigma) = \sqrt{2}\sin\varsigma$ and $\tau(\varsigma) = 0$ which shows that $\phi(\varsigma) = \phi_0$ is a steady. For

$$\mathfrak{l}(\varsigma) = \mathfrak{m}(\varsigma) = \mathfrak{n}(\varsigma) = 1,$$
$$\mathfrak{a}(t) = t,\ \mathfrak{b}(t) = t\beta(\varsigma)\sin\phi_0,$$
$$-\mathfrak{c}(t) = t\beta(\varsigma)\cos\phi_0,\ \beta(\varsigma) \neq 0.$$

The $\mathscr{SFM}$ on $\varkappa(\varsigma)$ is

$$\mathscr{M} : \mathbf{z}(\varsigma,t) = (\varsigma, 1+\sin\varsigma, \sin\varsigma) + t(1, -\beta(\varsigma)\sin\phi_0, \beta\cos\phi_0) \times \begin{bmatrix} 1 & \cos\varsigma & \cos\varsigma \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Likwise, let $r = \sqrt{2}$ in Eq. (7), we acquire $\widehat{\varkappa}(\varsigma) = (\varsigma, \sin\varsigma, \sin\varsigma - 1)$, and

$$\widehat{\mathbf{t}}_1(\varsigma) = (\cos\beta, \cos\beta\cos\varsigma + \frac{1}{\sqrt{2}}\sin\beta, \cos\beta\cos\varsigma - \frac{1}{\sqrt{2}}\sin\beta),$$
$$\widehat{\mathbf{t}}_2(\varsigma) = (0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}),$$
$$\widehat{\mathbf{t}}_3(\varsigma) = (-\sin\beta, -\sin\beta\cos\varsigma - \frac{1}{\sqrt{2}}\sin\beta, -\sin\beta\cos\varsigma - \frac{1}{\sqrt{2}}\sin\beta).$$

Comparably, we have

$$\widehat{\mathscr{M}} : \widehat{\mathbf{z}}(\varsigma,t) = (\varsigma, \sin\varsigma, \sin\varsigma - 1) + t(1, -\beta(\varsigma)\sin\phi_0, \beta(\varsigma)\cos\phi_0) \times \begin{bmatrix} \cos\beta & \cos\beta\cos\varsigma + \frac{1}{\sqrt{2}}\sin\beta & \cos\beta\cos\varsigma - \frac{1}{\sqrt{2}}\sin\beta \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\sin\beta & -\sin\beta\cos\varsigma - \frac{1}{\sqrt{2}}\sin\beta & -\sin\beta\cos\varsigma - \frac{1}{\sqrt{2}}\sin\beta \end{bmatrix}.$$

By $\beta(\varsigma) = 1$, $\phi_0 = \pi/4$, $-2 \leq t \leq 2$ and $0 \leq \varsigma \leq 2\pi$, then Eq. (19) is displaced. The parallel $\mathscr{SFM}$ is exhibited in Figure 3. Figure 4 exhibit the right $\mathscr{SFM}$. The blue curve draws $\varkappa(\varsigma)$ on $\mathscr{M}$ and the green curve is $\widehat{\varkappa}(\varsigma)$ on $\widehat{\mathscr{M}}$.

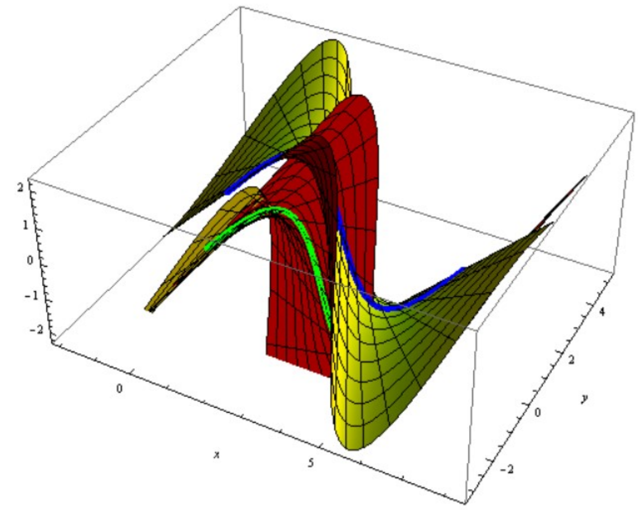

Fig. 3: Parallel $\mathscr{SFM}$.

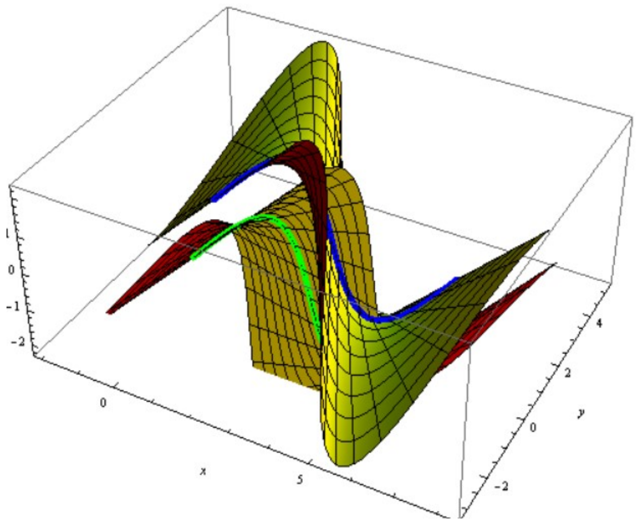

Fig. 4: Parallel $\mathscr{SFM}$.

3.1 $\mathscr{RSFM}$ with $\mathscr{BM}$ as common curvature lines

Suppose $\mathscr{M}_i : \mathbf{z}_i(\varsigma,t)$ is a $\mathscr{RS}$ with the directrix $\varkappa_i(\varsigma)$ and $\varkappa_i(\varsigma)$ is also an iso-parametric curve of $\mathbf{z}_i(\varsigma,t)$, then there exists t_0 such that $\mathbf{z}_i(\varsigma,t_0) = \varkappa_i(\varsigma)$. This shows that:

$$\mathscr{M}_i: \ \mathbf{z}_i(\varsigma,t) - \mathbf{z}_i(\varsigma,t_0) = (t-t_0)\mathbf{g}_i(\varsigma), \quad 0 \le \varsigma \le L, \text{ with } t,\ t_0 \in [0,T], \tag{24}$$

where $\mathbf{g}_i(\varsigma)(i = 1,2,3)$ demonstrates the direction along the rulings. In view of the Eq. (11), we earn

$$(t-t_0)\mathbf{g}_i(\varsigma) = \mathfrak{a}(\varsigma,t)\mathbf{t}_{1i}(\varsigma)+\mathfrak{b}(\varsigma,t)\mathbf{t}_{2i}(\varsigma)+\mathfrak{c}(\varsigma,t)\mathbf{t}_{3i}(\varsigma), \tag{25}$$

where $0 \le \varsigma \le L$, with $t,\ t_0 \in [0,T]$. In fact, Eq. (24) is a planning of equations with 3-unknown $\mathfrak{a}(\varsigma,t)$, $\mathfrak{b}(\varsigma,t)$, and $\mathfrak{c}(\varsigma,t)$. The decisions can be infrared as

$$\begin{array}{l} \mathfrak{a}(\varsigma,t) = (t-t_0) < \mathbf{g}_i(\varsigma), \mathbf{t}_{1i}(\varsigma) >, \\ \mathfrak{b}(\varsigma,t) = (t-t_0) < \mathbf{g}_i(\varsigma), \mathbf{t}_{2i}(\varsigma) >, \\ \mathfrak{c}(\varsigma,t) = (t-t_0) < \mathbf{g}_i(\varsigma), \mathbf{t}_{3i}(\varsigma) > . \end{array} \tag{26}$$

Via Eqs. (17), if $\varkappa(\varsigma)$ is a curvature line on $\mathscr{M}_i$, we acquire

$$\begin{array}{r} \beta(\varsigma)\sin\phi =< \mathbf{g}_i(\varsigma), \mathbf{t}_{2i}(\varsigma) >, \\ -\beta(\varsigma)\cos\phi =< \mathbf{g}_i(\varsigma), \mathbf{t}_{3i}(\varsigma) > . \end{array} \tag{27}$$

The above equations are distinctly the needful and enough events for which $\mathscr{M}_i$ is a $\mathscr{RS}$ with a directrix $\varkappa_i(\varsigma)$; $i = 1,2,3$.

In $\mathscr{G}_3$, there exist only 3-types of ruled surfaces demonstrated as follows [22,23]:
Type $\mathscr{I}$. Non-conoidal or conoidal $\mathscr{RS}$ for which the wrist (striction) curve does not lie in an Euclidean 2-plane $\mathscr{E}^2$.
Type $\mathscr{II}$. $\mathscr{RS}$ for which the wrist curve $\in \mathscr{E}^2$.
Type $\mathscr{III}$. Conoidal $\mathscr{RS}$ for which the absolute line is oriented line in infinity.

We promptly search if $\varkappa_i(\varsigma)$ is a principal line on these 3-types:
Type $\mathscr{I}$. $\varkappa_1(\varsigma) = (\varsigma, \varkappa_2(\varsigma), \varkappa_3(\varsigma))$ does not lie in $\mathscr{E}^2$ and $\mathbf{g}_1(\varsigma) = (1, g_2(\varsigma), g_3(\varsigma))$ is non-isotropic. Then,

$$\begin{aligned} \mathbf{t}_{11}(\varsigma) &= \left(1, \varkappa_2'(\varsigma), \varkappa_3'(\varsigma)\right), \\ \mathbf{t}_{21}(\varsigma) &= \frac{1}{\kappa(\varsigma)}\left(0, \varkappa_2''(\varsigma), \varkappa_3''(\varsigma)\right), \\ \mathbf{t}_{31}(\varsigma) &= \frac{1}{\kappa(\varsigma)}\left(0, -\varkappa_3''(\varsigma), \varkappa_2''(\varsigma)\right), \end{aligned} \tag{28}$$

where $\kappa(\varsigma) = \sqrt{\left(\varkappa_2''(\varsigma)\right)^2 + \left(\varkappa_3''(\varsigma)\right)^2}$. From Eqs. (1), (26), and (28), we find:

$$\mathfrak{a}(\varsigma,t) = (t-t_0),\ \mathfrak{b}(\varsigma,t) = \mathfrak{c}(\varsigma,t) = 0, \tag{29}$$

which does not accomplish Theorem 3.1.
Type $\mathscr{II}$. $\varkappa_2(\varsigma) = (0, \varkappa_2(\varsigma)), z(\varsigma))$ in $\mathscr{E}^2$ and $\mathbf{g}_2(u) = (1, g_2(u), g_3(u))$ is non-isotropic. Then,

$$\begin{aligned} \mathbf{t}_{12}(\varsigma) &= \left(0, \varkappa_2'(\varsigma), \varkappa_3'(\varsigma)\right), \\ \mathbf{t}_{22}(\varsigma) &= \frac{1}{\kappa(\varsigma)}\left(0, \varkappa_2''(\varsigma), \varkappa_3''(\varsigma)\right), \\ \mathbf{t}_{32}(\varsigma) &= \frac{1}{\kappa(\varsigma)}(0,0,0), \end{aligned} \tag{30}$$

where $\kappa(\varsigma) = \sqrt{\left(\varkappa_2''(\varsigma)\right)^2 + \left(\varkappa_3''(\varsigma)\right)^2}$. From Eqs. (1), (26), and (30), we find:

$$\mathfrak{a}(\varsigma,t) = \mathfrak{b}(\varsigma,t) = \mathfrak{c}(\varsigma,t) = 0, \tag{31}$$

which does not accomplish Theorem 3.1.

Corollary 3.2. In $\mathscr{G}_3$, there are no $\mathscr{RSFM}$ of Type $\mathscr{I}$ and Type $\mathscr{II}$ with $\mathscr{BM}$ as mutual curvature lines.
Type $\mathscr{III}$. Let $\varkappa_3(\varsigma) = (\varsigma, \varkappa_2(\varsigma), 0) \notin \mathscr{E}^2$ and $\mathbf{g}_3(\varsigma) = (0, g_2(\varsigma), g_3(\varsigma))$ is non-isotropic. Then,

$$\begin{aligned} \mathbf{t}_{13}(\varsigma) &= \left(1, \varkappa_2'(\varsigma), 0\right), \\ \mathbf{t}_{23}(\varsigma) &= \frac{1}{\kappa(\varsigma)}\left(0, \varkappa_2''(\varsigma)0\right), \\ \mathbf{t}_{33}(\varsigma) &= \frac{1}{\kappa(u)}\left(0, 0, \varkappa_2''(\varsigma)\right), \end{aligned} \tag{32}$$

where $\kappa(\varsigma) = \sqrt{\left(\varkappa_2''(\varsigma)\right)^2}$. From Eqs. (1), (26), and (31), we possess:

$$\left.\begin{array}{l} \mathfrak{a}(\varsigma,t) = 0,\ \mathfrak{b}(\varsigma,t) = \varepsilon(t-t_0)g_2(\varsigma), \\ \mathfrak{c}(\varsigma,t) = \varepsilon(t-t_0)g_3(\varsigma), \\ g_2(\varsigma) \neq 0,\ g_3(\varsigma) \neq 0,\ t_0 \neq 0, \end{array}\right\} \tag{33}$$

where

$$\varepsilon = \begin{cases} 1, \text{ if } \varkappa_2''(\varsigma) > 0. \\ -1, \text{ if } \varkappa_2''(\varsigma) < 0. \end{cases} \tag{34}$$

Eq. (33) accomplish Theorem 3.1. Suppose at each point on $\varkappa_3(\varsigma)$ the ruling $\mathbf{g}_3(\varsigma) \in Sp\{\mathbf{t}_{13}(\varsigma), \mathbf{t}_{23}(\varsigma), \mathbf{t}_{33}(\varsigma)\}$, then

$$\mathbf{g}_3(\varsigma) = \lambda(\varsigma)\mathbf{t}_{13}(\varsigma) + \sigma(\varsigma)\mathbf{t}_{23}(\varsigma) + \mu(\varsigma)\mathbf{t}_{33}(\varsigma), \tag{35}$$

for some functions $\lambda(\varsigma)$, $\sigma(\varsigma)$, and $\mu(\varsigma)$. Using it into the Eqs. (27), we bring

$$\sigma(\varsigma) = \beta(\varsigma)\sin\phi,\ \mu(\varsigma) = -\beta(\varsigma)\cos\phi,\ \lambda(\varsigma) = 0. \tag{36}$$

Then,

$$\mathbf{g}_3(\varsigma) = \beta(\varsigma)\sin\phi\,\mathbf{t}_{23}(\varsigma) - \beta(\varsigma)\cos\phi\,\mathbf{g}_{33}(\varsigma). \tag{37}$$

So, the $\mathscr{RSFM}$ of type $\mathscr{M}_3$ can be allocated as

$$\mathbf{z}_3(\varsigma,t) = \ \varkappa_3(\varsigma) + t\lambda\mathbf{t}_{13} + t\beta(\varsigma)(\sin\phi\,\mathbf{t}_{23}(\varsigma) - \cos\phi\,\mathbf{t}_{33}(\varsigma)), \quad 0 \le \varsigma \le L,\ 0 \le t \le T.$$

Then $\mathscr{RSFM}$ of Type $\widehat{\mathscr{M}_3}$ can be allocated as

$$\widehat{\mathbf{z}}_3(\varsigma,t) = \widehat{\varkappa}_3(\widehat{\varsigma}) + t\lambda\widehat{\mathbf{t}}_{13} + t\beta(\varsigma)(\sin\phi\widehat{\mathbf{t}}_{23}(\varsigma) - \cos\phi\widehat{\mathbf{t}}_{33}(\varsigma)),$$
$$0 \le \varsigma \le L,\ 0 \le t \le T.$$

The functions $\lambda(\varsigma)$ and $\beta(\varsigma)$ can control the shape of $\mathscr{M}_3$ and $\widehat{\mathscr{M}_3}$

Example 3.3. Via Example 3.1, we have:

$$\mathscr{M}_3 : \mathbf{z}_3(\varsigma,t) = (\varsigma + \lambda t, \sin\varsigma + t(\lambda\cos\varsigma - \beta), \cos\varsigma - t(\lambda\sin\varsigma + \beta\sin 2\varsigma))$$

and

$$\widehat{\mathscr{M}_3} : \widehat{\mathbf{z}}_3(\varsigma,t) = \begin{bmatrix} \varsigma + t(\lambda\cos\beta - \beta\sin\beta\cos\varsigma) \\ -\sin\varsigma + t[\lambda(\cos\beta + \sin\beta)\cos\varsigma - \beta\sin^2\varsigma - \beta\cos^2\varsigma(\cos\beta - \sin\beta)] \\ -\cos\varsigma + t[(-\lambda(\cos\beta + \sin\beta)\sin\varsigma - \beta\sin 2\varsigma(1 + (\sin\beta - \cos\beta)] \end{bmatrix}.$$

By making $\lambda(\varsigma) = \beta(\varsigma) = \varsigma$ and $-0.3 \le t \le 0.3$, $0 \le \varsigma \le 2\pi$, the parallel $\mathscr{SFM}$ is exhibited in Figure 5. Figure 6 exhibit the right $\mathscr{SFM}$. The blue curve is on $\mathscr{M}$ and the green curve is on $\widehat{\mathscr{M}}$.

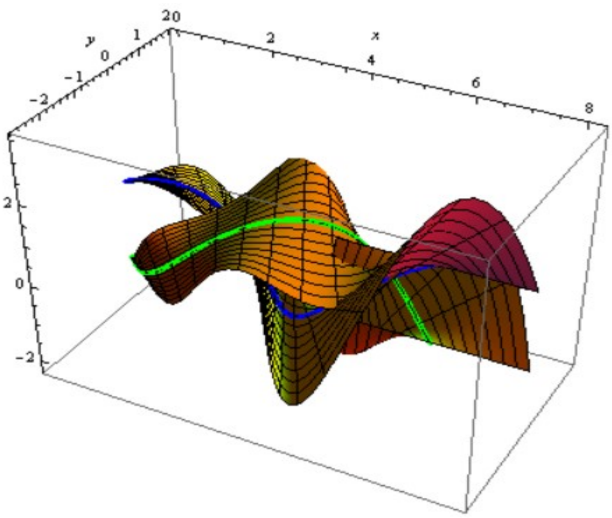

Fig. 5: Parallel $\mathscr{RSFM}$.

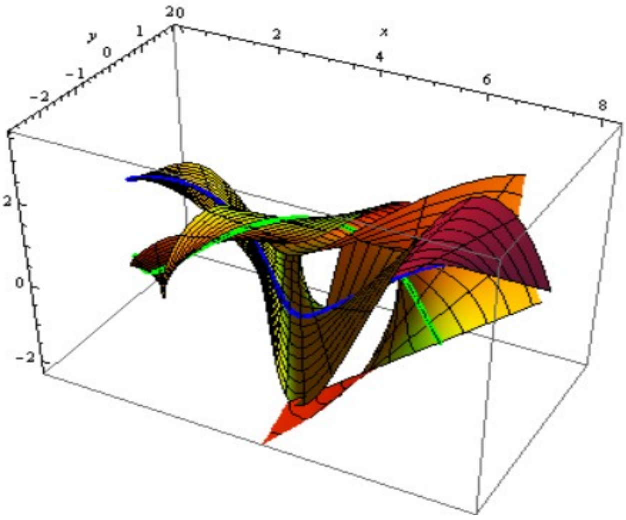

Fig. 6: Right $\mathscr{RSFM}$.

4 Conclusion

This paper considered curvature lines and their related surfaces in Galilean 3-space. Given a 3D curve, we demand surfaces that are privileged with this curve as a characteristic curve. The article, in different manner than thus in [33], displayed the $\mathscr{BM}$ as curvature lines and obtains a $\mathscr{SFM}$ with a $\mathscr{BM}$ as common curvature lines. Then, adequate and needful events for an permissible curve to be an iso-parametric curvature line on $\mathscr{S}$ are latterly extracted in this case. Kinds of the $\mathscr{SFM}$ with a $\mathscr{BM}$ as mutual curvature lines are plotted. Our consequences in this article participate to the works cited in [26–33]. We aim that the results will be helpful for investigators operating on production evolution procedure in the manufacturing industry. The discernment of align the techniques used here to numerous spaces such as general relativity theory, pseudo-Galilean space, and Heisenberg space is already an investigation topic. We will discuss this problem in the future.

Conflict of Interest

The authors declare that they have no conflict.

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